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On Ordered (p, q)-Lateral Ideals in Ordered Ternary Semigroups

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Abstract

In this paper, we study some useful results of ordered (p, q)-lateral ideals in ordered ternary semigroups. Also, some properties of (p, q)-lateral simple ordered ternary semigroup have been examined. Further, we characterize the relationship between minimal (resp., maximal) ordered (p, q)-lateral ideals and (p, q)-lateral simple ordered ternary semigroups.

Keywords: Ordered ternary semigroup; Ordered (p, q)-lateral ideal; Minimal ordered (p, q)-lateral ideals; Maximal ordered (p, q)-lateral ideals; Total ordered ternary semigroup; (p, q)-lateral simple ordered ternary semigroup; (p, q)-lateral simple elements

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1. Introduction

The idea of investigation of n-ary algebras i.e. the sets with one n-ary operation was given by Kasner (1904). In particular, n-ary semigroups are known as ternary semigroups for n=3 which was introduced by Lehmer (1932) with one associative operation. Kerner (2000) expressed many applications of ternary structures in physics. Now ternary have structures become a highly active area of research. A number of different ternary structures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. For instance, Akram et al. (2015) introduced a ternary structure known as bi Γ-ternary semigroup as a generalization of Γ-semigroup. Akram et al. (2016, 2017) defined N-fuzzy sets and N- intuitionistic fuzzy sets in bi Γ-ternary semigroups. Yiarayong (2019) introduced the notion of ternary left almost semigroups. Ali et al. (2019) introduced po-bi-ternary Γ-semigroups and studied the relationship between minimal po-bi $(\alpha, (\xi, \zeta), \beta)$-quasi-$\Gamma$-ideals and $(\alpha, (\xi, \zeta), \beta)$-quasi simple po-bi-ternary Γ-semigroups.

Ideal theory in ternary semigroup was studied by Sioson (1965). He also defined regular ternary semigroups. The properties of quasi ideals and bi-ideals in ternary semigroups was studied by Dixit and Diwan (1995). Dubey and Anuradha (2014) defined m-right, (p, q)-lateral and n-left ideals in ternary semigroup and gave their characterizations.

Iampan (2009) gave the definition of ordered ternary semigroup and characterized the minimality and maximality concept in ordered ternary semigroups using ordered lateral ideals. Abbasi and Khan (2017) studied generalised ideals in ternary semigroups.

2. Preliminaries

To start with we need the following.

A non-empty set S with a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1, x_2, x_3]$, is called a ternary semigroup if it satisfies the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in S$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [[x_1x_2[x_3x_4x_5]].$$

For non-empty subsets A, B and C of a ternary semigroup S, let

$$[ABC] := \{[abc] : a \in A, b \in B, \text{and } c \in C\}.$$

If $A = \{a\}$, then we write $[\{a\}BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. For the sake of simplicity, we write $[x_1x_2x_3]$ as $x_1x_2x_3$ and $[ABC]$ as ABC.

Definition 2.1.

By Dixit and Diwan (1995), a non-empty subset T of a ternary semigroup S is called a ternary subsemigroup of S if $TTT \subseteq T$.

By Sioson (1965), for any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, x_3 ....x_{2n}$ and x_{2n+1} of a ternary semigroup, we can write

$$[x_1x_2x_3.........x_{2n+1}] = [x_1, x_2, x_3..[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}]...x_{2n+1}].$$

Definition 2.2.

By Iampan (2009), a ternary semigroup S is called a partially ordered ternary semigroup if there exits a partially ordered relation $\leq$ such that for any $a, b, x, y \in S$, $a \leq b \Rightarrow axy \leq bxy, xay \leq xby$, and $xya \leq xyb$.

Example 2.3.

Let us define the following S=$\left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{N}_0 \right\}$. Here $\mathbb{N}_0$, the set of all non-negative integers is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}_0}$ is "less than or equal to". Now we define partial order relation $\leq_S$ on S by, for any A, $B \in S$,

$$A \leq_S B \text{ if and only if } a_{ij} \leq_{\mathbb{N}_0} b_{ij}, \text{ for all } i \text{ and } j.$$

Then, it is easy to verify that S is an ordered ternary semigroup under usual multiplication of matrices over $\mathbb{N}_0$ with partial order relation $\leq_S$.

For $H \subseteq S$ we denote $(H]$ the subset of S defined by

$$(H] = \{s \in S \mid s \leq h, \text{ for some } h \in H\}.$$

Theorem 2.4.

Let S be an ordered ternary semigroup. Then the following hold:

1) $A \subseteq (A]$, for all $A \subseteq S$.

2) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

3) $((A]]$ = $(A]$, for all $A \subseteq S$.

4) $(A](B](C] \subseteq (ABC]$, for all A, B, $C \subseteq S$.

Definition 2.5.

By Iampan (2009), a non-empty subset I of an ordered ternary semigroup S is called an ordered lateral (respectively, ordered left, ordered right) ideal of S if

(1) $SIS \subseteq I$ (resp., $SSI \subseteq I$, $ISS \subseteq I$), and

(2) $(I] = I$.

Example 2.6.

Consider Example 2.3. Let $M_l = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{N}_0 \right\}$. Then, M_l is an ordered lateral ideal of S.

Remark 2.7.

If a non-empty subset M of an ordered ternary semigroup S is an ordered right as well as ordered left ideal of S, then M need not be an ordered lateral ideal of S.

Example 2.8.

Let us define $S = \left\{ \begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix} : a, b, c, d \in \mathbb{N}_0 \right\}$. Then, S is an ordered ternary semigroup under the usual multiplication of matrices over $\mathbb{N}_0$ with partial order relation $\leq_S$, as defined in Example 2.3. Let $M = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{N}_0 \right\}$. Then, we can easily verify that M is an ordered right as well as ordered left ideal of S but M is not an ordered lateral ideal of S.

Proposition 2.9.

Let S be an ordered ternary semigroup and $a \in S$. Then, the principal ordered lateral ideal generated $'a'$ is given by $M(a) = (SaS \cup SSaSS \cup \{a\}]$.

Definition 2.10.

By Kellil (2016), an ideal I of an ordered ternary semigroup S is called idempotent if $I^3 = I$.

3. Main Results

Definition 3.1.

Let S be an ordered ternary semigroup. Then, an ordered ternary subsemigroup M is called an ordered (p, q)-lateral ideal of S if

(1) $(S^p M S^q \cup S^p S M S S^q) \subseteq M$, and

(2) $(M] = M$, where p, q are positive integers and $p + q$ is an even positive integer.

Remark 3.2.

Every ordered lateral ideal of an ordered ternary semigroup S is an ordered (p, q)-lateral ideal, but every ordered (p, q)-lateral ideal of an ordered ternary semigroup S need not be an ordered lateral ideal.

Example 3.3.

Let S be a set of all strictly upper triangular matrices of order 6 over $\mathbb{N}_0$, the set of all non-negative integers, i.e.

$$S = \{(a_{ij})_{6\times 6} \mid a_{ij} = 0 \text{ if } i \geq j \text{ and } a_{ij} \in \mathbb{N}_0 \text{ if } i < j\},$$

where $\mathbb{N}_0$ is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}_0}$ is "less than or equal to". Then, S is an ordered ternary semigroup under the usual multiplication of matrices over $\mathbb{N}_0$ with partial order relation $\leq_S$, as defined in Example 2.3.

Consider

$$M_{lgen} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{N}_0 \right\}.$$

Then, it is easy to see that M_{lgen} is an ordered ternary subsemigroup of S and M_{lgen} is an ordered $(1,3)$-lateral ideal of S. Now,

$$SM_{lgen}S \bigcup SSM_{lgen}SS = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c, d \in \mathbb{N}_0 \right\} \nsubseteq M_{lgen}.$$

Therefore, M_{lgen} is not an ordered lateral ideal of S.

Lemma 3.4.

Let $\{M_i : i \in I\}$ be a family of ordered (p,q)-lateral ideals of an ordered ternary semigroup S. Then, $\bigcap_{i\in I} M_i$ is also an ordered (p,q)-lateral ideal of S if $\bigcap_{i\in I} M_i \neq \emptyset$.

Theorem 3.5.

Let S be an ordered ternary semigroup. Then, every ordered (p,q)-lateral ideal is an ordered $(p+p_1, q+q_1)$-lateral ideal of S, where p_1 and q_1 are non-negative integers and p_1+q_1 is even.

Proof:

Let M be an ordered (p,q)-lateral ideal of S. Then, $(S^pMS^q \cup S^pSMSS^q) \subseteq M$ and $(M] = M$. Now we have following two cases.

Case 1: If p_1 and q_1 are odd, then $p_1 = 2p_2+1$ and $q_1 = 2q_2+1$, where p_2 and q_2 are non-negative integers. It follows that $S^{p_1} = S^{2p_2+1} \subseteq S$ and $S^{q_1} = S^{2q_2+1} \subseteq S$. Now,

$$\begin{aligned} S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1}SMSS^{q+q_1} &= S^{p+2p_2+1}MS^{q+2q_2+1} \cup S^{p+2p_2+1}SMSS^{q+2q_2+1} \\ &= S^pS^{2p_2+1}MS^{2q_2+1}S^q \cup S^pS^{2p_2+1}SMSS^{2q_2+1}S^q \\ &\subseteq S^pSMSS^q \cup S^pSSMSSS^q \\ &= S^pSMSS^q \cup S^{p-1}SSSMSSSS^{q-1} \\ &\subseteq S^pSMSS^q \cup S^{p-1}SMSS^{q-1} \\ &= S^pSMSS^q \cup S^pMS^q \\ &\subseteq M. \end{aligned}$$

Case 2: If p_1 and q_1 are even, then $p_1 = 2p_3$ and $q_1 = 2q_3$, where p_3 and q_3 are non-negative integers. It follows that $S^{p_1} = S^{2p_3}$ and $S^{q_1} = S^{2q_3}$. Then, we have

$$\begin{aligned} S^{p+p_1}MS^{q+q_1} \cup S^{p+p_1}SMSS^{q+q_1} &= S^{p+2p_3}MS^{q+2q_3} \cup S^{p+2p_3}SMSS^{q+2q_3} \\ &= S^pS^{2p_3}MS^{2q_3}S^q \cup S^pS^{2p_3}SMSS^{2q_3}S^q \\ &\subseteq S^{p-1}S^{2p_3+1}MS^{2q_3+1}S^{q-1} \cup S^pS^{2p_3+1}MS^{2q_3+1}S^q \\ &\subseteq S^{p-1}SMSS^{q-1} \cup S^pSMSS^q \\ &\subseteq S^pMS^q \cup S^pSMSS^q \\ &\subseteq M. \end{aligned}$$

Hence, in all the two cases and by assumption $(M]= M$, M is an ordered $(p + p_1, q + q_1)$-lateral ideal of S. ■

Corollary 3.6.

Let S be an ordered ternary semigroup and A be an ordered ternary subsemigroup of S. If A is an ordered (p, q)-lateral ideal of S, then for any positive integer n:
(1) A will be an ordered (np, nq)-lateral ideal of S,
(2) A will be an ordered (p^n, q^n)-lateral ideal of S.

Lemma 3.7.

For any non-empty subset A of an ordered ternary semigroup S,
(1) $(S^pSASS^q \cup S^pAS^q]$ is an ordered (p, q)-lateral ideal of S,
(2) $(\bigcup_{i=1}^{p+q-1} A^i$ (skip A^i, when i is even) $\cup$ $(S^pSASS^q \cup S^pAS^q]$ is the smallest ordered (p, q)-lateral ideal of S containing A.

Proof:

It is easy to verify that $(S^pSASS^q \cup S^pAS^q]$ is an ordered ternary subsemigroup of S. To show $(S^pSASS^q \cup S^pAS^q]$ is an ordered (p, q)-lateral ideal of S, we have

$$\begin{aligned} &S^p((S^pSASS^q \cup S^pAS^q])S^q \cup S^pS((S^pSASS^q \cup S^pAS^q])SS^q \\ \subseteq\ &(S^p]((S^pSASS^q \cup S^pAS^q])(S^q] \cup (S^p](S]((S^pSASS^q \cup S^pAS^q])(S](S^q] \\ \subseteq\ &(S^p(S^pSASS^q \cup S^pAS^q)S^q \cup S^pS(S^pSASS^q \cup S^pAS^q)SS^q] \\ =\ &(S^pS^pSASS^qS^q \cup S^pS^pAS^qS^q \cup S^pSS^pSASS^qSS^q \cup S^pSS^pAS^qSS^q]. \end{aligned} \tag{1}$$

As we know that, here $p + q$ is even, thus we arises following two cases.

Case 1: If p and q are odd, then $S^p \subseteq S$ and $S^q \subseteq S$. Now from Equation (1)

$$\begin{aligned}
&S^p((S^pSASS^q \cup S^pAS^q])S^q \cup S^pS((S^pSASS^q \cup S^pAS^q])SS^q \\
&\subseteq (S^pS^pSASS^qS^q \cup S^pS^pAS^qS^q \cup S^pSS^pSASS^qSS^q \cup S^pSS^pAS^qSS^q] \\
&\subseteq (S^pSSASSS^q \cup S^pSASS^q \cup S^pSSSASSSS^q \cup S^pSSASSS^q] \\
&= (S^{p-1}SSSASSSS^{q-1} \cup S^pSASS^q \cup S^pSSSASSSS^q \cup S^{p-1}SSSASSSS^{q-1}] \\
&\subseteq (S^{p-1}SASS^{q-1} \cup S^pSASS^q \cup S^pSASS^q \cup S^{p-1}SASS^{q-1}] \\
&= (S^pAS^q \cup S^pSASS^q \cup S^pSASS^q \cup S^pAS^q] \\
&= (S^pAS^q \cup S^pSASS^q].
\end{aligned}$$

Case 2: If p and q are even, then $S^p = S^{2m}$ and $S^q = S^{2n}$, where m and n are positive integers. Now again from Equation (1), we have

$$\begin{aligned}
&S^p((S^pSASS^q \cup S^pAS^q])S^q \cup S^pS((S^pSASS^q \cup S^pAS^q])SS^q \\
&\subseteq (S^pS^pSASS^qS^q \cup S^pS^pAS^qS^q \cup S^pSS^pSASS^qSS^q \cup S^pSS^pAS^qSS^q] \\
&= (S^pS^{2m+1}AS^{2n+1}S^q \cup S^pS^{2m}AS^{2n}S^q \cup S^pSS^{2m+1}AS^{2n+1}SS^q \cup S^pS^{2m+1}AS^{2n+1}S^q] \\
&\subseteq (S^pSASS^q \cup S^{p-1}SS^{2m}AS^{2n}SS^{q-1} \cup S^pSSASSS^q \cup S^pSASS^q] \\
&= (S^pSASS^q \cup S^{p-1}S^{2m+1}AS^{2n+1}S^{q-1} \cup S^{p-1}SSSASSSS^{q-1} \cup S^pSASS^q] \\
&\subseteq (S^pSASS^q \cup S^{p-1}SASS^{q-1} \cup S^{p-1}SASS^{q-1} \cup S^pSASS^q] \\
&= (S^pSASS^q \cup S^pAS^q \cup S^pAS^q \cup S^pSASS^q] \\
&= (S^pSASS^q \cup S^pAS^q].
\end{aligned}$$

Therefore, in all these cases $(S^pSASS^q \cup S^pAS^q]$ is an ordered (p,q)-lateral ideal of S.

(2) Let M = $(\bigcup_{i=1}^{p+q-1} A^i \cup (S^pSASS^q \cup S^pAS^q)]$ and x, y, $z \in M$. Clearly, $A \subseteq M$. Now we have following two cases.

Case 1: x, y, $z \in (\bigcup_{i=1}^{p+q-1} A^i]$. Then there exists a, b, $c \in (\bigcup_{i=1}^{p+q-1} A^i)$ such that $x \leq a$, $y \leq b$ and $z \leq c$. It implies $xyz \leq abc$ and $abc \in A^m$. If $m \leq p+q-1$, we have $abc \in (\bigcup_{i=1}^{p+q-1} A^i)$ and then $xyz \in (\bigcup_{i=1}^{p+q-1} A^i]$. If $m > p+q-1$, then $abc \in (S^pSASS^q \cup S^pAS^q)$. It implies $xyz \in (S^pSASS^q \cup S^pAS^q]$.

Case 2: x, y, $z \in (S^pSASS^q \cup S^pAS^q]$. It is easy to show that $xyz \in (S^pSASS^q \cup S^pAS^q]$.

Therefore, $(\bigcup_{i=1}^{p+q-1} A^i \cup (S^pSASS^q \cup S^pAS^q]$ is an ordered ternary subsemigroup of S.

Using part (1), it is easy to verify that $(\bigcup_{i=1}^{p+q-1} A^i \cup (S^pSASS^q \cup S^pAS^q]$ is an ordered (p,q)-lateral ideal of S.

Finally, it remains to prove that M is the smallest ordered (p,q)-lateral ideal of S containing A. For this, suppose that M_1 is an ordered (p,q)-lateral ideal of S containing A. Then,

$$\begin{aligned}
M &= (\textstyle\bigcup_{i=1}^{p+q-1} A^i \cup (S^pSASS^q \cup S^pAS^q)] \\
&\subseteq (\textstyle\bigcup_{i=1}^{p+q-1} M_1^i \cup (S^pSM_1SS^q \cup S^pM_1S^q)] \\
&\subseteq M_1 \cup M_1 \\
&\subseteq M_1.
\end{aligned}$$

Hence, $(\bigcup_{i=1}^{p+q-1} A^i \cup (S^pSASS^q \cup S^pAS^q)]$ is the smallest ordered (p,q)-lateral ideal of S containing A. ∎

Furthermore, for any $a \in S$, M(a)= $((S^pSaSS^q \cup S^paS^q) \cup \{a, a^3, ...a^{p+q-1}\}]$ (skip a^i, if i is even).

Theorem 3.8.

Let A and B be two ordered ternary subsemigroups of an ordered ternary semigroup S such that $A \subseteq B$ and $B^3 = B$. If A is an ordered (p,q)-lateral ideal of S. Then, A will be an ordered lateral ideal of B.

Proof:

Suppose A and B are two ordered ternary subsemigroups of S such that $A \subseteq B$ and $B^3 = B$. If A is an ordered (p,q)-lateral ideal of S, then $S^pSASS^q \cup S^pAS^q \subseteq A$ and $(A] = A$. Now we have $BAB \cup BBABB = BAB^3 \cup B^3BABB^3$ or $B^3AB \cup B^3BABB^3$. Proceeding in this way, we get $BAB \cup BBABB = B^pAB^q \cup B^pBABB^q$. Now,

$$\begin{aligned} BAB \cup BBABB &= B^pAB^q \cup B^pBABB^q \\ &\subseteq S^pAS^q \cup S^pSASS^q \\ &\subseteq A. \end{aligned}$$

This shows that A is an ordered lateral ideal of B. ∎

Definition 3.9.

An ordered ternary semigroup S is total if any element of S can be written as the product of three elements of S, that is, $S^3 = S$.

Corollary 3.10.

If S is a total ordered ternary semigroup, then every ordered (p,q)-lateral ideal of S will be an ordered lateral ideal of S.

Corollary 3.11.

Let S be an ordered ternary semigroup. If an ordered (p,q)-lateral ideal I of S is idempotent, then I will be an ordered lateral ideal of S.

Definition 3.12.

An ordered (p,q)-lateral ideal of an ordered ternary semigroup S is called minimal ordered (p,q)-lateral ideal of S if it does not properly contain any ordered (p,q)-lateral ideal of S.

Definition 3.13.

An ordered (p,q)-lateral ideal of an ordered ternary semigroup S is called maximal ordered (p,q)-lateral ideal of S if it is not contained in any other proper ordered (p,q)-lateral ideal of S.

Definition 3.14.

Let S be an ordered ternary semigroup. Then, S is called an (p,q)-lateral simple if S is a unique ordered (p,q)-lateral ideal of S.

Example 3.15.

Consider an ordered ternary semigroup S as given in Example 2.8. Then, S is an (p,q)-lateral simple ordered ternary semigroup.

Theorem 3.16.

Let S be an ordered ternary semigroup. Then, an ordered (p,q)-lateral ideal M is minimal if and only if $(S^paS^q \cup S^pSaSS^q] = M$ for all $a \in M$.

Proof:

Suppose an ordered (p,q)-lateral ideal M is minimal. Let $a \in M$. Then, $(S^pSaSS^q \cup S^paS^q] \subseteq (S^pSMSS^q \cup S^pMS^q] \subseteq (M] \subseteq M$. By Lemma 3.7, we have $(S^pSaSS^q \cup S^paS^q]$ is an ordered (p,q)-lateral ideal of S. As M is minimal ordered (p,q)-lateral ideal of S, therefore, $(S^pSaSS^q \cup S^paS^q] = M$.

Conversely, suppose that $(S^pSaSS^q \cup S^paS^q] = M$ for all $a \in M$. Let M' be any ordered (p,q)-lateral ideal of S contained in M. Let $m \in M'$. Then $m \in M$. By assumption, we have $(S^pSmSS^q \cup S^pmS^q] = M$ for all $m \in M$. Now, $M = (S^pSmSS^q \cup S^pmS^q] \subseteq (S^pSM'SS^q \cup S^pM'S^q] \subseteq M'$. This implies $M \subseteq M'$. Thus, M= M'. Hence, M is minimal ordered (p,q)-lateral ideal of S. ■

Theorem 3.17.

Let S be an ordered ternary semigroup. Then, S is an (p,q)-lateral simple if and only if $(S^paS^q \cup S^pSaSS^q] = S$ for all $a \in S$.

Proof:

Assume that S is an (p,q)-lateral simple ordered ternary semigroup. We have that S is a minimal ordered (p,q)-lateral ideal of S. By Theorem 3.16, $(S^paS^q \cup S^pSaSS^q] = S$ for all $a \in S$.

Conversely, suppose that $(S^paS^q \cup S^pSaSS^q] = S$ for all $a \in S$. By Theorem 3.16, S is a minimal ordered (p,q)-lateral ideal of S, and therefore S is an (p,q)-lateral simple. ■

Definition 3.18.

Let S be an ordered ternary semigroup. An element a which satisfy the condition $(S^paS^q \cup S^pSaSS^q] = S$ is called an (p,q)-lateral simple element.

Theorem 3.19.

Let M be the set of all (p,q)-lateral simple elements of an ordered ternary semigroup S. If $S \setminus M$

is non-empty, then $S \setminus M$ is the maximal ordered (p, q)-lateral ideal of S.

Proof:

Let $S \setminus M = M'$ such that $M' \neq \emptyset$. We have to show that M' is a maximal ordered (p, q)-lateral ideal of S. On the contrary, suppose that M' is not an ordered (p, q)-lateral ideal of S. Then, there exists $m \in M'$, $s_1 \in S^p$ and $s_2 \in S^q$ such that $s_1ms_2 \notin M'$. It implies $s_1ms_2 \in M$ and so $(S^p(s_1ms_2)S^q \cup S^pS(s_1ms_2)SS^q] = S$.

Now

$$\begin{aligned} S &= (S^p(s_1ms_2)S^q \cup S^pS(s_1ms_2)SS^q] \\ &\subseteq (S^pS^pmS^qS^q \cup S^pSS^pmS^qSS^q] \\ &\subseteq (S^pmS^q \cup S^pSmSS^q]. \end{aligned}$$

This shows that $m \in M$, which is a contradiction. Thus, $s_1ms_2 \in M'$.

Let $m \in M'$ and $s \in S$ such that $s \leq m$. If $s \in M$, then $S = (S^psS^q \cup S^pSsSS^q] \subseteq (S^pmS^q \cup S^pSmSS^q]$ and so $m \in M$, which is a contradiction and hence $s \in M'$. Therefore, M' is an ordered (p, q)-lateral ideal of S.

Now let M'' be any ordered (p, q)-lateral ideal of S such that M' is properly contained in M''. Then, there exists $m \in M'' \setminus M'$ such that $(S^pmS^q \cup S^pSmSS^q] = S$, so $S = (S^pmS^q \cup S^pSmSS^q] \subseteq M''$. Therefore, $M'' = S$ and hence, M' is a maximal ordered (p, q)-lateral ideal of S. ∎

Lemma 3.20.

If $M_{(p,q)}$ is an ordered (p, q)-lateral ideal of S and B is an ordered ternary subsemigroup of S and if B is an ordered (p, q)-lateral simple such that $B \cap M_{(p,q)} \neq \emptyset$, then $B \subseteq M_{(p,q)}$.

Proof:

Suppose that B is an ordered (p, q)-lateral simple such that $B \cap M_{(p,q)} \neq \emptyset$. Let $a \in B \cap M_{(p,q)}$. By Lemma 3.7, we have $(B^paB^q \cup B^pBaBB^q] \cap B$ is an ordered (p, q)-lateral ideal of B. This implies that $(B^paB^q \cup B^pBaBB^q] \cap B = B$. Hence, $B \subseteq (B^paB^q \cup B^pBaBB^q] \subseteq (B^pM_{(p,q)}B^q \cup B^pBM_{(p,q)}BB^q] \subseteq M_{(p,q)}$, so $B \subseteq M_{(p,q)}$. ∎

Theorem 3.21.

Let S be an ordered ternary semigroup. If an ordered (p, q)-lateral ideal $M_{(p,q)}$ of S is an ordered (p, q)-lateral simple ordered ternary semigroup, then $M_{(p,q)}$ is a minimal ordered (p, q)-lateral ideal of S.

Proof:

Suppose that $M_{(p,q)}$ is an ordered (p, q)-lateral simple. Let $A_{(p,q)}$ be an ordered (p, q)-lateral ideal of S such that $A_{(p,q)} \subseteq M_{(p,q)}$. Then, $A_{(p,q)} \cap M_{(p,q)} \neq \emptyset$. Hence, from Lemma 3.20, we have $M_{(p,q)} \subseteq A_{(p,q)}$. Therefore $A_{(p,q)} = M_{(p,q)}$, so $M_{(p,q)}$ is a minimal ordered (p, q)-lateral ideal of S.

Alternative method. Let $M_{(p,q)}$ be an ordered (p,q)-lateral simple. By Theorem 3.17, we have $(M^p_{(p,q)}aM^q_{(p,q)} \cup M^p_{(p,q)}M_{(p,q)}aM_{(p,q)}M^q_{(p,q)}] = M_{(p,q)}$ for all $a \in M_{(p,q)}$. For every $a \in M_{(p,q)}$, we have $M_{(p,q)} = (M^p_{(p,q)}aM^q_{(p,q)} \cup M^p_{(p,q)}M_{(p,q)}aM_{(p,q)}M^q_{(p,q)}] \subseteq (S^paS^q \cup S^pSaSS^q] \subseteq (S^pM_{(p,q)}S^q \cup S^pSM_{(p,q)}SS^q] \subseteq M_{(p,q)}$. Then, $(S^paS^q \cup S^pSaSS^q] = M_{(p,q)}$ for all $a \in M_{(p,q)}$. By Theorem 3.16, $M_{(p,q)}$ is minimal. ∎

4. Conclusion

In this paper, we study some results and properties of ordered (p,q)-lateral ideals and (p,q)-lateral simple ordered ternary semigroups. Based on the results of this paper some further work can be done on (p,q)-lateral ideals and (p,q)-lateral hyperideals in other ternary structures like bi Γ-ternary semigroups and ternary semihypergroups.

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