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## Spectral tau-Jacobi algorithm for space fractional advection-dispersion problem

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### Abstract

In this paper, we use the shifted Jacobi polynomials to approximate the solution of the space fractional advection-dispersion. The method is based on the Jacobi operational matrices of fractional derivative and integration. A double shifted Jacobi expansion is used as an approximating polynomial. We apply this method to solve linear and nonlinear term FDEs by using initial and boundary conditions.

**Keywords:** Caputo derivative; Jacobi polynomials; Spectral method; Space fractional differential equations; Advection-dispersion equations

**MSC 2010 No.:** 65M70, 33C45, 34A08

### 1. Introduction

Ordinary and partial differential equations of integer order are special cases of fractional differential equations. Recently, linear and nonlinear fractional differential equations (FDEs) attracted the interest of Scientists. Many researchers describe the applications in

fluid, biology, optics, mechanics, engineering, physics, mathematics and other fields of science by using fractional ordinary and partial differential equations.

The fractional calculus is very important for investigating the properties of derivatives and integrals of non-integer orders. There are several analytical methods for solving FDEs, such as the sub-equation method Zhang and Zhang (2011), the first integral method Lu (2012), the extended tanh method Abdou (2007), Shukri and Al-Khaled (2010), Kudryashov method Serife and Emine (2014), the Exp-function method Zhang et al. (2010), the  $(G'/G)$ -Expansion method Wanga et al. (2008). Most of these methods depend on the balance principle.

Several numerical methods, such as, finite difference method Gao et al. (2012), Sweilam et al. (2011), finite element method Deng (2008), Yingjun and Jingtang (2011), adomain decomposition method Hu et al. (2008), Shawagfeh (2002), variational iteration method Mehmet et al. (2012), Yasir et al. (2011), homotopy analysis method Dehghan et al. (2010), spectral methods Doha et al. (2018), Ortiz and Samara (1983).

The spectral methods are used to introduce the approximate solutions for the fractional differential equations. Spectral version are the Galerkin, Petrov-Galerkin, Collocation, Tau methods, Abd-Elhameed et al. (2013), Abd-Elhameed and Youssri, (2019) Youssri and Abd-Elhameed (2018) and wavelets methods Sweilam et al. (2017), Sweilam et al. (2017). The problem handled in this paper is important in FDEs, hence, we note that it was solved using different methods and different numerical techniques such as Pang et al. (2015) using the Kansa method, Shen and Liu (2011), Jiang et al. (2012) derived for the multi-term time-space Caputo-Riesz FADE on a finite domain. The adomain decomposition method is applied in Doha et al. (2017), Hikal and Abu Ibrahim (2015).

In this work, we apply shifted Jacobi polynomials for solving the fractional order initial and boundary value problem with variable coefficients. The main idea of this paper is the numerical solution for this equation by spectral shifted Jacobi tau method. We get a system of equations which are solved using the Gaussian elimination technique.

This paper is arranged as follows. In Section 2 we introduce some definitions and properties of fractional calculus and Shifted Jacobi polynomials. In Section 3 we explain the algorithm for solving the initial and boundary FDE with variable coefficients using shifted Jacobi polynomial. In Section 4 we give some numerical experiments. Finally, in Section 5 we will present some conclusions of our work.

## **2. Important properties and definitions**

In this section, we introduce some important definitions and properties for fractional calculus Igor Podlubny (1999), Kenneth and Ross (1993), and shifted Jacobi polynomials Doha et al. (2012), Doha et al. (2014), Rainville (1960).

**Definition 1.**

The fractional integral of order  $\beta$  ( $\beta > 0$ ) according to Riemann-Liouville is

$$I^\beta f(y) = \frac{1}{\Gamma(\beta)} \int_0^y (y-t)^{\beta-1} f(t) dt, \quad \beta > 0, y > 0, I^0 f(y) = f(y),$$

and  $I^\beta$  satisfies the following properties:

$$\begin{aligned} I^\beta I^\gamma &= I^{\beta+\gamma}, \\ I^\beta I^\gamma &= I^\gamma I^\beta, \\ I^\beta y^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+\beta+1)} y^{\nu+\beta}. \end{aligned} \tag{2}$$

**Definition 2.**

The fractional derivative of order  $\beta$  according to Caputo

$$D^\beta f(y) = I^{m-\beta} D^m f(y) = \frac{1}{\Gamma(m-\beta)} \int_0^y (y-t)^{m-\beta-1} f^{(m)}(t) dt, \tag{3}$$

where  $m - 1 < \beta \leq m$ , and  $D^\beta$  satisfies the following properties:

$$\begin{aligned} (D^\beta I^\beta f)(y) &= f(y), \\ D^\beta y^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\beta+1)} y^{\nu-\beta}. \end{aligned} \tag{4}$$

**2.1. Important properties of Shifted Jacobi polynomials**

Let  $P_n^{(\alpha,\beta)}(y): y \in [-1,1]$  denote the standard Jacobi polynomials which are orthogonal with weight function  $w(y) = (1-y)(1+y)$

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(y) P_m^{(\alpha,\beta)}(y) w^{(\alpha,\beta)}(y) dy = \delta_{nm} \lambda_m^{(\alpha,\beta)}, \tag{5}$$

where

$$\lambda_m^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1} \Gamma(m+\alpha+1) \Gamma(m+\beta+1)}{(2m+\alpha+\beta+1) m! \Gamma(m+\alpha+\beta+1)},$$

Jacobi polynomials can be generated by using the recurrence relation

$$P_m^{(\alpha,\beta)}(y) = \frac{(\alpha+\beta+2m-1)(\alpha^2-\beta^2+y(\alpha+\beta+2m)(\alpha+\beta+2m-2))}{2m(\alpha+\beta+m)(\alpha+\beta+2m-2)} P_{m-1}^{(\alpha,\beta)}(y),$$

$$- \frac{(\alpha+\beta-1)(\beta+m-1)(\alpha+\beta+2m)}{m(\alpha+\beta+m)(\alpha+\beta+2m-2)} P_{m-2}^{(\alpha,\beta)}(y),$$

we have

$$P_0^{(\alpha,\beta)}(y) = 1, P_1^{(\alpha,\beta)}(y) = \frac{\alpha+\beta+2}{2}(y) + \frac{\alpha-\beta}{2}.$$

If we denote  $y = \frac{2z}{h} - 1$ , we obtain the Shifted Jacobi polynomials in the interval  $[0, h]$ . Suppose that

$$P_m^{(\alpha,\beta)}\left(\frac{2z}{h} - 1\right) = P_{h,m}^{(\alpha,\beta)}(z).$$

They are orthogonal with the weight function  $w_h^{(\alpha,\beta)}(z)$

$$\int_0^h P_{h,n}^{(\alpha,\beta)}(z) P_{h,m}^{(\alpha,\beta)}(z) w_h^{(\alpha,\beta)}(z) dz = L_{h,m}^{(\alpha,\beta)}, \tag{6}$$

where

$$L_{h,m}^{(\alpha,\beta)} = \left(\frac{h}{2}\right)^{\alpha+\beta+1} \delta_{nm} \quad \lambda_m^{(\alpha,\beta)} = \frac{h^{\alpha+\beta+1} \Gamma(m+\alpha+1) \Gamma(m+\beta+1)}{(2m+\alpha+\beta+1) m! \Gamma(m+\alpha+\beta+1)}.$$

The shifted Jacobi polynomials  $P_{h,m}^{(\alpha,\beta)}(z)$  has the form

$$P_{h,m}^{(\alpha,\beta)}(z) = \sum_{k=0}^m \frac{(-1)^{m-k} \Gamma(m+\beta+1) \Gamma(m+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(m+\alpha+\beta+1) (m-k)! k! h^k} z^k, \tag{7}$$

where

$$P_{h,m}^{(\alpha,\beta)}(0) = \frac{(-1)^m \Gamma(m+\beta+1)}{\Gamma(\beta+1) m!}, \quad P_{h,m}^{(\alpha,\beta)}(h) = \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1) m!}.$$

Suppose that a function  $V(z)$  can be expanded in terms of Shifted Jacobi polynomials

$$V(z) = \sum_{m=0}^{\infty} c_m P_{h,m}^{(\alpha,\beta)}(z),$$

therefore

$$c_m = \frac{1}{L_{h,m}^{(\alpha,\beta)}} \int_0^h w_h^{(\alpha,\beta)}(z) V(z) P_{h,m}^{(\alpha,\beta)}(z) dz. \tag{8}$$

We can write this relation by using only the first  $(M + 1)$  terms

$$\sum_{m=0}^M c_m P_{h,m}^{(\alpha,\beta)}(z) = C^T \psi_{h,M}(z), \tag{9} \quad V_M(z) =$$

$$\begin{cases} C^T = [c_0, c_1, \dots, c_M], \\ \psi_{h,M}(z) = [P_{h,0}^{(\alpha,\beta)}(z), P_{h,1}^{(\alpha,\beta)}(z), \dots, P_{h,M}^{(\alpha,\beta)}(z)]^T. \end{cases} \quad (10)$$

If  $V(z, t)$  of two variables can be expanded in terms of double Shifted Jacobi polynomials

$$V_{N,M}(z) = \sum_{n=0}^N \sum_{m=0}^M c_{nm} P_{\ell,n}^{(\alpha,\beta)}(t) P_{h,m}^{(\alpha,\beta)}(z) = \psi_{\ell,N}^T(t) A \psi_{h,M}(z), \quad (11)$$

where  $A$  is the Shifted Jacobi coefficient matrix is defined by

$$A = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0M} \\ a_{10} & a_{11} & \dots & a_{1M} \\ \vdots & \vdots & \dots & \vdots \\ a_{N0} & a_{N1} & \dots & a_{NM} \end{bmatrix},$$

where

$$a_{nm} = \frac{1}{L_{\ell,n}^{(\alpha,\beta)} L_{h,m}^{(\alpha,\beta)}} \int_0^\ell \int_0^h V(z, t) P_{\ell,n}^{(\alpha,\beta)}(t) P_{h,m}^{(\alpha,\beta)}(z) w_\ell^{(\alpha,\beta)}(t) w_h^{(\alpha,\beta)}(z) dz dt \quad (12)$$

If we denote the first integration of  $\psi_{\ell,N}(t)$  by

$$I^1 \psi_{\ell,N}(t) = p^{(1)} \psi_{\ell,N}(t), \quad (13)$$

where  $p^{(1)}$  is an  $(N + 1) \times (N + 1)$  matrix of first integration by using Riemann-Liouville as:

$$p^{(1)} = \begin{bmatrix} \Lambda_1(0,0, \alpha, \beta) & \Lambda_1(0,1, \alpha, \beta) & \dots & \Lambda_1(0, N, \alpha, \beta) \\ \Lambda_1(1,0, \alpha, \beta) & \Lambda_1(1,1, \alpha, \beta) & \dots & \Lambda_1(1, N, \alpha, \beta) \\ \vdots & \vdots & \dots & \vdots \\ \Lambda_1(N, 0, \alpha, \beta) & \Lambda_1(N, 1, \alpha, \beta) & \dots & \Lambda_1(N, N, \alpha, \beta) \end{bmatrix}, \quad (14)$$

$$\begin{aligned} \Lambda_1(n, m, \alpha, \beta) &= \sum_{k=0}^n \frac{(-1)^{n-k} \Gamma(n+\beta+1) \Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \Gamma(m+\alpha+\beta+1) (n-k)! \Gamma(k+2)} \\ &\times \sum_{f=0}^m \frac{(-1)^{m-f} \Gamma(m+f+\alpha+\beta+1) \Gamma(\alpha+1) \Gamma(f+k+\beta+2) (2m+\alpha+\beta+1) m!}{\Gamma(m+\alpha+1) \Gamma(f+\beta+1) (m-f)! f! \Gamma(f+k+\alpha+\beta+3)}. \end{aligned}$$

The fractional derivative of  $\psi_{h,M}(z)$  written as

$$D^{(\gamma)} \psi_{h,M}(z), \quad (15) \quad D^\gamma \psi_{h,M}(z) =$$

where  $D^{(\gamma)}$  is an  $(M + 1) \times (M + 1)$  matrix of fractional order derivative by using Caputo as

$$D^{(\gamma)} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ \Omega_\gamma([\gamma], 0) & \Omega_\gamma([\gamma], 1) & \dots & \Omega_\gamma([\gamma], M) \\ \vdots & \vdots & \dots & \vdots \\ \Omega_\gamma(M, 0) & \Omega_\gamma(M, 1) & \vdots & \Omega_\gamma(M, M) \end{bmatrix}, \tag{16}$$

$$\Omega_\gamma(n, m) = \sum_{k=[\gamma]}^n \frac{(-1)^{n-k} h^{\alpha+\beta-\gamma+1} \Gamma(m+\beta+1) \Gamma(n+\beta+1) \Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(m+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) (n-k)! \Gamma(k-\gamma+1)},$$

$$\times \sum_{q=0}^m \frac{(-1)^{m-q} \Gamma(m+q+\alpha+\beta+1) \Gamma(\alpha+1) \Gamma(q+k+\beta-\gamma+1)}{\Gamma(q+\beta+1) (m-q)! q! \Gamma(q+k+\alpha+\beta-\gamma+2)}, \tag{17}$$

### 3. Solution of space fractional advection-dispersion problem

Consider the space Riemann-Liouville fractional advection-dispersion problem

$$\frac{\partial V(z,t)}{\partial t} + v(z) D^\beta V(z,t) - K(z)D^\gamma V(z,t) = q(z,t), \tag{18}$$

$$0 < \beta < 1, 1 < \gamma < 2, (z,t) \in \Omega: (0,h)x(0,\ell),$$

with the non-homogenous boundary and initial conditions

$$\begin{aligned} V(0,t) &= V_0(t), V(h,t) = V_h(t) 0 < t < \ell, \\ V(z,0) &= f_0(z) 0 < z < h. \end{aligned} \tag{19}$$

We approximate  $V(z,t)$ ,  $v(z)$ ,  $K(z)$ ,  $q(z,t)$  and  $f_0(z)$  by the Shifted Jacobi polynomials as:

$$\begin{aligned} V_{N,M}(z,t) &= \psi_{\ell,N}^T(t) A \psi_{h,M}(z), \\ v_M(z) &= v^T \psi_{h,M}(z), \\ K_M(z) &= K^T \psi_{h,M}(z), \\ q_{N,M}(z,t) &= \psi_{\ell,N}^T(t) Q \psi_{h,M}(z), \\ f_0(z) &= \psi_{\ell,N}^T(t) F \psi_{h,M}(z), \end{aligned} \tag{20}$$

where  $v^T$ ,  $K^T$ ,  $Q$  and  $F$  can be written as

$$v^T = [v_0, v_1, \dots, v_M], K^T = [K_0, K_1, \dots, K_M], Q = \begin{bmatrix} q_{00} & q_{01} & \dots & q_{0M} \\ q_{10} & q_{11} & \dots & q_{1M} \\ \vdots & \vdots & \dots & \vdots \\ q_{N0} & q_{N1} & \dots & q_{NM} \end{bmatrix}, F = \begin{bmatrix} f_0 & f_1 & \dots & f_M \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{21}$$

By integrating equation (17)

$$V(z, t) - f_0(z) = - \int_0^t v(z) D^\beta V(z, t) dt + \int_0^t K(z) D^\gamma V(z, t) dt + \int_0^t q(z, t) dt, \tag{22}$$

using (12),(14) and (19)

$$\int_0^t v(z) D^\beta V(z, t) dt = v^T \psi_{h,M}(z) \left( \int_0^t \psi_{\ell,N}^T(t) dt \right) AD^{(\beta)} \psi_{h,M}(z),$$

$$= v^T \psi_{h,M}(z) \psi_{\ell,N}^T(t) p^T A D^{(\beta)} \psi_{h,M}(z),$$

$$\psi_{\ell,N}^T(t) p^T A D^{(\beta)} \psi_{h,M}(z) \psi_{h,M}^T(z) v. \tag{23}$$

Let

$$\psi_{h,M}(z) \psi_{h,M}^T(z) v = H^T \psi_{h,M}(z),$$

(24)

where is an  $(M + 1) \times (M + 1)$  matrix. Equation (23) can write in the form

$$\sum_{k=0}^M v_k P_{h,k}^{(\alpha,\beta)}(z) P_{h,j}^{(\alpha,\beta)}(z) = \sum_{k=0}^M H^T \psi_{h,M}(z).$$

Multiply both sides by  $P_{h,m}^{(\alpha,\beta)}(z) w_h^{(\alpha,\beta)}(z)$ ,  $m = 0, 1, \dots, M$  and integrating from 0 to  $h$ ,

$$H_{nm} = \frac{1}{I_{h,m}^{(\alpha,\beta)}} \sum_{k=0}^M v_k \int_0^h P_{h,k}^{(\alpha,\beta)}(z) P_{h,m}^{(\alpha,\beta)}(z) P_{h,n}^{(\alpha,\beta)}(z) w_h^{(\alpha,\beta)}(z) dz, \quad n, m = 0, 1, \dots, M. \tag{25}$$

From (23) into (22)

$$\int_0^t v(z) D^\beta V(z, t) dt = \psi_{\ell,N}^T(t) p^T A D^{(\beta)} H^T \psi_{h,M}(z), \tag{26}$$



also we get,

$$\psi_{\ell,N}^T(t) p^T A \quad D^{(\gamma)} G^T \quad \psi_{h,M}(z), \quad \int_0^t K(z) \quad D^\gamma V(z, t) \quad dt = \quad (27)$$

where

$$\psi_{h,M}(z) \quad \psi_{h,M}^T(z) \quad K = G^T \quad \psi_{h,M}(z), \quad (28)$$

$$G_{nm} = \frac{1}{L_{h,m}^{(\alpha,\beta)}} \sum_{k=0}^M K_k \int_0^h P_{h,k}^{(\alpha,\beta)}(z) P_{h,m}^{(\alpha,\beta)}(t) \quad P_{h,n}^{(\alpha,\beta)}(z) w_h^{(\alpha,\beta)}(z) dz, \quad (29)$$

and

$$\psi_{\ell,N}^T(t) \quad p^T \quad Q \quad \psi_{h,M}(z), \quad \int_0^t q(z, t) \quad dt = \quad (30)$$

apply (25), (26) and (29) in (21). Then, the residual  $R_{M,N}(z, t)$  for (21) can be written as

$$R_{M,N}(z, t) = \psi_{\ell,N}^T(t) [A - F + p^T A \quad D^{(\beta)} H^T - p^T A \quad D^{(\gamma)} G^T - -p^T Q] \psi_{h,M}(z), \quad (31)$$

$$= \psi_{\ell,N}^T(t) \quad E \quad \psi_{h,M}(z),$$

where

$$E = A - F + p^T A \quad D^{(\beta)} H^T - p^T A \quad D^{(\gamma)} G^T - p^T Q,$$

we approximate  $E$  by the Shifted Jacobi polynomials as the following equations

$$E = (E_{mn}) = 0, m = 0, 1, \dots, M, n = 0, 1, \dots, N - 2, \quad (32)$$

are the elements in  $E$ . With respect to the boundary conditions

$$\begin{aligned} \psi_{\ell,N}^T(t) \quad A \quad \psi_{h,M}(0) &= V_0(t), \\ \psi_{\ell,N}^T(t) \quad A \quad \psi_{h,M}(h) &= V_h(t). \end{aligned} \quad (33)$$

We use the Gaussian elimination technique to solve the resulted algebraic system.

#### 4. Numerical results

In this section, we obtain some numerical results for three examples

**Example 1.**

Consider the following equation:

$$\frac{\partial V(z,t)}{\partial t} + \Gamma(5 - \beta)z^\beta D^\beta V(z,t) - \Gamma(\gamma + 1)zD^\gamma V(z,t) = (z^3 - z^2)e^{-t},$$

subject to the boundary conditions

$$V(0,t) = t, V(1,t) = e^{-t}, 0 < t < 1,$$

and the initial condition

$$V(z,0) = z^4, 0 < z < 1,$$

the exact solution for this equation has the form  $V(z,t) = z^3e^{-t} - z^2e^{-t}$ , we have two tables for  $t = 1$  and  $t = \frac{1}{2}$  corresponding to different values of  $(\alpha, \beta)$ .

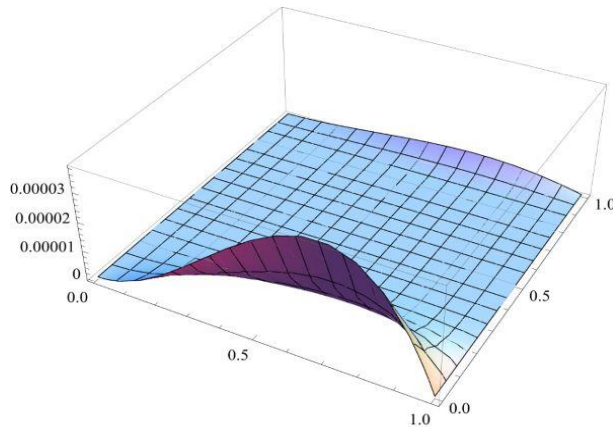
**Table 1:** Max. absolute errors of Example 1 in case ( $t = 1$ ) for different values of  $(\alpha, \beta)$

$z$	$(\alpha, \beta) = (0,0)$	$(\alpha, \beta) = (\frac{-1}{2}, \frac{-1}{2})$	$(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$	$(\alpha, \beta) = (\frac{-1}{2}, \frac{1}{2})$	$(\alpha, \beta) = (\frac{1}{2}, \frac{-1}{2})$
0	0.	0.	0.	0.	0.
0.1	$2.81 \times 10^{-7}$	$2.81 \times 10^{-7}$	$2.81 \times 10^{-7}$	$2.81 \times 10^{-7}$	$2.81 \times 10^{-7}$
0.2	$1.01 \times 10^{-6}$	$1.01 \times 10^{-6}$	$1.01 \times 10^{-6}$	$1.01 \times 10^{-6}$	$1.01 \times 10^{-6}$
0.3	$1.99 \times 10^{-6}$	$1.99 \times 10^{-6}$	$1.99 \times 10^{-6}$	$1.99 \times 10^{-6}$	$1.99 \times 10^{-6}$
0.4	$3.02 \times 10^{-6}$	$3.01 \times 10^{-6}$	$3.02 \times 10^{-6}$	$3.02 \times 10^{-6}$	$3.02 \times 10^{-6}$
0.5	$3.90 \times 10^{-6}$	$3.90 \times 10^{-6}$	$3.90 \times 10^{-6}$	$3.90 \times 10^{-6}$	$3.90 \times 10^{-6}$
0.6	$4.44 \times 10^{-6}$	$4.44 \times 10^{-6}$	$4.44 \times 10^{-6}$	$4.44 \times 10^{-6}$	$4.44 \times 10^{-6}$
0.7	$4.45 \times 10^{-6}$	$4.45 \times 10^{-6}$	$4.45 \times 10^{-6}$	$4.45 \times 10^{-6}$	$4.45 \times 10^{-6}$
0.8	$3.77 \times 10^{-6}$	$3.77 \times 10^{-6}$	$3.77 \times 10^{-6}$	$3.77 \times 10^{-6}$	$3.77 \times 10^{-6}$
0.9	$2.29 \times 10^{-6}$	$2.29 \times 10^{-6}$	$2.29 \times 10^{-6}$	$2.29 \times 10^{-6}$	$2.29 \times 10^{-6}$
1	0.	0.	0.	0.	0.

**Table 2:** Max. absolute errors of example 1 in case ( $t = \frac{1}{2}$ ) for different values of  $(\alpha, \beta)$

$z$	$(\alpha, \beta) = (0,0)$	$(\alpha, \beta) = (\frac{-1}{2}, \frac{-1}{2})$	$(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$	$(\alpha, \beta) = (\frac{-1}{2}, \frac{1}{2})$	$(\alpha, \beta) = (\frac{1}{2}, \frac{-1}{2})$
0	0.	0.	0.	0.	0.

0.1	$5.69 \times 10^{-9}$	$5.69 \times 10^{-9}$	$5.69 \times 10^{-9}$	$5.69 \times 10^{-9}$	$5.69 \times 10^{-9}$
0.2	$2.04 \times 10^{-8}$	$2.04 \times 10^{-8}$	$2.04 \times 10^{-8}$	$2.04 \times 10^{-8}$	$2.04 \times 10^{-8}$
0.3	$4.08 \times 10^{-8}$	$4.08 \times 10^{-8}$	$4.08 \times 10^{-8}$	$4.08 \times 10^{-8}$	$4.08 \times 10^{-8}$
0.4	$6.31 \times 10^{-8}$	$6.31 \times 10^{-8}$	$6.31 \times 10^{-8}$	$6.31 \times 10^{-8}$	$6.31 \times 10^{-8}$
0.5	$8.36 \times 10^{-8}$	$8.36 \times 10^{-8}$	$8.36 \times 10^{-8}$	$8.36 \times 10^{-8}$	$8.36 \times 10^{-8}$
0.6	$9.81 \times 10^{-8}$	$9.81 \times 10^{-8}$	$9.81 \times 10^{-8}$	$9.81 \times 10^{-8}$	$9.81 \times 10^{-8}$
0.7	$1.02 \times 10^{-8}$	$1.02 \times 10^{-7}$	$1.02 \times 10^{-7}$	$1.02 \times 10^{-7}$	$1.02 \times 10^{-7}$
0.8	$9.10 \times 10^{-8}$	$9.10 \times 10^{-8}$	$9.10 \times 10^{-8}$	$9.10 \times 10^{-8}$	$9.10 \times 10^{-8}$
0.9	$5.92 \times 10^{-8}$	$5.92 \times 10^{-8}$	$5.92 \times 10^{-8}$	$5.92 \times 10^{-8}$	$5.92 \times 10^{-8}$
1	0.	0.	0.	0.	0.



**Figure 1:** The absolute error of Example 1

### Example 2.

Consider the following equation:

$$\frac{\partial V(z,t)}{\partial t} + D^\beta V(z,t) - D^\gamma V(z,t) = (z - z^2) \sinh(z + t),$$

subject to the boundary conditions

$$V(0,t) = 0, V(1,t) = 0, 0 < t < 1,$$

and the initial condition

$$V(z,0) = z^2 - z^3, 0 < z < 1,$$

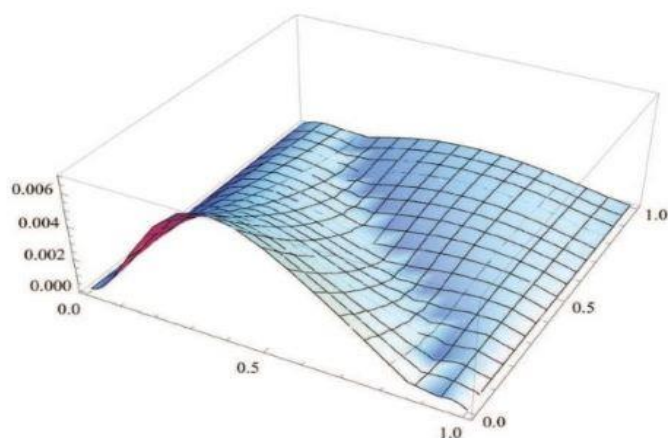
the exact solution for this equation has the form  $V(z,t) = (z - z^2)\sinh(z + t)$ , we have two tables for  $t = 1$  and  $t = \frac{1}{2}$  corresponding to different values of  $(\alpha, \beta)$ ,

**Table 3:** Max. absolute errors of Example 2 in case ( $t = 1$ ) for different values of  $(\alpha, \beta)$

$z$	$(\alpha, \beta) = (0,0)$	$(\alpha, \beta) = \left(\frac{-1}{2}, \frac{-1}{2}\right)$	$(\alpha, \beta) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$(\alpha, \beta) = \left(\frac{-1}{2}, \frac{1}{2}\right)$	$(\alpha, \beta) = \left(\frac{1}{2}, \frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.
0.1	$3.38 \times 10^{-4}$	$3.38 \times 10^{-4}$	$3.38 \times 10^{-4}$	$3.38 \times 10^{-4}$	$3.38 \times 10^{-4}$
0.2	$7.01 \times 10^{-5}$	$7.01 \times 10^{-5}$	$7.01 \times 10^{-5}$	$7.01 \times 10^{-5}$	$7.01 \times 10^{-5}$
0.3	$5.99 \times 10^{-4}$	$5.99 \times 10^{-4}$	$5.99 \times 10^{-4}$	$5.99 \times 10^{-4}$	$5.99 \times 10^{-4}$
0.4	$9.85 \times 10^{-4}$	$9.85 \times 10^{-4}$	$9.85 \times 10^{-4}$	$9.85 \times 10^{-4}$	$9.85 \times 10^{-4}$
0.5	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$
0.6	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$	$1.17 \times 10^{-3}$
0.7	$1.01 \times 10^{-3}$	$1.01 \times 10^{-3}$	$1.00 \times 10^{-3}$	$1.00 \times 10^{-3}$	$1.00 \times 10^{-3}$
0.8	$6.95 \times 10^{-4}$	$6.95 \times 10^{-4}$	$6.95 \times 10^{-4}$	$6.95 \times 10^{-4}$	$6.95 \times 10^{-4}$
0.9	$2.88 \times 10^{-4}$	$2.88 \times 10^{-4}$	$2.88 \times 10^{-4}$	$2.88 \times 10^{-4}$	$2.88 \times 10^{-4}$
1	0.	0.	0.	0.	0.

**Table 4:** Max. absolute errors of Example 2 in case  $(t = \frac{1}{2})$  for different values of  $(\alpha, \beta)$

$z$	$(\alpha, \beta) = (0,0)$	$(\alpha, \beta) = \left(\frac{-1}{2}, \frac{-1}{2}\right)$	$(\alpha, \beta) = \left(\frac{1}{2}, \frac{1}{2}\right)$	$(\alpha, \beta) = \left(\frac{-1}{2}, \frac{1}{2}\right)$	$(\alpha, \beta) = \left(\frac{1}{2}, \frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.
0.1	$1.42 \times 10^{-3}$	$1.42 \times 10^{-3}$	$1.42 \times 10^{-3}$	$1.42 \times 10^{-3}$	$1.42 \times 10^{-3}$
0.2	$1.76 \times 10^{-3}$	$1.76 \times 10^{-3}$	$1.76 \times 10^{-3}$	$1.76 \times 10^{-3}$	$1.76 \times 10^{-3}$
0.3	$1.41 \times 10^{-3}$	$1.41 \times 10^{-3}$	$1.41 \times 10^{-3}$	$1.41 \times 10^{-3}$	$1.41 \times 10^{-3}$
0.4	$7.53 \times 10^{-4}$	$7.53 \times 10^{-4}$	$7.53 \times 10^{-4}$	$7.53 \times 10^{-4}$	$7.53 \times 10^{-4}$
0.5	$8.42 \times 10^{-5}$	$8.42 \times 10^{-5}$	$8.42 \times 10^{-5}$	$8.42 \times 10^{-5}$	$8.42 \times 10^{-5}$
0.6	$4.30 \times 10^{-4}$	$4.30 \times 10^{-4}$	$4.30 \times 10^{-4}$	$4.30 \times 10^{-4}$	$4.30 \times 10^{-4}$
0.7	$7.22 \times 10^{-4}$	$7.22 \times 10^{-4}$	$7.22 \times 10^{-4}$	$7.22 \times 10^{-4}$	$7.22 \times 10^{-4}$
0.8	$7.81 \times 10^{-4}$	$7.81 \times 10^{-4}$	$7.81 \times 10^{-4}$	$7.81 \times 10^{-4}$	$7.81 \times 10^{-4}$
0.9	$5.83 \times 10^{-4}$	$5.83 \times 10^{-4}$	$5.83 \times 10^{-4}$	$5.83 \times 10^{-4}$	$5.83 \times 10^{-4}$
1	0.	0.	0.	0.	0.



**Figure 2:** The absolute error of Example 2

## 5. Conclusion

The basic idea of this article is to solve the space fractional advection-dispersion problem (17) and (18) using tau method based on shifted Jacobi polynomials. We use the Caputo sense to evaluate the fractional derivatives. Two examples are solved in this article and errors are obtained.

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