Applications and Applied Mathematics: An International Journal (AAM)

Volume 15 | Issue 2 | Article 7

12-2020

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 830 - 845

The Odd Inverse Rayleigh Family of Distributions: Simulation & Application to Real Data

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Received: March 30, 2019; Accepted: December 12, 2019

Abstract

A new family of inverse probability distributions named inverse Rayleigh family is introduced to generate many continuous distributions. The shapes of probability density and hazard rate functions are investigated. Some Statistical measures of the new generator including moments, quantile and generating functions, entropy measures and order statistics are derived. The Estimation of the model parameters is performed by the maximum likelihood estimation method. Furthermore, a simulation study is used to estimate the parameters of one of the members of the new family. The data application shows that the new family models can be useful to provide better fits than other lifetime models.

Keywords: Inverse Rayleigh distribution; Quantile function; Reliability; Entropy; Mean Residual Life, Parameters estimation, Simulation.

MSC 2010 No.: 60E05, 62F10

1. Introduction

In fact, many classical distributions used for modeling data in several lifetime data analysis. Recent developments stress on the definition of the new families of distributions that extend most probability distributions and feed enormous flexibility in modeling data. Hence, some generated families of distributions have been presented by adding one or more parameters to generate new distributions. Eugene *et al.* (2002) studied beta-G (B-G), Zografos and

Balakrishanan (2009) discussed gamma-G and transformed-transformer (T-X) investigated by Alzaatreh *et al*. (2013). Recently, Bourguignon *et al.* (2014) introduced Weibull-G and Burr generalized family of distributions is discussed by Alizadeh *et al.* (2017), Odd Frèchet-G family by Haq and Elgarhy (2018), generalized odd Burr III family by Haq et al. (2019).

This paper is classified as follows: In Section 1, the inverse Rayleigh generated (IR) family of probability distributions is discussed. Three models from our new family are discussed in Section 3. In Section 4, some important mathematical formulas of the IR family are derived. Distribution of order statistics is deduced in Section 5 and the estimation of the model parameters is performed in Section 6. In Section 7, a simulation study is organized to the IR-W model to evaluate the adequacy of estimates as a member of the newly generated family. In Section 8, an application to real data investigates the performance of the new models. Some conclusions are introduced in Section 9.

2. The IR Family

In this Section, The IR family is provided. The PDF of the IR family is derived. Additionally, reliability, hazard rate, and cumulative hazard rate functions are obtained and studied the behaviors and curved shapes of some models from the new family.

The inverse Rayleigh (IR) distribution was presented by Voda (1972), the estimation of negative moments to IR distribution investigated by Mohsin and Shahbaz (2005) and different methods of estimation have been numerically by Soliman *et al.* (2010). Recently, Ahmed (2014) studied a transmuted inverse Rayleigh distribution, Rehman and Dar (2015) introduced exponentiated inverse Rayleigh distribution and Khan and King (2017) suggested transmuted new generalized inverse Weibull.

The PDF $r(\mathrm{t};\theta)$ and cdf $R(\mathrm{t};\theta)$ of IR distribution with one parameter θ are given by

$$r(t;\theta) = 2\theta t^{-3} e^{-\frac{\theta}{t^2}}; t > 0, \theta \in \mathbb{R}, \tag{1}$$

$$R(t;\theta) = e^{-\frac{\theta}{t^2}}. \tag{2}$$

The Inverse Rayleigh distribution is used in enormous applications, particularly survival analysis and it is a popular model used in economics. Since IR distribution has only one parameter and so it does not present extreme flexibility for analyzing different types of lifetime data. Our study aims to deduce a new family of distributions using the IR family.

Depending on the transformer (T-X) generator (Alzaatreh *et al.* (2013)) and by integrating the PDF of IR distribution as the following:

$$F(x;\theta,\eta) = \int_0^{\frac{G(x;\eta)}{1-G(x;\eta)}} 2\theta x^{-3} e^{-\frac{\theta}{x^2}} dt = exp\{-\theta([G(x;\eta)]^{-1} - 1)^2\}, \tag{3}$$

then the IR PDF function will be

$$f(x;\theta,\eta) = 2\theta \frac{g(x;\eta)}{G^2(x;\eta)}([G(x;\eta)]^{-1}-1)\, exp\{-\theta([G(x;\eta)]^{-1}-1)^2\}$$
$$;x \geq 0, \theta, \eta > 0. \tag{4}$$

The reliability function is defined as

$$S(x;\theta,\eta) = 1 - exp\{-\theta([G(x;\eta)]^{-1}-1)^2\}. \tag{5}$$

Also, the hazard rate and cumulative hazard rate functions are

$$h(x;\theta,\eta) = \frac{2\theta \frac{g(x;\eta)}{G^2(x;\eta)}([G(x;\eta)]^{-1}-1)\, exp\{-\theta([G(x;\eta)]^{-1}-1)^2\}}{1 - exp\{-\theta([G(x;\eta)]^{-1}-1)^2\}}, \tag{6}$$

$$H(x;\theta,\eta) = -\, ln(1 - exp\{-\theta([G(x;\eta)]^{-1}-1)^2\}).$$

3. Special Models

IR-uniform, IR-Weibull, and IR- Fréchet are discussed in this Section as special cases from our family.

3.1. IR-Uniform

By taking the baseline distribution is the uniform, then the Probability density function (PDF) and hazard rate functions (HRF) of the IR-uniform (IRU) distribution will be

$$f(x;\theta,b) = \frac{2\theta b}{x^2}\left(\frac{b}{x}-1\right) exp\left\{-\theta\left(\frac{b}{x}-1\right)^2\right\}; 0 < x < b < \infty, \theta > 0,$$

$$h(x;\theta,\alpha) = \frac{\frac{2\theta\alpha}{x^2}\left(\frac{\alpha}{x}-1\right) exp\left\{-\theta\left(\frac{\alpha}{x}-1\right)^2\right\}}{1-exp\left\{-\theta\left(\frac{\alpha}{x}-1\right)^2\right\}}.$$

3.2. IR -Weibull

Replacing the uniform distribution in the previous Section by the Weibull model, then the PDF and HRF of IR-Weibull (IRW) are acquired, respectively as

$$f(x;,\theta,\lambda,\alpha) = 2\theta\lambda\alpha x^{\alpha-1}e^{-\lambda x^{\alpha}}(-e^{-\lambda x^{\alpha}})^{-2}\left(\frac{1}{1-e^{-\lambda x^{\alpha}}}-1\right) exp\left\{-\theta\left(\frac{1}{1-e^{-\lambda x^{\alpha}}}-1\right)^2\right\}$$
$$;x > 0, \theta, \lambda, \alpha > 0.$$

$$h(x;\theta,\lambda,\alpha) = \frac{2\theta\lambda\alpha x^{\alpha-1}e^{-\lambda x^{\alpha}}\left(\frac{1}{1-e^{-\lambda x^{\alpha}}}-1\right) exp\left\{-\theta\left(\frac{1}{1-e^{-\lambda x^{\alpha}}}-1\right)^2\right\}}{\left[1-exp\left\{-\theta\left(\frac{1}{1-e^{-\lambda x^{\alpha}}}-1\right)^2\right\}\right](1-e^{-\lambda x^{\alpha}})^2}.$$

3.3. IR-Fréchet

Considering the prior distribution is the Fréchet (Fréchet (1927)) as in Sections 2.2 and 2.3, the PDF is defined as

$$f(x;\theta,\alpha,\beta) = 2\theta\alpha^{-3}\beta^{-3}x^{3(\alpha+1)}e^{2\beta x^{-\alpha}}\left(1 - \alpha\beta x^{-\alpha-1}e^{-\beta x^{-\alpha}}\right)exp\left\{-\theta\left(\frac{1}{\alpha\beta x^{-\alpha-1}e^{-\beta x^{-\alpha}}}-1\right)^{2}\right\}; 0,\alpha,\beta,\theta > 0.$$

and its HRF is

$$h(x;\theta,\alpha,\beta) = \frac{2\theta\alpha^{-3}\beta^{-3}x^{3(\alpha+1)}e^{2\beta x^{-\alpha}}\left(1-\alpha\beta x^{-\alpha-1}e^{-\beta x^{-\alpha}}\right)exp\left\{-\theta\left(\frac{1}{\alpha\beta x^{-\alpha-1}e^{-\beta x^{-\alpha}}}-1\right)^{2}\right\}}{\left[1-exp\left\{-\theta\left(\frac{1}{\alpha\beta x^{-\alpha-1}e^{-\beta x^{-\alpha}}}-1\right)^{2}\right\}\right]}.$$

The IRU, IRW and IRF models are illustrated by Figures 1, 2 and 3.

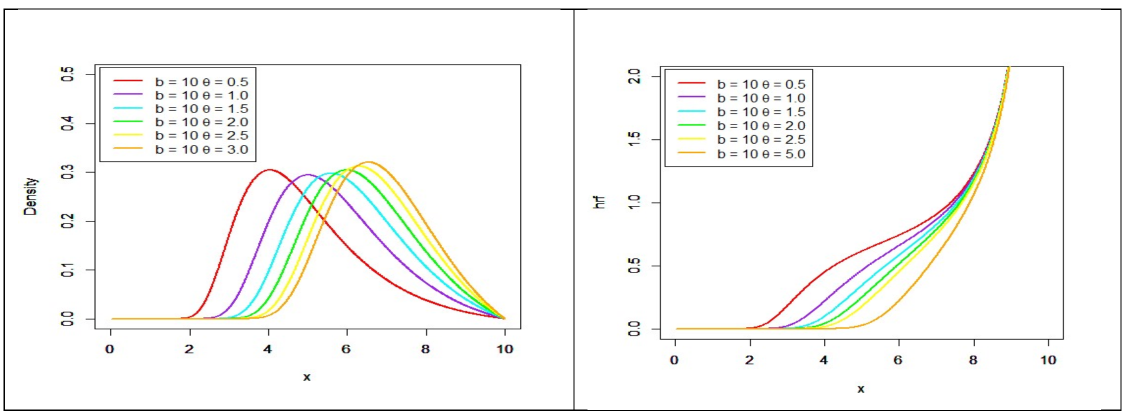

Figure 1: The PDF and HRF of IRU model

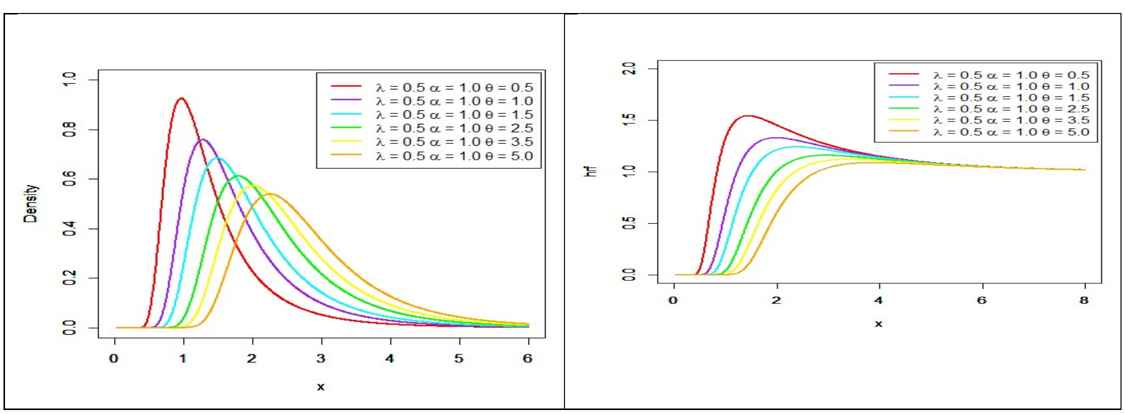

Figure 2: The PDF and HRF of IRW model

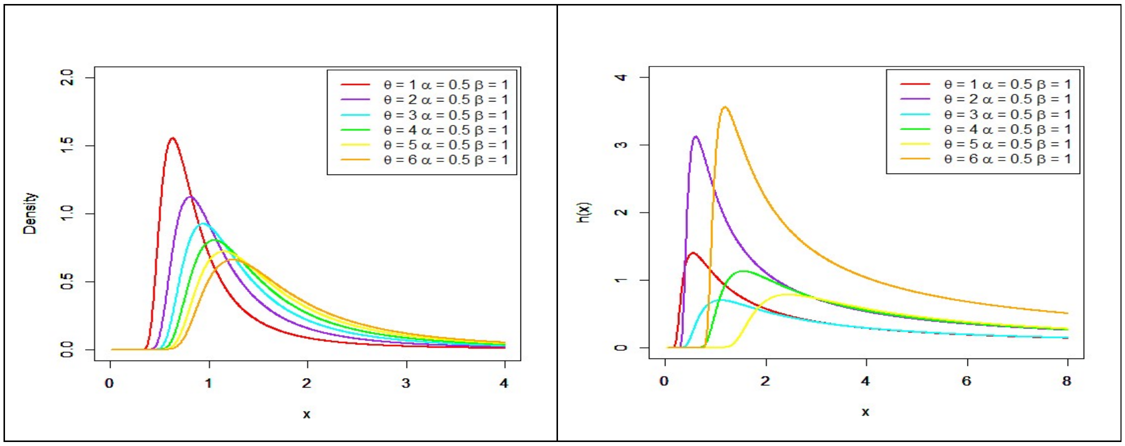

Figure 3: The PDF and HRF of IRF model

4. Mathematical properties

Some mathematical formulas of IR family are computed in this Section.

4.1. Mathematical Expansions

The PDF, cdf, and reliability are deduced in closed forms in this sub-Section.
Using the rule of exponential function as a power series

$$exp\{-\theta([G(x;\eta)]^{-1}-1)^2\} = \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!} ([G(x;\eta)]^{-1}-1)^{2i}. \tag{7}$$

The cdf of the IR family can be written as

$$F(x;\theta,\eta) = \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!} ([G(x;\eta)]^{-1}-1)^{2i}. \tag{8}$$

By Substituting from (7) into PDF (4), we can get to

$$f(x;\theta,\eta) = \sum_{i=0}^{\infty} \frac{2(-1)^i \theta^{i+1}}{i!} ([G(x;\eta)]^{-1}-1)^{2i+2} \frac{g(x;\eta)}{G^2(x;\eta)}; x \geq 0, \theta > 0, \eta > 0. \tag{9}$$

By applying the binomial theorem

$$\left([G(x;\eta)]^{-1}-1\right)^{2i+2} = \sum_{j=0}^{2i+2} \binom{2i+2}{j} \left(G(x;\eta)\right)^{-j}. \tag{10}$$

Substituting from (10) into (9), the PDF will be

$$f(x;\theta,\eta) = \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} W_{i,j}\, g(x;\eta) \left(G(x;\eta)\right)^{-(j+2)} ; x \geq 0, \theta, \eta > 0, \tag{11}$$

where $W_{i,j} = \frac{2(-1)^i \theta^{i+1}}{i!}\binom{2i+2}{j}$, also we can rewrite the cdf in (8) as follows

$$F(x;\theta,\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\binom{2i}{j}\frac{(-1)^i \theta^i}{i!}\left(G(x;\eta)\right)^{-j}, \tag{12}$$

and

$$F(x;\theta,\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\omega_{i,j}\left(G(x;\eta)\right)^{-j}, \tag{13}$$

where $\omega_{i,j} = \binom{2i}{j}\frac{(-1)^i \theta^i}{i!}$.

By differentiating Equation (13), we obtain

$$f(x;\theta,\eta) = \sum_{i=0}^{\infty}\sum_{j=0}^{2i}\omega_{i,j}\, h_{(-j)}(x;\eta). \tag{14}$$

Here, $h_{(-j)}(x;\eta)$ denotes the PDF of the exponential model with power parameter *(-j)*. Equation (14) expresses that the IR density can be written as a mixture of exponential densities. So, several mathematical properties of the new family can be obtained based on the properties of the exponential distribution.

4.2. The Quantile function and Median

Quantile functions are used to obtain the percentiles of the model. The quantile function of the IR family $Q(u)$ is computed by inverting Equation (3).

Since

$$Q(u) = F^{-1}(u), \text{ then } u = 1 - e^{-\theta\left(\frac{1}{x_G}-1\right)^2}.$$

Doing some mathematical simplifications, the previous Equation is

$$x_G = \left(\left(-\frac{\ln(1-\mathrm{u})}{\theta}\right)^{\frac{1}{2}} + 1\right)^{-1}. \tag{15}$$

By solving Equation (15) numerically the median (*Med=Q(0.5))* of IR family of distributions can be accessed as the following;

$$Med = \left(\left(-\frac{\ln(0.5)}{\theta}\right)^{\frac{1}{2}} + 1\right)^{-1}.$$

Example 1.

Consider the IRU distribution discussed in subSection (4.1). The quantile function of IRU is

$$Q(u) = \alpha\left(1 + \left(\frac{-\ln u}{\theta}\right)^{\frac{1}{2}}\right)^{-1}, \tag{16}$$

and the median is

$$Med_{IRU} = Q(0.5) = \alpha\left(1 + \left(\frac{-\ln 0.5}{\theta}\right)^{\frac{1}{2}}\right)^{-1}.$$

4.3. Skewness & Kurtosis

Here, skewness and kurtosis measures based on quantile function for the IRU are deduced. Kenney and Keeping (1962) presented skewness called Bowley skewness which is characterized by the following Equation

$$Sk = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{4}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}, \tag{17}$$

and Moors (1988) determined the kurtosis coefficient by the relation

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}. \tag{18}$$

Considering the IRU distribution in example 1, the skewness and kurtosis of IRU distribution can be derived by Substituting from (16) into (17) and (18).

4.4. Moments

Moments are an important technique using to study the most characteristics and statistical properties of probability distribution like the measures of tendency and the measure of dispersion. The *nth* moment of IR family about the origin will be deduced. The moments (μ_n') of random variable X can be acquired from PDF (14) as follows

$$\mu_n' = E(x^n) = \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} \omega_{ij} \int_0^{\infty} x^n h_{(-j)}(x;\eta)\, dx,\ \mu_n' = \omega_{i,j}\psi_{i,j,n};\ n = 1,2,\ldots, \tag{19}$$

and

$$\psi_{i,j,n} = \sum_{i,j,=0}^{\infty} \omega_{i,j} \int_0^{\infty} x^n\, h_{(-j)}(x;\eta) dx.$$

Specifically, the mean of the IR family is computed as follows:

$$\mu = \omega_{i,j}\,\psi_{i,j,1},$$

where

$$\psi_{i,j,1} = \sum_{i,j=0}^{\infty} \omega_{ij} \int_0^{\infty} x\, h_{(-j)}(x;\eta)\, dx$$

and the variance is

$$Var(X) = \omega_{i,j}\,\psi_{i,j,2} - \left[\omega_{ij}\,\psi_{i,j,1}\right]^2,$$

where

$$\psi_{i,j,2} = \sum_{i,j=0}^{\infty} \int_0^{\infty} x^2\, h_{(-j)}(x;\eta)\, dx.$$

Based on (19) skewness and kurtosis coefficients of IR family are

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{\left(\mu_2' - \mu_1'^2\right)^{3/2}},$$

and

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{\left(\mu_2' - \mu_1'^2\right)^2}.$$

The moment generating function of the IR family is characterized by

$$\xi_X(t) = \sum_{n=0}^{\infty} \frac{t^r\,\mu_n'}{n!}.$$

Using (19)

$$\xi_X(t) = \sum_{n=1}^{\infty} \frac{t^n\,\omega_{ij}\,\psi_{i,j,n}}{n!}\ ;\ n = 1,2,3,\ldots$$

Example 2.

Consider the PDF and cdf of IRU distribution that is discussed in sub-Section (4.1). The *p-th* moment of IRU is obtained from (11) as follows

$$\mu'_p = 2\theta\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{2i+2} D_{i,j}\int_0^{\infty} x^{p-2}\left(\frac{\alpha}{x}-1\right)exp\left\{\theta(j+1)\left(\frac{\alpha}{x}-1\right)^2\right\}\mathrm{d}\,x,$$

and

$$\mu'_p = 2\theta\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{2i+2} D_{i,j}\left[\int_0^{\infty}\alpha\, x^{p-3}\exp\left\{-\theta(-j-1)\left(\frac{\alpha}{x}-1\right)^2\right\}dx - \int_0^{\infty} x^{p-2}\exp\left\{-\theta(-j-1)\left(\frac{\alpha}{x}-1\right)^2\right\}dx\right],$$

where

$$D_{i,j} = \frac{2(-1)^i\theta^{i+1}}{i!}\binom{2i+2}{j}.$$

The mean and standard deviation of the IR-U distribution could be simply deduced using the above Equation.

4.5. Incomplete moments

The incomplete moments, $m_p^I(t)$ of the IR distribution is defined by

$$m_p^I(t) = \int_0^t x^r f(x;\Phi)dx.$$

Substituting from Equation (11), then

$$m_p^I(t) = \int_0^t x^p f(x;\Phi)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{2i+2} D_{i,j}\int_0^t x^p g(x;\eta)\left[G(x;\eta)\right]^{-(j+2)}dx.$$

Example 3.

The incomplete *r-th* moment of the IRU distribution that is introduced in sub-Section (4.1) is calculated by the following relation

$$m_p^I(t) = 2\theta\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{2i+2} D_{i,j}\int_0^t x^{p-2}\left(\alpha x^{-1}-1\right)\exp\left\{\theta(j+1)\left(\alpha x^{-1}-1\right)^2\right\}\mathrm{d}\,x.$$

4.6. The Residual Lifetime

Hazard rate, mean residual life and left truncated mean are some functions related to the residual lifetime of a unit. These functions acquire the cumulative distribution function $F(X)$ (Zoroa *et al.* (1990)).

Let X be a random variable alluding to the lifetime of a unit aged t. Then $X_t = X - t \mid X >$ t refers to the remaining lifetime past that age t. The cdf $F(x)$ is determined by the *r-th* moment of the residual life of X (Navarro *et al.* (1998)) as follows

$$m_p(t;\Phi) = \frac{1}{\left(1 - F(t;\Phi)\right)} \int_t^{\infty} (x-t)^p f(x;\Phi) dx.$$

Specifically, if $r = 1$, then $m_1(t)$ represents the mean residual life (MRL) function that demonstrates the expected life time for a unit that is awake at age t. The MRL function has several applications, for examples; in reliability analysis, production technology and quality control.

Example 4.

Considering the IRU distribution and using Equation (11) we can compute the *p-th* moment of the residual life of X (for $p = 1, 2$ ...) as follows

$$\int_t^{\beta} (x-t)^p f(x;\Phi) dx = \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} W_{i,j} \int_t^{\beta} (x-t)^p g(x;\eta) \left[G(x;\eta)\right]^{-(j+2)} dx$$
$$= 2\theta\alpha \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} W_{i,j} \int_0^{t} (x-t)^p x^{-2} \left(\alpha x^{-1} - 1\right) exp\left\{\theta(j+1)\left(\alpha x^{-1} - 1\right)^2\right\} \mathrm{d}x.$$

4.7. Entropies

The entropy of a random variable X is the measure the uncertainty (Rényi (1961)). Two types of entropy measures; Rényi and p entropies are discussed. The Rényi entropy $I_{Ren}(S)$ of a random variable X is defined as

$$I_{\mathrm{Re}n}(\mathrm{s}) = \frac{1}{1-s} \log \int_0^{\infty} f^s(x) dx;\ s > 0,\ s \neq 1.$$

Based on Equation (11), the $I_{\mathrm{Re}n}(\mathrm{s})$ takes the following form

$$I_{\mathrm{Re}n}(\mathrm{s}) = \frac{1}{1-s} \log\left[\int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} \left(W_{i,j}\right)^s g^s(x;\eta) \left(G(x;\eta)\right)^{-s(j+2)} \mathrm{d}x\right].$$

The p-entropy, say $I(p)$, is determined by

$$I(p) = \frac{1}{p-1} \log\left[1 - \int_0^{\infty} f^p(x) dx\right];\ p > 0, p \neq 1.$$

Using Equation (11),

$$I(p) = \frac{1}{p-1} \log\left[1 - \int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{2i+2} \left(W_{i,j}\right)^p g^p(x;\eta) \left(G(x;\eta)\right)^{-p(j+2)} dx\right].$$

From Equation (5), the p-entropy is obtained as

$$I(q)=\frac{1}{p-1}\log\left[1-\sum_{i=0}^{\infty}\int_0^{\infty}\varepsilon^{i+q}\, b^p\, a^{b(i+p)}\, g(x;\zeta)\left[\bar{G}(x;\zeta)\right]^{-p}\left[-\log\bar{G}(x;\zeta)\right]^{-p(b+1)-bi}dx\right].$$

4.8. Useful Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from IR family and $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ indicate the order statistics obtained from the sample. The PDF of $X_{r:n}$ is obtained through the following

$$f_{q:n}(x;\Phi)=\frac{1}{B(q,n-q+1)}\sum_{s=0}^{n-q}(-1)^s\binom{n-q}{s}\left[F(x;\Phi)\right]^{q+s-1}f(x;\Phi),$$

where, $\Phi=(\theta,\eta)$. By using Equations (3) and (4), the PDF of q-th order statistic takes the following form

$$f_{q:n}(x;\Phi)=\frac{2\theta}{B(q,n-q+1)}\sum_{s=0}^{n-q}(-1)^s\binom{n-q}{s}\frac{g(x;\eta)}{G^2(x;\eta)}\left(\left[G(x;\eta)\right]^{-1}-1\right)$$
$$\times\exp\left\{-\theta(q+s)\left(\left[G(x;\eta)\right]^{-1}-1\right)^2\right\}. \tag{20}$$

Since,

$$\exp\left\{-\theta(q+s)\left(\left[G(x;\eta)\right]^{-1}-1\right)^2\right\}=\sum_{j=0}^{\infty}\frac{(-1)^j\,(q+\theta)^j\,\theta^j}{j!}\left(\left[G(x;\eta)\right]^{-1}-1\right)^{2j},$$

then (20) will be

$$f_{q:n}(x;\Phi)=\frac{2}{B(q,n-q+1)}\sum_{s=0}^{n-q}\sum_{j=0}^{\infty}\frac{(-1)^{s+j}\,(n-q)!(\mathrm{r}+\theta)^j\,\theta^{j+1}}{s!\,j!\,(\mathrm{n}-q-\mathrm{s})!}\frac{g(x;\eta)}{G^2(x;\eta)}\left(\left[G(x;\eta)\right]^{-1}-1\right)^{2j+1}.$$

Using the binomial expansion

$$\left(\left[G(x;\eta)\right]^{-1}-1\right)^{2j+1}=\sum_{m=0}^{2j+1}(-1)^{2j+m+1}\binom{2j+1}{m}\left(G(x;\eta)\right)^{-m}.$$

Therefore,

$$f_{q:n}(x;\Phi)=\frac{2}{B(q,n-q+1)}\sum_{s=0}^{n-q}\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\frac{(-1)^{3j+m+s+1}\,(\mathrm{r}+\theta)^j\,\theta^{j+1}}{j!}\binom{2j+1}{m}\binom{n-q}{j}$$
$$\times g(x;\eta)\left(G(x;\eta)\right)^{-m-2},$$

$$f_{q:n}(x;\Phi) = \sum_{s=0}^{n-q}\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1} \Lambda_{i,j,m,q}\; g(x;\eta)\big(G(x;\eta)\big)^{-m-2}, \tag{21}$$

and

$$\Lambda_{i,j,m,q} = \frac{2}{B(q,n-q+1)}\sum_{s=0}^{n-q}\sum_{j=0}^{\infty}\sum_{m=0}^{2j+1}\frac{(-1)^{3j+m+s+1}\,(\mathrm{r}+\theta)^{j}\,\theta^{j+1}}{j!}\binom{2j+1}{m}\binom{n-q}{j}.$$

Specifically, the PDF of the smallest and the largest order statistics $X_{1:n}$ and $X_{n:n}$ are gotten from (19) by substituting $r = 1$ and $r = n$.

5. Estimation of Parameters

In this Section, the maximum likelihood estimation method are applied for the parameters of IR generated family from complete samples. Let $X_1, X_2, \dots, X_n$ be a simple random sample from IR family with observed values $x_1, x_1, \dots x_n$. The log-likelihood function of (4) is defined as follows

$$\begin{aligned} ln\,L(\Phi) = n\,ln(2\theta) - \theta\sum_{i=1}^{n}\Big(\big[G(x_i;\eta)\big]^{-1}-1\Big)^2 + \sum_{i=1}^{n} ln\big(\mathrm{g}(x_i;\eta)\big) \\ -2\sum_{i=1}^{n} ln\big(G(x_i;\eta)\big) + \sum_{i=1}^{n} ln\Big(\big[G(x_i;\eta)\big]^{-1}-1\Big). \end{aligned}$$

Differentiating $ln\,L(\Phi)$ with respect to (θ,β) and equating the result by zero, the maximum likelihood estimators (MLEs) will be obtained.

$$\frac{\partial ln\,L(\Phi)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\Big(\big[G(x_i;\eta)\big]^{-1}-1\Big)^2.$$

$$\begin{aligned} \frac{\partial ln\,L(\Phi)}{\partial\eta} &= 2\theta\frac{g(x_i;\eta)}{G^2(x_i;\eta)}\Big(\big[G(x_i;\eta)\big]^{-1}-1\Big) \\ &\quad + \sum_{i=1}^{n}\frac{g'(x_i;\eta)}{g(x_i;\eta)} - 2\sum_{i=1}^{n}\frac{g(x_i;\eta)}{G(x_i;\eta)} - \sum_{i=1}^{n}\frac{g(x_i;\eta)}{G^2(x_i;\eta)}\Big(\big[G(x_i;\eta)\big]^{-1}-1\Big)^{-1}. \end{aligned}$$

By solving the non-linear Equations $\frac{\partial\ell}{\partial\theta}=0, \frac{\partial\ell}{\partial\eta}=0$ numerically using one of the statistical software packages, the MLEs of will be computed.

6. Simulation Technique

A simulation study is conducted by IR-W model in this Section. Samples of sizes $n = 50, 100, 300$ are generated from the distribution and MLEs of the parameters are determined. 1000 repetitions are applied to compute the mean square error (MSE) and bias of estimators using the following relations;

$$Bias\left(\hat{\delta}\right) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\delta} - \delta\right) \text{ and } MSE\left(\hat{\delta}\right) = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\delta} - \delta\right)^2 \text{ for any parameter } \delta.$$

Table 1. MLEs, Mean, Bias and MSE of IRW

n	**Parameter**	**Mean**	**Bias**	**MSE**	**Mean**	**Bias**	**MSE**
		Set 1-(0.5,0.5,0.5)			Set 2-(0.5,1.5,0.5)		
50	Θ	0.641515	0.141515	13.08120	0.214455	-0.285545	18.64590
	A	0.509938	0.009938	0.005625	1.530950	0.030953	0.049971
	Λ	0.521497	0.021497	0.012623	0.541766	0.041766	0.030885
100	Θ	0.543004	0.043004	0.063241	0.559704	0.059704	0.622025
	A	0.505667	0.005666	0.002677	1.516470	0.016468	0.023628
	Λ	0.509311	0.009311	0.005619	0.518396	0.018396	0.011354
300	Θ	0.510720	0.010720	0.006869	0.516131	0.016131	0.009403
	A	0.502033	0.002033	0.000859	1.50391	0.003911	0.007675
	Λ	0.502737	0.002737	0.001758	0.505899	0.005899	0.003158
n	**Parameter**	**Mean**	**Bias**	**MSE**	**Mean**	**Bias**	**MSE**
		Set 3-(0.5,0.5,1.5)			Set 4-(1.5,0.5,0.5)		
50	Θ	1.255790	0.755793	3.478940	0.678165	-0.821835	17.78630
	A	0.510197	0.010197	0.005473	0.510478	0.010478	0.005468
	Λ	0.642117	-0.857883	8.358350	1.565730	1.065730	1.241970
100	Θ	1.522560	1.022560	7.732070	0.543122	-0.956878	1.337380
	A	0.505131	0.005131	0.002567	0.506074	0.006074	0.002585
	Λ	0.568834	-0.931166	1.566560	1.532040	1.032040	1.115370
300	Θ	1.532910	1.032910	1.128360	0.510828	-0.989172	0.985217
	A	0.501933	0.001933	0.000853	0.502084	0.002084	0.000862
	Λ	0.518184	-0.981816	0.974280	1.510980	1.010980	1.037740

From Table 1, we notice that:

- The bias decreases when the sample size increases to the most values and that shows the accuracy of the MLEs of IRW distribution.
- When the sample size n increases, the MSE decreases. That investigates consistency of the MLEs of the parameters.

7. Data Application

A real data is provided to investigate the applicability of inverse Rayleigh Weibull (IRW) model in practice. The IRW distribution will be fitted to real data and the results with transmuted Rayleigh (Merovci (2013)); (TR), exponentiated inverse Rayleigh (Rehman *et al.* (2015)); (EIR), inverse Fréchet Weibull (IFW), inverse Weibull (Mahmoud *et al.* (2003)); (IW), transmuted generalized Rayleigh (Merovci (2014)); (TGR) and Kumaraswamy inverse Lindley (Sharma *et al.* (2016)); (KuIL) distributions will be compared.

The data are 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020 (see, Bader and Priest (1982)).

The estimated parameters for each distribution will be obtained using the maximum-likelihood method. The values of statistics: (A^{*}) and (W^{*}), Akaike information criterion (AIC) and Bayesian information criterion (BIC) are calculated. In general, according to Akaike (1974), the smaller values of these statistics give better fit to real data. Table 2 shows the maximum likelihood estimates (MLEs) of the model parameters and its standard error (S.E) and Table 3 shows the goodness measures for estimates of the real data.

Table 2. Estimated parameters for the real data

Models	**Estimated Parameters**		
TR (β, λ)	1.82396	0.9000	-
EIL$(\alpha., \theta)$	1.96116	1.50033	-
IFW(α, λ)	175.287	16.811	-
IW(α, β)	2.72143	5.43377	-
TGR(α, β, λ)	6.2143	0.5021	0.1207
IRW(λ,θ, α)	0.312765	8.92154	1.46555
KuIL(θ, α, β)	0.270381	127.501	59.1817

Table 3. Goodness measures for estimates

Models	**AIC**	**BIC**	-lnL	A^*	W^*
IRW(λ,θ,α)	119.092	125.521	56.5458	0.37789	0.073435
TR (β, λ)	155.572	159.858	75.7858	5.41110	0.906164
EIL$(\alpha., \theta)$	321.923	326.209	158.961	18.2200	3.704280
IFW(α, λ)	119.450	123.737	57.7252	0.426601	0.078992
IW(α, β)	121.804	126.091	58.9021	0.622595	0.101452
TGR(α, β, λ)	122.640	126.926	59.3200	0.38047	0.084888
KuIL(θ, α, β)	121.864	182.082	57.9320	0.44866	0.095544

Figure 4 shows the estimated density and empirical cdf of the IRW model for the real data. Likewise, from Figure 4 one can see that: the IRW model is more applicable to fitting the data. Obviously from these figures; the fitted density for the IRW model is **closest to** the empirical cumulative distribution for the real data.

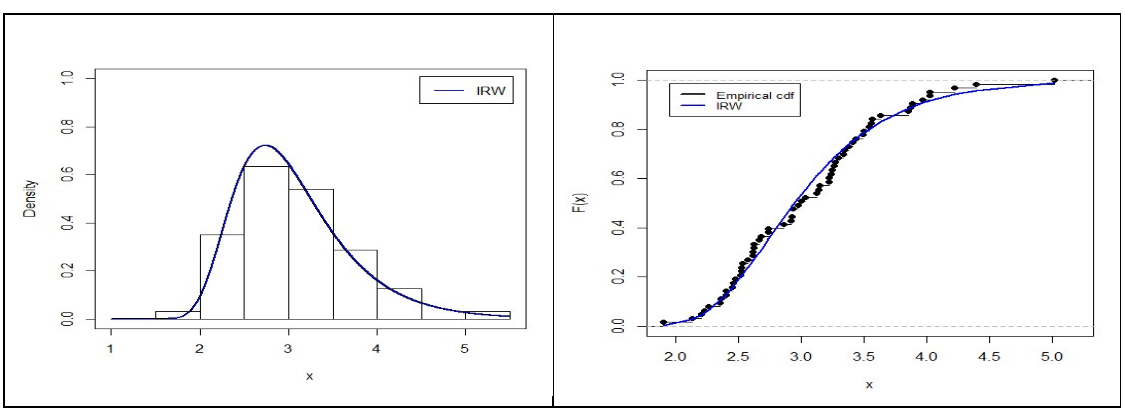

Figure 4. Estimated density and empirical cdf of the IRW model

8. **Conclusion**

A recently generated family of probability distributions called inverse Rayleigh family is proposed. Some of its mathematical-statistical properties including an expansion for the density function and explicit expressions for moments, moment generating function, Median, quantile function, skewness, and kurtosis are derived. The mean residual life and order statistics of the new model are derived. The maximum likelihood method of estimation is embedded for estimating the model parameters. We utilized a simulation study to evaluate the finite sample behavior. Real data is applying to show the convenience of the proposed distribution. The inverse Rayleigh Weibull distribution provides enough flexibility for analyzing different types of lifetime data than transmuted Rayleigh, exponentiated inverse Rayleigh, inverse Fréchet Weibull, inverse Weibull, and transmuted generalized Rayleigh and Kumaraswamy inverse Lindley distributions.

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