A Fuzzy Two-warehouse Inventory Model for Single Deteriorating Item with Selling-Price-Dependent Demand and Shortage under Partial-Backlogged condition

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Abstract

In this paper we have developed an inventory model for a single deteriorating item with two separate storage facilities (one is owned warehouse (OW) and the other a rented warehouse (RW)) and in which demand is selling- price dependent. Shortage is allowed and is partially backlogged with a rate dependent on the duration of waiting time up to the arrival of next lot. It is assumed that the holding cost of the rented warehouse is higher than that of owned warehouse. As demand, selling- price, holding- cost, shortage, lost- sale, deterioration- rate are uncertain in nature, we consider them as triangular fuzzy numbers and developed the model for fuzzy total cost function and is defuzzified by using Signed Distance and Centroid methods. In order to validate the proposed model, we compare the results of crisp and fuzzy models through a numerical example and based on the example the effect of different parameters have been rigorously studied by sensitivity analysis taking one parameter at a time keeping the other parameters unchanged.

1. Introduction

Traditionally, inventory models are mostly developed with single warehouse facility. But practically it is almost impossible for big shops or showrooms placed in important market places of town or municipality area having a bigger warehouse due to unavailability of space. Even if they get it, they have to pay very high rents. Moreover, in the area of inventory management, when a supplier or organization purchases (or produces) a huge amount of units of item for the future requirement of demand, the items cannot be stored in their existing own- warehouse at the market place. On the other hand the excess units are stocked in a rented warehouse (RW) with an infinite capacity, i.e. it is as large as it may be required as per the time. Since the holding cost per unit in RW is much higher in compared with the OW, the units of items placed in the rented warehouse are first exhausted fully. Thus, it is necessary to study the significance of storage capability in various inventory policy issues.

In the last few decades, two-warehouse inventory models have been widely applied in business world. Such type of model was first developed by Hartely in 1976, in which the transport-cost from rented warehouse to owned warehouse was not considered. Subsequently, by introducing the transport-cost, Sarma (1983) extended Hartely's (1976) model. Again Murdeswar and Sathe (1985) draw-out this model to the case of finite refilling rate. Dave (1988) corrected the error in models of Murdeswar and Sathe (1985) mentioning the case of bulk unleash pattern for each finite and infinite refilling rates. Several other researchers had attempted to extend their works to various realistic situations. Goswami and Chaudhuri (1992) provided an economic order quantity model for items with two-level of storage for linear demand. Benkherouf (1997) developed a deterministic inventory model for deteriorating items with two-storage facilities. For knowing more research works in this field, one can see Yang (2004), Huang(2006), Lee and Hsu (2009), Liang and Zhou (2011), Yang and Chang (2013), Jaggi et al. (2013, 2017), Bhunia et al. (2014, 2015), Xu et al. (2016), Mandal and Giri (2017), Sheikh and Patel (2017), Saha, Sen and Nath (2018) etc.

Demand plays an important role in inventory management. There are many types of demands like price-dependent demand, time-dependent demand, quantity-discount demand, ramp-type demand etc. A large number of research papers have been published in price-dependent demand rate. Rong et al. (2008) provided a two-warehouse inventory model of deteriorating items with selling-price-dependent demand and shortages under partially or fully backlogged condition. Jaggi and Verma (2008) considered a two warehouse inventory model of non-deteriorating items with selling-price-dependent demand and shortages under fully backlogged. However, in this price- dependent demand rate Jaggi et al. (2010) developed an inventory model for deteriorating items with limited capacity and time-proportional backlogging rate. Mishra et al. (2018) have developed a model with retailer's joint ordering, pricing, and preservation technology investment policies for a deteriorating item under permissible delay in payments. Prior to this, they (Mishra et al. (2017)) have developed an inventory model under price and stock dependent demand for controllable deterioration rate with shortage and preservation technology investment under shortage.

Generally, while modeling an inventory problem, researchers assumed that the system parameters like demand, holding-cost, deterioration-rate, shortages, etc. are certain or fixed. However, in real life situations all of them probably will have some little fluctuations or vague in nature. So in practical situations, we may treat these parameters as fuzzy variables which will be more realistic.

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Zadeh (1965) was the first person to introduce the concept of fuzziness in inventory model problem. Zadeh and Bellman (1970) proposed an inventory model on decision making in fuzzy approach. Many authors developed inventory models in fuzzy environment. Roy and Maiti (1998) derived a multi-item inventory model of deteriorating items in fuzzy environment. Since total-average-cost, inventory-cost, warehouse space, purchasing and selling prices are vague and imprecise; in their model they considered these parameters as fuzzy linear membership function and inventory costs and prices as triangular fuzzy numbers. They solved the model by using fuzzy non-linear programming method. Maiti and Maiti (2006) provided a two warehouse multi item inventory model with advertisement, price and displayed inventory level dependent demand as fuzzy and purchase cost, investment amount and store house capacity were considered as trapezoidal fuzzy numbers. The problem was solved by goal programming method. A two-warehouse inventory model for a deteriorating item with partially/fully backlogged shortage in fuzzy approach was developed by Rong et al.(2008). They used the nearest interval approximation method and defuzzified the total cost function by global criteria method. Roy et al. (2009) presented a production inventory model with remanufacturing for defective and usable items in fuzzy environment. In their study, they developed a genetic algorithm with Roulette Wheel Selection, Arithmetic Crossover, Random Mutation and applied to get the maximum total profit.

Yadav et al. (2012) developed a two-warehouse inventory model of deteriorating items with stock dependent demand using genetic algorithm in fuzzy environment. In their study, purchase cost, investment amount and storage capacity were considered as fuzzy. Malik and Singh(2013) developed a fuzzy based two-warehouse inventory model with linear demand pattern in which deterioration rate was considered different in both the warehouses. Holding-cost, ordering-cost and deterioration-cost were taken as triangular fuzzy numbers and shortages were not allowed. In this context they defuzzified the total fuzzy cost by using signed distance method. Two-warehouse inventory model of deteriorating items with three-component demand rate and time proportional backlogging rate in fuzzy environment was developed by Kumar et al. (2013) in which own warehouse (OW) was assumed as finite dimension and rented warehouse (RW) was in fuzzy sense. The demand rate of items was dependent on the selling price. As total revenue and shortages cost are impreciseness in nature, these were considered as vague value. They used the fuzzy goal programming method for optimization of the given fuzzy model where they converted the multi-objective problem in to a single objective function.

Singh and Anuradha (2014) developed a two storage economic order quantity inventory model for deteriorating items under fuzzy environment in which demand increases with respect to time and shortages are partially backlogged. Due to uncertainty of system parameters, capacity of own warehouses, holding-cost, unit-cost, shortages and opportunity-cost are treated as triangular fuzzy numbers. They applied graded mean integration representation method for defuzzification of the total-cost- function.

Deterioration and demand are taken as the most important factors while modeling an inventory model. Most of the researchers in the above discussion assumed that deterioration rate and the demand rate as constant quantities, but in actual practice these quantities are not stable. Under this condition Shabani et al. (2015) developed a two-warehouse inventory model with fuzzy deterioration rate and fuzzy demand rate under conditionally permissible delay in payments. Max-mini principle was used to minimize the fuzzy total cost function. Mandal and Islam (2015)

developed a fuzzy two-warehouse inventory model for Weibull deteriorating items with constant demand, shortages under fully backlogged in which they considered cost components such as holding-cost, shortage-cost and deterioration-cost as triangular fuzzy numbers. They applied graded mean integration representation method for defuzzification of total-cost-function. Yadav et al. (2017) discussed a fuzzy based two-warehouse inventory model for non-instantaneous deteriorating items with conditionally permissible delay in payment. Cost components (holding-cost, purchase-cost and selling-price etc.) and demand rate were considered as triangular fuzzy numbers. Signed distance method was used to minimize the total cost function. Very recently Indrajitsingha et al. (2018) developed a fuzzy inventory model for deteriorating items with stock - dependent demand-rate in which graded mean integration representation method; signed distance method and Centroid method were used for defuzzification.

A large number of research papers have been developed in the area of two-warehouse inventory model in crisp approach. However, a very few two-warehouse inventory models have been developed in uncertainty. In this literature review, we discussed most of the fuzzy two-warehouse inventory models. Customer-satisfaction plays a crucial role for an organization in the present competitive market scenario to maximizing the profit. In this context, the inventory level should be properly set as to meet the customer's expectations. With a lost-sale, the customer's needs for the item are filled by a competitor who is assumed as loss of profit in sales. On the other way, the organization not only loses the customer but also lose the customers goodwill. Therefore stock out cost from the total profit should not be excluded. No organization ignores the effect of demand in his business. There are many types of inventory as per time, price, variable, ramp type, stock etc. At the end of each calendar year for a product, demand is same among the customer. It is observed that when the scarcity of the products occurs in the market, the demand increases. Thus demand depends upon the selling-price. As we increase the selling- price, the demand decreases and vice-versa. By considering the above parameters in account, in the proposed model we have developed a two-warehouse inventory model for deteriorating items with price dependent demand where shortages are partially backlogged. In this model we have considered two warehouses (one own warehouse (OW) and other rented warehouse (RW)). The holding cost of rented warehouse (RW) is higher than that of own warehouse (OW). The cost components (demand, holding-cost, shortage-cost, and lost-sales and deterioration rates for two warehouses are assumed as triangular fuzzy numbers.

2. Definitions and Preliminaries

In order to establish the model we require the following definitions:

Definition 2.1.

Let X be a space of points and $\mu: X \longrightarrow [0,1]$ be such that for every $x \in X$, $\mu(x)$ is a real number in the interval [0, 1]. We define a fuzzy set $\tilde{A}$ in X as the order pair $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)): x \in X\}$, where x is called a generic element and $\mu_{\tilde{A}}(x)$ a membership function.

Definition 2.2.

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A fuzzy set, $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \subseteq X$, is called a convex fuzzy set, if all $\tilde{A}_x$ are convex sets for every $x \in X$. That is, for every pair of elements $x_1, x_2 \in A_\alpha$ and $\alpha \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$, $\forall \lambda \in [0,1]$.

Definition 2.3.

Let $a, b \in R$ such that $a < b$. Then, for $0 \le \alpha \le 1$, the fuzzy set $[a_\alpha, b_\alpha]$ is called a fuzzy interval, if its membership functions is

$$\mu_{[a_\alpha, b_\alpha]} = \begin{cases} \alpha, & a \le x \le b, \\ 0, & otherwise. \end{cases}$$

Definition 2.4.

Let $a, b, c \in R$ such that $a < b < c$. Then the fuzzy number $\tilde{A} = (a, b, c)$, is called a triangular fuzzy number if its membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b, \\ \dfrac{c-x}{c-b}, & b \le x \le c, \\ 0, & otherwise. \end{cases}$$

In particular, when $a = b = c$, $(c, c, c) = c$, is called a fuzzy point. The family of all triangular fuzzy numbers on R is usually denoted as

$$F_N = \{(a, b, c): a < b < c \forall a, b, c \in R \}.$$

The $\alpha - cut$ of $\widetilde{A} = (a, b, c) \in F_N$, $0 \le \alpha \le 1$, usually denoted by $A(\alpha)$, is defined as $A(\alpha) = [A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = a + (b-a)\alpha$ and $A_R(\alpha) = c - (c-b)\alpha$, are the left and right endpoints of $A(\alpha)$ respectively.

Definition 2.5.

If $\tilde{A} = (a, b, c)$ is a triangular fuzzy number, then the signed distance of $\tilde{A}$ is defined as

$$d(\tilde{A}, \tilde{0}) = \int_0^1 d([A_L(\alpha), A_R(\alpha)], \tilde{0}) = \frac{1}{4}(a + 2b + c).$$

Definition 2.6.

The Centroid method on the triangular fuzzy number $\tilde{A} = (a, b, c)$ is defined as

$$C(\tilde{A}) = \frac{a + b + c}{3}.$$

3. Assumptions

The mathematical model in this paper is developed on the basis of following assumptions:

    i.   The inventory system involves only one product.
    ii.   The replenishment occurs instantaneously at infinite rate.
    iii.   The lead time is negligible.
    iv.   The demand rate is a function of selling price.
    v.   The shortages are allowed and partially backlogged.
    vi.   The owned warehouse (OW) has a limited capacity of W units.
    vii. The rented warehouse (RW) has unlimited capacity calculated per day basis.
    viii. The holding unit cost of RW is greater than that of OW.
    ix.   The items assumed in this model are deteriorating in nature.
    x.   Higher powers of θ are neglected.
    xi.   The items are kept in OW first.
    xii. The items stored in RW will be consumed first.

4. Notations

The following notations are used throughout the manuscript:

$I_r(t)$	:	Inventory level at time t in RW, $t \geq 0$.
$I_o(t)$	:	Inventory level at time t in OW, $t \geq 0$.
θ	:	Rate of deterioration.
α	:	Initial demand rate.
β	:	Positive demand parameter.
t_1	:	Time point when stock level of RW reaches to zero.
t_2	:	Time point when stock level of OW reaches to zero.
W	:	Storage capacity of OW.
C_1	:	Selling price ($/unit/year).
S	:	Initial stock level.
q_1	:	Backorder quantity during stock out.
T	:	Cycle time.
p	:	Purchasing cost ($/unit/day).
k	:	Rate of backlogging.
h_r	:	Holding cost ($/unit/year) in RW.
h_o	:	Holding cost ($/unit/year) in OW.
d	:	Unit deterioration cost ($/unit/day).
C_2	:	Unit shortage cost ($/unit/day).
C_3	:	Unit lost sale cost ($/unit/day).
$TAC(t_1, t_2)$	:	Total average cost ($/unit/day).
$\tilde{\alpha}$	:	Fuzzy initial demand rate.
$\tilde{\beta}$	:	Fuzzy positive demand parameter.
$\tilde{C}_1$	:	Fuzzy selling price ($/unit/day).

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$\tilde{k}$	:	Fuzzy backlogging rate (\$/unit/day).
$\tilde{h}_r$	:	Fuzzy holding cost (\$/unit/year) in RW.
$\tilde{h}_o$	:	Fuzzy holding cost (\$/unit/day) in OW.
$\tilde{C}_2$	:	Fuzzy shortage (\$/unit/day).
$\tilde{C}_3$	:	Fuzzy opportunity cost due to lost sale (\$/unit/day).
$\widetilde{TAC}(t_1, t_2)$	:	Fuzzy total cost (\$/unit/day).
$\widetilde{TAC}_S(t_1, t_2)$	:	Defuzzified value of $\widetilde{TAC}(t_1, t_2)$ by applying Signed Distance Method (\$/unit/day).
$\widetilde{TAC}_C(t_1, t_2)$	:	Defuzzified value of $\widetilde{TAC}(t_1, t_2)$ by applying Centroid Method (\$/unit/day).

5. Mathematical Formulation

Suppose, there are q units of items in the stock at the beginning from which q_1 units are utilized to satisfy backlogged demand and S units are the initial stock level. Clearly $S = q - q_1$. Suppose, W units of material stored in OW and the rest $(S - W)$ units are stored in a RW. Since holding cost of RW is greater than the holding cost of OW, the items in RW are consumed first. During the consumption period of RW, the inventory level of OW is decreased due to deterioration only. Suppose, at time $t = t_1$, the inventory level of RW becomes zero due to demand and deterioration. During the time period $[t_1, t_2]$, stock is available only in OW. At time $t = t_2$, inventory level of OW depletes to zero due to demand and deterioration and after that shortage occurs. This is shown in the Figure. 1.
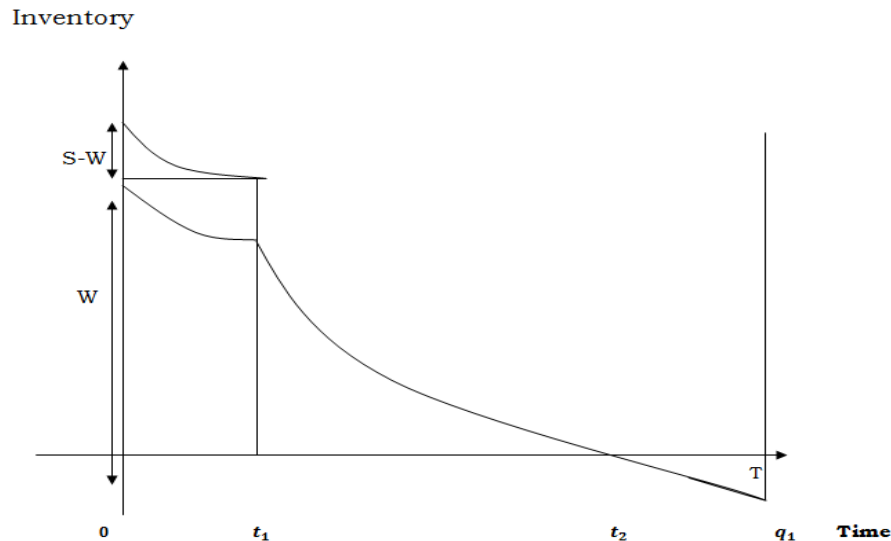

Figure 1. Inventory time graph for two-warehouse

5.1. Crisp Model

The differential equations governing the system for RW and OW during the period $0 \leq t \leq T$ are described as follows:

$$\frac{dI_r(t)}{dt} = -\theta I_r(t) - (\alpha - \beta C_1), \quad 0 \leq t \leq t_1, \tag{1}$$

with $I_r(t_1) = 0$.

$$\frac{dI_o(t)}{dt} = -\theta I_o(t), \qquad\qquad 0 \leq t \leq t_1, \tag{2}$$

with $I_o(0) = W$.

$$\frac{dI_o(t)}{dt} = -\theta I_o(t) - (\alpha - \beta C_1), \quad t_1 \leq t \leq t_2, \tag{3}$$

with $I_o(t_2) = 0$.

The solutions of the equations (1), (2) and (3) are given by

$$I_r(t) = \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \quad 0 \leq t \leq t_1, \tag{4}$$

$$I_o(t) = We^{-\theta t}, \qquad\qquad 0 \leq t \leq t_1, \tag{5}$$

$$I_o(t) = \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta(t_2 - t)} - 1\right), \quad t_1 \leq t \leq t_2. \tag{6}$$

From (5), we have

$$I_r(0) = S - W,$$

$$S = W + \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta t_1} - 1\right). \tag{7}$$

At $t = t_1$, equations (5) and (6) yield

$$We^{-\theta t_1} = \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta(t_2 - t_1)} - 1\right)$$

$$W = \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta t_2} - e^{\theta t_1}\right). \tag{8}$$

With the above data, the following parameters are calculated as follows:

Purchasing- cost (PC)

$$PC = (S + q_1)p,$$

where

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$$q_1 = \int_{t_2}^{T} k(\alpha - \beta C_1)dt = (\alpha - \beta C_1)k(T - t_2).$$

Then,

$$PC = \left\{ \left(W + \frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta t_1} - 1\right)\right) + (\alpha - \beta C_1)k(T - t_2)\right\}p \ . \tag{9}$$

Holding- cost (HC)

$$HC = HC_r + HC_o,$$

where

$$HC_r = h_r \int_0^{t_1} I_r(t)dt = h_r \frac{(\alpha - \beta C_1)}{\theta}\left(\frac{e^{\theta t_1} - 1}{\theta} - t_1\right) \tag{10}$$

and

$$HC_o = h_o \left\{\int_0^{t_1} I_o(t)dt + \int_{t_1}^{t_2} I_o(t)dt\right\}$$

$$= \frac{Wh_o}{\theta}\left(1 - e^{-\theta t_1}\right) + h_o \frac{(\alpha - \beta C_1)}{\theta}\left(\frac{e^{\theta(t_2 - t_1)} - 1}{\theta} + (t_1 - t_2)\right). \tag{11}$$

Deterioration- cost (DC)

$$DC = DC_r + DC_o,$$

where

$$DC_r = d\left\{\frac{(\alpha - \beta C_1)}{\theta}\left(e^{\theta t_1} - 1\right) - (\alpha - \beta C_1)t_1\right\} \tag{12}$$

and

$$DC_o = d\{W - (\alpha - \beta C_1)(t_2 - t_1)\} \ . \tag{13}$$

Shortage -cost (SC)

$$SC = C_2 \int_{t_2}^{T}(\alpha - \beta C_1)dt = C_2(\alpha - \beta C_1)(T - t_2) \ . \tag{14}$$

Lost- sale- cost (L.C)

$$LC = C_3 \int_{t_2}^{T}(1 - k)(\alpha - \beta C_1)dt = C_3(1 - k)(\alpha - \beta C_1)(T - t_2) \ . \tag{15}$$

Total average cost $TAC(t_1, t_2)$ for this model during a cycle is given by

$$TAC(t_1, t_2) = \frac{1}{T}[PC + HC + DC + SC + LC]$$

$$= \frac{1}{T}\left[\left\{\left(W + \frac{(\alpha - \beta C_1)}{\theta}(e^{\theta t_1} - 1)\right) + (\alpha - \beta C_1)k(T - t_2)\right\}p \right.$$

$$+ h_r \frac{(\alpha - \beta C_1)}{\theta}\left(\frac{e^{\theta t_1} - 1}{\theta} - t_1\right) + \frac{Wh_o}{\theta}(1 - e^{-\theta t_1})$$

$$+ h_o \frac{(\alpha - \beta C_1)}{\theta}\left(\frac{e^{\theta(t_2 - t_1)} - 1}{\theta} + (t_1 - t_2)\right)$$

$$+ d\left\{\frac{(\alpha - \beta C_1)}{\theta}(e^{\theta t_1} - 1) - (\alpha - \beta C_1)t_1 + W - (\alpha - \beta C_1)(t_2 - t_1)\right\}$$

$$\left. + C_2(\alpha - \beta C_1)(T - t_2) + C_3(1 - k)(\alpha - \beta C_1)(T - t_2)\right].$$

$$(16)$$

To minimize the total cost function $TAC(t_1, t_2)$ per unit time, the values of t_1 and t_2 can be obtained by solving the equations

$$\frac{\partial TAC(t_1, t_2)}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial TAC(t_1, t_2)}{\partial t_2} = 0. \qquad (17)$$

Equations in (17) are equivalent to

$$\left[\frac{\left\{\begin{matrix}\left(de^{\theta t_1}(\alpha - \beta C_1) + e^{\theta t_1}p(\alpha - \beta C_1) + Wh_o e^{-\theta t_1}\right) \\ +h_o(1 - e^{\theta(t_2 - t_1)})(\alpha - \beta C_1) + \frac{h_r(e^{\theta t_1} - 1)(\alpha - \beta C_1)}{\theta^2}\end{matrix}\right\}}{T}\right] = 0$$

and

$$\left[\frac{\left\{\begin{matrix}-kp(\alpha - \beta C_1) + d(-\alpha + \beta C_1) - C_2(\alpha - \beta C_1) \\ -C_3(1 - k)(\alpha - \beta C_1) + \frac{h_o(e^{\theta(t_2 - t_1)} - 1)(\alpha - \beta C_1)}{\theta}\end{matrix}\right\}}{T}\right] = 0.$$

Thus, the values of t_1 and t_2 obtained from the above equations will minimize the total cost function, if they satisfy the equations

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$$\frac{\partial^2 TAC(t_1,t_2)}{\partial t_1^2} > 0 \ , \ \frac{\partial^2 TAC(t_1,t_2)}{\partial t_2^2} > 0 \quad \text{and}$$

$$\left(\frac{\partial^2 TAC(t_1,t_2)}{\partial t_1^2}\right)\left(\frac{\partial^2 TAC(t_1,t_2)}{\partial t_2^2}\right) - \left(\frac{\partial^2 TAC(t_1,t_2)}{\partial t_1^2 \partial t_2^2}\right)^2 > 0 \ . \tag{18}$$

5.2. Fuzzy Model

Due to uncertainty, it is not easy to define all the system of parameters exactly. Subsequently, we assume them as fuzzy parameters, namely $\tilde{\alpha}, \tilde{\beta}, \tilde{k}, \tilde{C}_1 \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$. These parameters may change within some limits.

Let $\tilde{\alpha} = (a_1, a_2, a_3)$, $\tilde{\beta} = (b_1, b_2, b_3)$, $\tilde{k} = (k_1, k_2, k_3)$, $\tilde{C}_1 = (m_1, m_2, m_3)$, $\tilde{h}_r = (r_1, r_2, r_3)$, $\tilde{h}_o = (O_1, O_2, O_3)$, $\tilde{\theta} = (\theta_1, \theta_2, \theta_3)$, $\tilde{C}_2 = (n_1, n_2, n_3)$, and $\tilde{C}_3 = (l_1, l_2, l_3)$ be considered as triangular fuzzy numbers.

Then, the total average cost is given by

$$\widetilde{TAC}(t_1,t_2) = \frac{1}{T}\begin{bmatrix} \left\{\left(W + \frac{(\tilde{\alpha}-\tilde{\beta}\tilde{C}_1)}{\tilde{\theta}}\left(e^{\tilde{\theta}t_1} - 1\right)\right) + (\tilde{\alpha} - \tilde{\beta}\tilde{C}_1)\tilde{k}(T - t_2)\right\}p \\ +\tilde{h}_r\frac{(\tilde{\alpha}-\tilde{\beta}\tilde{C}_1)}{\tilde{\theta}}\left(\frac{e^{\tilde{\theta}t_1}-1}{\tilde{\theta}} - t_1\right) + \frac{W\tilde{h}_o}{\tilde{\theta}}\left(1 - e^{-\tilde{\theta}t_1}\right) \\ +\tilde{h}_o\frac{(\tilde{\alpha}-\tilde{\beta}\tilde{C}_1)}{\tilde{\theta}}\left(\frac{e^{\tilde{\theta}(t_2-t_1)}-1}{\tilde{\theta}} + (t_1 - t_2)\right) \\ +d\left\{\frac{(\tilde{\alpha}-\tilde{\beta}\tilde{C}_1)}{\tilde{\theta}}\left(e^{\tilde{\theta}t_1} - 1\right) - (\tilde{\alpha} - \tilde{\beta}\tilde{C}_1)t_1 + W - (\tilde{\alpha} - \tilde{\beta}\tilde{C}_1)(t_2 - t_1)\right\} \\ +\tilde{C}_2(\tilde{\alpha} - \tilde{\beta}\tilde{C}_1)(T - t_2) + \tilde{C}_3(1 - \tilde{k})(\tilde{\alpha} - \tilde{\beta}\tilde{C}_1)(T - t_2) \end{bmatrix} . \tag{19}$$

We defuzzify the fuzzy total cost function $\widetilde{TAC}(t_1,t_2)$ by Signed Distance method as follows:

$$\widetilde{TAC}_S(t_1,t_2) = \frac{1}{4}\left[\widetilde{TAC}_{S1}(t_1,t_2), \widetilde{TAC}_{S2}(t_1,t_2), \widetilde{TAC}_{S3}(t_1,t_2)\right],$$

where

$$\widetilde{TAC}_{S1}(t_1,t_2) = \frac{1}{T}\begin{bmatrix} \left\{\left(W + \frac{(a_1-b_1m_1)}{\theta_1}\left(e^{\theta_1 t_1} - 1\right)\right) + (a_1 - b_1m_1)k_1(T - t_2)\right\}p \\ +r_1\frac{(a_1-b_1m_1)}{\theta_1}\left(\frac{e^{\theta_1 t_1}-1}{\theta_1} - t_1\right) + \frac{WO_1}{\theta_1}\left(1 - e^{-\theta_1 t_1}\right) \\ +O_1\frac{(a_1-b_1m_1)}{\theta_1}\left(\frac{e^{\theta_1(t_2-t_1)}-1}{\theta_1} + (t_1 - t_2)\right) \\ +d\left\{\frac{(a_1-b_1m_1)}{\theta_1}\left(e^{\theta_1 t_1} - 1\right) - (a_1 - b_1m_1)t_1 + W - (a_1 - b_1m_1)(t_2 - t_1)\right\} \\ +n_1(a_1 - b_1m_1)(T - t_2) + l_1(1 - k_1)(a_1 - b_1m_1)(T - t_2) \end{bmatrix},$$

$$\widehat{TAC}_{S2}(t_1, t_2) =$$

$$\frac{1}{T}\begin{bmatrix} \left\{\left(W + \frac{(a_2-b_2m_2)}{\theta_2}\left(e^{\theta_2 t_1} - 1\right)\right) + (a_2 - b_2m_2)k_2(T - t_2)\right\}p \\ +r_2\frac{(a_2-b_2m_2)}{\theta_2}\left(\frac{e^{\theta_2 t_1}-1}{\theta_2} - t_1\right) + \frac{WO_2}{\theta_2}\left(1 - e^{-\theta_2 t_1}\right) \\ +O_2\frac{(a_2-b_2m_2)}{\theta_2}\left(\frac{e^{\theta_2(t_2-t_1)}-1}{\theta_2} + (t_1 - t_2)\right) \\ +d\left\{\frac{(a_2-b_2m_2)}{\theta_2}\left(e^{\theta_2 t_1} - 1\right) - (a_2 - b_2m_2)t_1 + W - (a_2 - b_2m_2)(t_2 - t_1)\right\} \\ +n_2(a_2 - b_2m_2)(T - t_2) + l_2(1 - k_2)(a_2 - b_2m_2)(T - t_2) \end{bmatrix}$$

and

$$\widehat{TAC}_{S3}(t_1, t_2) =$$

$$\frac{1}{T}\begin{bmatrix} \left\{\left(W + \frac{(a_3-b_3m_3)}{\theta_3}\left(e^{\theta_3 t_1} - 1\right)\right) + (a_3 - b_3m_3)k_3(T - t_2)\right\}p \\ +r_3\frac{(a_3-b_3m_3)}{\theta_3}\left(\frac{e^{\theta_3 t_1}-1}{\theta_3} - t_1\right) + \frac{WO_3}{\theta_3}\left(1 - e^{-\theta_3 t_1}\right) \\ +O_3\frac{(a_3-b_3m_3)}{\theta_3}\left(\frac{e^{\theta_3(t_2-t_1)}-1}{\theta_3} + (t_1 - t_2)\right) \\ +d\left\{\frac{(a_3-b_3m_3)}{\theta_3}\left(e^{\theta_3 t_1} - 1\right) - (a_3 - b_3m_3)t_1 + W - (a_3 - b_3m_3)(t_2 - t_1)\right\} \\ +n_3(a_3 - b_3m_3)(T - t_2) + l_3(1 - k_3)(a_3 - b_3m_3)(T - t_2) \end{bmatrix}.$$

$$\widehat{TAC}_S(t_1, t_2) = \frac{1}{4}\left[\widehat{TAC}_{S1}(t_1, t_2) + 2\,\widehat{TAC}_{S2}(t_1, t_2) + \widehat{TAC}_{S3}(t_1, t_2)\right]. \tag{20}$$

To minimize the total cost function $\widehat{TAC}_S(t_1, t_2)$ per unit time, the value of t_1 and t_2 can be obtained by solving the equations

$$\frac{\partial \widehat{TAC}_S(t_1,t_2)}{t_1} = 0 \quad \text{and} \quad \frac{\partial \widehat{TAC}_S(t_1,t_2)}{t_2} = 0. \tag{21}$$

Equations in (21) are equivalent to

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$$
\left[
\begin{array}{c}
\left\{
\dfrac{
\begin{array}{c}
d(a_1 - b_1 m_1)e^{\theta_1 t_1} + p(a_1 - b_1 m_1)e^{\theta_1 t_1} + WO_1 e^{-\theta_1 t_1} \\[4pt]
+ \dfrac{(e^{\theta_1 t_1}-1)(a_1-b_1 m_1)r_1}{\theta_1^2} + \dfrac{O_1(a_1-b_1 m_1)(1-e^{\theta_1(t_2-t_1)})}{\theta_1}
\end{array}
}{4T}
\right\} \\[30pt]
+2\left\{
\dfrac{
\begin{array}{c}
de^{\theta_2 t_1}(a_2 - b_2 m_2) + p(a_2 - b_2 m_2)e^{\theta_2 t_1} + WO_2 e^{-\theta_2 t_1} \\[4pt]
+ \dfrac{(e^{\theta_2 t_1}-1)(a_2-b_2 m_2)r_2}{\theta_2^2} + \dfrac{O_2(a_2-b_2 m_2)(1-e^{\theta_2(t_2-t_1)})}{\theta_2}
\end{array}
}{T}
\right\} \\[30pt]
+\left\{
\dfrac{
\begin{array}{c}
de^{\theta_3 t_1}(a_3 - b_3 m_3) + p(a_3 - b_3 m_3)e^{\theta_3 t_1} + WO_3 e^{-\theta_3 t_1} \\[4pt]
+ \dfrac{(e^{\theta_3 t_1}-1)(a_3-b_3 m_3)r_3}{\theta_3^2} + \dfrac{O_3(a_3-b_3 m_3)(1-e^{\theta_3(t_2-t_1)})}{\theta_3}
\end{array}
}{T}
\right\}
\end{array}
\right] = 0
$$

and

$$
\left[
\begin{array}{c}
\left\{
\dfrac{
\begin{array}{c}
-k_1 p(a_1-b_1 m_1)+d(b_1 m_1-a_1)-(1-k_1)(a_1-b_1 m_1)l_1 \\[4pt]
-(a_1-b_1 m_1)n_1+\dfrac{\left(e^{\theta_1(t_2-t_1)}-1\right)(a_1-b_1 m_1)O_1}{\theta_1}
\end{array}
}{4T}
\right\} \\[30pt]
+2\left\{
\dfrac{
\begin{array}{c}
-k_2 p(a_2-b_2 m_2)+d(b_2 m_2-a_2)-(1-k_2)(a_2-b_2 m_2)l_2 \\[4pt]
-(a_2-b_2 m_2)n_2+\dfrac{\left(e^{\theta_2(t_2-t_1)}-1\right)(a_2-b_2 m_2)O_2}{\theta_2}
\end{array}
}{T}
\right\} \\[30pt]
+\left\{
\dfrac{
\begin{array}{c}
-k_3 p(a_3-b_3 m_3)+d(b_3 m_3-a_3)-(1-k_3)(a_3-b_3 m_3)l_3 \\[4pt]
-(a_3-b_3 m_3)n_3+\dfrac{\left(e^{\theta_3(t_2-t_1)}-1\right)(a_3-b_3 m_3)O_3}{\theta_3}
\end{array}
}{T}
\right\}
\end{array}
\right] = 0..
$$

Thus, the values of t_1 and t_2 obtained from the above equations will minimize the fuzzy total cost function, if those values satisfy the equations

$$
\frac{\partial^2 \widetilde{TAC}_S(t_1,t_2)}{\partial t_1^2} > 0 \ , \ \frac{\partial^2 \widetilde{TAC}_S(t_1,t_2)}{\partial t_2^2} > 0 \quad \text{and}
$$

$$
\left(\frac{\partial^2 \widetilde{TAC}_S(t_1,t_2)}{\partial t_1^2}\right)\left(\frac{\partial^2 \widetilde{TAC}_S(t_1,t_2)}{\partial t_2^2}\right) - \left(\frac{\partial^2 \widetilde{TAC}_S(t_1,t_2)}{\partial t_1^2 \partial t_2^2}\right)^2 > 0. \tag{22}
$$

We defuzzified the fuzzy total cost function $\widetilde{TAC}(t_1, t_2)$ by Centroid method as follows:

$$\widetilde{TAC}_C(t_1, t_2) = \frac{1}{3}\left[\widetilde{TAC}_{C1}(t_1, t_2), \widetilde{TAC}_{C2}(t_1, t_2), \widetilde{TAC}_{C3}(t_1, t_2)\right],$$

where

$\widetilde{TAC}_{C1}(t_1, t_2)$

$$= \frac{1}{T}\left[\begin{array}{l} \left\{\left(W + \frac{(a_1 - b_1 m_1)}{\theta_1}\left(e^{\theta_1 t_1} - 1\right)\right) + (a_1 - b_1 m_1)k_1(T - t_2)\right\}p \\[2mm] +r_1 \frac{(a_1 - b_1 m_1)}{\theta_1}\left(\frac{e^{\theta_1 t_1} - 1}{\theta_1} - t_1\right) + \frac{WO_1}{\theta_1}\left(1 - e^{-\theta_1 t_1}\right) \\[2mm] +O_1 \frac{(a_1 - b_1 m_1)}{\theta_1}\left(\frac{e^{\theta_1(t_2 - t_1)} - 1}{\theta_1} + (t_1 - t_2)\right) \\[2mm] +d\left\{\frac{(a_1 - b_1 m_1)}{\theta_1}\left(e^{\theta_1 t_1} - 1\right) - (a_1 - b_1 m_1)t_1 + W - (a_1 - b_1 m_1)(t_2 - t_1)\right\} \\[2mm] +n_1(a_1 - b_1 m_1)(T - t_2) + l_1(1 - k_1)(a_1 - b_1 m_1)(T - t_2) \end{array}\right],$$

$\widetilde{TAC}_{C2}(t_1, t_2)$

$$= \frac{1}{T}\left[\begin{array}{l} \left\{\left(W + \frac{(a_2 - b_2 m_2)}{\theta_2}\left(e^{\theta_2 t_1} - 1\right)\right) + (a_2 - b_2 m_2)k_2(T - t_2)\right\}p \\[2mm] +r_2 \frac{(a_2 - b_2 m_2)}{\theta_2}\left(\frac{e^{\theta_2 t_1} - 1}{\theta_2} - t_1\right) + \frac{WO_2}{\theta_2}\left(1 - e^{-\theta_2 t_1}\right) \\[2mm] +O_2 \frac{(a_2 - b_2 m_2)}{\theta_2}\left(\frac{e^{\theta_2(t_2 - t_1)} - 1}{\theta_2} + (t_1 - t_2)\right) \\[2mm] +d\left\{\frac{(a_2 - b_2 m_2)}{\theta_2}\left(e^{\theta_2 t_1} - 1\right) - (a_2 - b_2 m_2)t_1 + W - (a_2 - b_2 m_2)(t_2 - t_1)\right\} \\[2mm] +n_2(a_2 - b_2 m_2)(T - t_2) + l_2(1 - k_2)(a_2 - b_2 m_2)(T - t_2) \end{array}\right]$$

and

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$$\widetilde{TAC}_{C3}(t_1, t_2) =$$

$$\frac{1}{T}\begin{bmatrix} \left\{\left(W + \frac{(a_3-b_3m_3)}{\theta_3}\left(e^{\theta_3 t_1} - 1\right)\right) + (a_3 - b_3m_3)k_3(T - t_2)\right\}p \\ +r_3\frac{(a_3-b_3m_3)}{\theta_3}\left(\frac{e^{\theta_3 t_1}-1}{\theta_3} - t_1\right) + \frac{WO_3}{\theta_3}\left(1 - e^{-\theta_3 t_1}\right) \\ +O_3\frac{(a_3-b_3m_3)}{\theta_3}\left(\frac{e^{\theta_3(t_2-t_1)}-1}{\theta_3} + (t_1 - t_2)\right) \\ +d\left\{\frac{(a_3-b_3m_3)}{\theta_3}\left(e^{\theta_3 t_1} - 1\right) - (a_3 - b_3m_3)t_1 + W - (a_3 - b_3m_3)(t_2 - t_1)\right\} \\ +n_3(a_3 - b_3m_3)(T - t_2) + l_3(1 - k_3)(a_3 - b_3m_3)(T - t_2) \end{bmatrix}.$$

Then,

$$\widetilde{TAC}_C(t_1, t_2) = \frac{1}{3}\left[\widetilde{TAC}_{C1}(t_1, t_2) + \widetilde{TAC}_{C2}(t_1, t_2) + \widetilde{TAC}_{C3}(t_1, t_2)\right]. \qquad (23)$$

To minimize the total cost function $\widetilde{TAC}_S(t_1, t_2)$ per unit time, the values of t_1 and t_2 can be obtained by solving the equations

$$\frac{\partial \widetilde{TAC}_C(t_1, t_2)}{t_1} = 0 \quad \text{and} \quad \frac{\partial \widetilde{TAC}_C(t_1, t_2)}{t_2} = 0 \qquad . \qquad (24)$$

Equations in (24) are equivalent to

$$\begin{bmatrix} \left\{\frac{\begin{pmatrix} d(a_1 - b_1m_1)e^{\theta_1 t_1} + p(a_1 - b_{1\ \ 1})e^{\theta_1 t_1} + WO_1 e^{-\theta_1 t_1} \\ +\frac{(e^{\theta_1 t_1}-1)(a_1-b_1m_1)r_1}{\theta_1^2} + \frac{O_1(a_1-b_1m_1)(1-e^{\theta_1(t_2-t_1)})}{\theta_1} \end{pmatrix}}{3T}\right\} \\ +\left\{\frac{\begin{pmatrix} de^{\theta_2 t_1}(a_2 - b_2m_2) + p(a_2 - b_2m_2)e^{\theta_2 t_1} + WO_2 e^{-\theta_2 t_1} \\ +\frac{(e^{\theta_2 t_1}-1)(a_2-b_2m_2)r_2}{\theta_2^2} + \frac{O_2(a_2-b_2m_2)(1-e^{\theta_2(t_2-t_1)})}{\theta_2} \end{pmatrix}}{T}\right\} = 0 \\ +\left\{\frac{\begin{pmatrix} de^{\theta_3 t_1}(a_3 - b_3m_3) + p(a_3 - b_3m_3)e^{\theta_3 t_1} + WO_3 e^{-\theta_3 t_1} \\ +\frac{(e^{\theta_3 t_1}-1)(a_3-b_3m_3)r_3}{\theta_3^2} + \frac{O_3(a_3-b_3m_3)(1-e^{\theta_3(t_2-t_1)})}{\theta_3} \end{pmatrix}}{T}\right\} \end{bmatrix}$$

and

$$\left[\begin{array}{c} \left\{\dfrac{\begin{array}{c} -k_1p(a_1-b_1m_1)+d(b_1m_1-a_1)-(1-k_1)(a_1-b_1m_1)l_1 \\ -(a_1-b_1m_1)n_1+\dfrac{\left(e^{\theta_1(t_2-t_1)}-1\right)(a_1-b_1m_1)O_1}{\theta_1} \end{array}}{3T}\right\} \\[4em] +2\left\{\dfrac{\begin{array}{c} -k_2p(a_2-b_2m_2)+d(b_2m_2-a_2)-(1-k_2)(a_2-b_2m_2)l_2 \\ -(a_2-b_2m_2)n_2+\dfrac{\left(e^{\theta_2(t_2-t_1)}-1\right)(a_2-b_2m_2)O_2}{\theta_2} \end{array}}{T}\right\} \\[4em] +\left\{\dfrac{\begin{array}{c} -k_3p(a_3-b_3m_3)+d(b_3m_3-a_3)-(1-k_3)(a_3-b_3m_3)l_3 \\ -(a_3-b_3m_3)n_3+\dfrac{\left(e^{\theta_3(t_2-t_1)}-1\right)(a_3-b_3m_3)O_3}{\theta_3} \end{array}}{T}\right\} \end{array}\right] = 0.$$

Thus, the values of t_1 and t_2 obtained from the above equations will minimize the total cost function, if they satisfy the equations

$$\frac{\partial^2 \widehat{TAC}_C(t_1,t_2)}{\partial t_1^2} > 0 \; , \; \frac{\partial^2 \widehat{TAC}_C(t_1,t_2)}{\partial t_2^2} > 0 \quad \text{and}$$

$$\left(\frac{\partial^2 \widehat{TAC}_C(t_1,t_2)}{\partial t_1^2}\right)\left(\frac{\partial^2 \widehat{TAC}_C(t_1,t_2)}{\partial t_2^2}\right) - \left(\frac{\partial^2 \widehat{TAC}_C(t_1,t_2)}{\partial t_1^2 \partial t_2^2}\right)^2 > 0. \tag{25}$$

6. Numerical Example

To illustrate the result of the proposed model, we consider a numerical example of the inventory system with the following parametric values:

6.1. Crisp Model

Let us suppose, $A = 60$ units, $\beta = 0.5$, $C_1 = \$30$/unit/day, $k = 0.7$ unit, $C_2 = \$10$/unit/day, $C_3 = \$16$/unit/day, $p = \$15$/ unit/day, $\theta = 0.006$, $W = 100$ units, $d = 16$ unit, $\tilde{h}_r = \$. \ 0.07/$ unit/day, $\tilde{h}_o = \$ \ 0.06$/unit/day, T=365 days. The values of different parameters considered here are realistic, though these are not taken from any case study. Corresponding to these input values, $t_1 = 47.4072$ days, $t_2 = 319.925$ days, TAC will be minimize and the minimum value is $\$ 2022.03$. To show the convexity of cost function $TAC(t_1, t_2)$, we plot a 3D graph. A three dimensional graph is shown in the Figure 2.
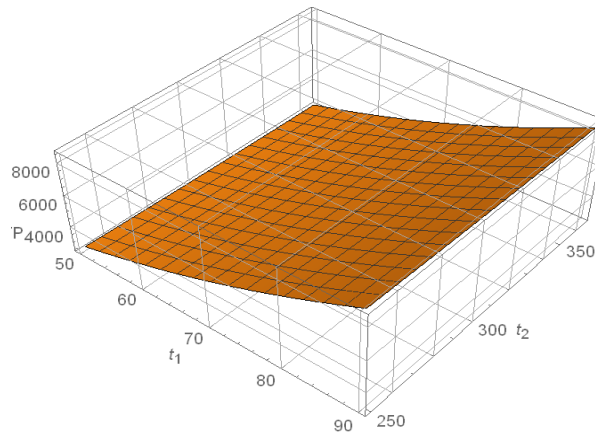
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Figure 2. Graphical representation of optimal result

6.2. Fuzzy model

Let us consider

$$\tilde{\alpha} = (50, 60, 70), \tilde{\beta} = (0.4, 0.5, 0.6), \ \tilde{h}_o = (0.05, 0.06, 0.07), \tilde{C}_1 = (25, 30, 35), \tilde{\theta} = (0.005, 0.006, 0.007), \tilde{C}_2 = (8, 10, 12), \tilde{h}_r = (0.06, 0.07, 0.08) \text{ and } \tilde{C}_3 = (14, 16, 18)$$

as triangular fuzzy numbers and $p = \$ 15$/unit/day, $W = 100$ units, $d = 16$ units, T= 365 days (1 year). The values of different parameters considered here are realistic, though these are not taken from any case study. Then, the fuzzy total average cost, determined by the Signed Distance Method, is $\widehat{TAC}_S(t_1, t_2) = \$1968.695$, with $t_1 = 46.6338$ days, $t_2 = 321.9077$ days. By Centroid Method, it is $\widehat{TAC}_C(t_1, t_2) = \$1950.9166$, with $t_1 = 46.3761$ days, $t_2 = 322.5686$ days.

7. Sensitivity analysis

A sensitivity analysis is carried out to study the effect of changes in the system parameters $\tilde{\alpha}$, $\tilde{\beta}, \tilde{k}, \tilde{C}_1 \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$. We use Mathematica11.1software for the calculation of the total cost function.

7.1. By Signed Distance Method (SDM)

   i.   When $\tilde{\alpha}, \tilde{\beta}, \tilde{k}, \tilde{C}_1, \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$ are all triangular fuzzy numbers, then value of $t_1 = 46.6338$ days, $t_2 = 321.9077$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$1968.695$.
   ii. When $\tilde{\beta}, \tilde{k}, \tilde{C}_1, \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.6385$days, $t_2 = 321.9122$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$ 1930.935$.
   iii. When $\tilde{k}, \tilde{C}_1, \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.6417$days, $t_2 = 321.9155$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$1952.955$.
   iv. When $\tilde{\beta}, \tilde{k}, \tilde{h}_r, \tilde{h}_o, \tilde{C}_2, \tilde{C}_3, \tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.6409$days, $t_2 = 321.9147$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$1962.3975$.

v. When $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 47.0171$ days, $t_2 = 322.593$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$1992.6925$.

vi. When $\tilde{h}_r$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 47.0052$ days, $t_2 = 320.025$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$1998.1525$.

vii. When $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 47.0067$ days, $t_2 = 320.025$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$2001.41$.

viii. When $\tilde{C}_2$ and $\tilde{C}_3$ are triangular fuzzy numbers, then value of $t_1 = 47.2392$ days, $t_2 = 319.65$ days with minimum total cost $\widehat{TAC}_S(t_1, t_2) = \$2060.495$.

7.2. By Centroid Method (CM)

i. When $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{k}$, $\tilde{C}_1$, $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are all triangular fuzzy numbers, then value of $t_1 = 46.3761$ days, $t_2 = 322.5686$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1950.9166$.

ii. When $\tilde{\beta}$, $\tilde{k}$, $\tilde{C}_1$, $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.6385$ days, $t_2 = 322.5746$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1899.89$.

iii. When $\tilde{k}$, $\tilde{C}_1$, $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.3865$ days, $t_2 = 322.579$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1929.93$.

iv. When $\tilde{\beta}$, $\tilde{k}$, $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.3855$ days, $t_2 = 322.578$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1942.52$.

v. When $\tilde{h}_r$, $\tilde{h}_o$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.8870$ days, $t_2 = 323.4823$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1982.9133$.

vi. When $\tilde{h}_r$, $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.8733$ days, $t_2 = 316.726$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1990.1933$.

vii. When $\tilde{C}_2$, $\tilde{C}_3$, $\tilde{\theta}$ are triangular fuzzy numbers, then value of $t_1 = 46.8733$ days, $t_2 = 320.0583$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$1994.5366$.

viii. When $\tilde{C}_2$ and $\tilde{C}_3$ are triangular fuzzy numbers, then value of $t_1 = 47.1833$ days, $t_2 = 319.5583$ days with minimum total cost $\widehat{TAC}_C(t_1, t_2) = \$2073.3166$.

In Table 1 to Table 8, we have analyzed the system parameters with different values in fuzzy sense, keeping some other parameters in its original values.

Table 1. Sensitivity analysis on initial demand parameter $(\tilde{\alpha})$

	SDM			CM		
$\tilde{\alpha}$	t_1	t_2	$\widehat{TAC}_S$	t_1	t_2	$\widehat{TAC}_C$
(50,60,70)	47.3965	319.9142	2022.0275	47.3930	319.9106	2022.0266
(60,70,80)	47.4754	320.1138	2476.4600	47.4735	319.9913	2476.4700
(70,80,90)	47.5288	320.0465	2930.8925	47.5276	320.0453	2930.8933

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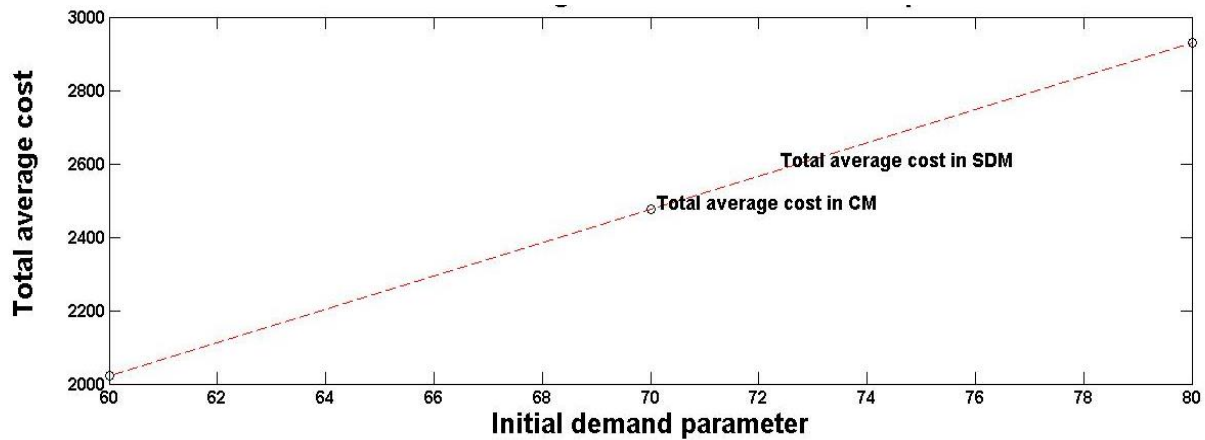AAM: Intern. J., Vol. 14, Issue 1 (June 2019)   529

Figure 3. Variation of total avg. cost w.r.t demand parameter

Table 2. Sensitivity analysis on positive demand parameter $(\tilde{\beta})$

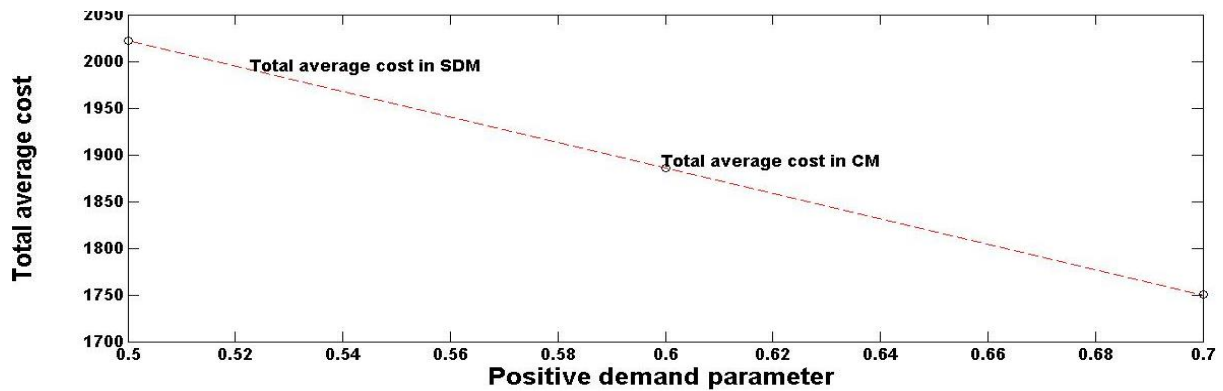| $\tilde{\beta}$	SDM			CM		
	t_1	t_2	$\widehat{TAC_S}$	t_1	t_2	$\widehat{TAC_C}$
(0.4,0.5,0.6)	47.4062	319.9240	2022.0250	47.4059	319.9236	2022.0233
(0.5,0.6,0.7)	47.3770	319.8947	1885.6950	47.3766	319.8943	1885.6966
(0.6,0.7,0.8)	47.3731	319.8607	1749.3675	47.3427	319.8603	1750.6866

Figure 4. Variation of total avg. cost w.r.t positive demand parameter

Table 3. Sensitivity analysis on backlogging rate $(\tilde{k})$

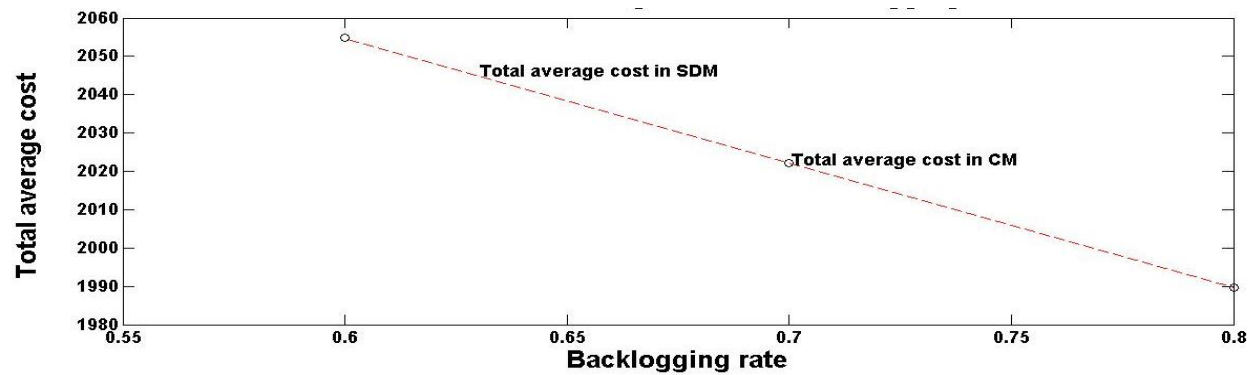| $\tilde{k}$	SDM			CM		
	t_1	t_2	$\widehat{TAC_S}$	t_1	t_2	$\widehat{TAC_C}$
(0.5,0.6,0.7)	47.8119	320.6537	2054.7375	47.8119	320.6536	2054.7566
(0.6,0.7,0.8)	47.4069	319.9245	2022.0850	47.4068	319.9243	2022.1033
(0.7,0.8,0.9)	47.0009	319.1935	1989.6575	47.0008	319.1933	1989.6766

Figure 5. Variation of total avg. cost w.r.t backlogging rate

Table 4. Sensitivity analysis on selling price $(\tilde{C}_1)$

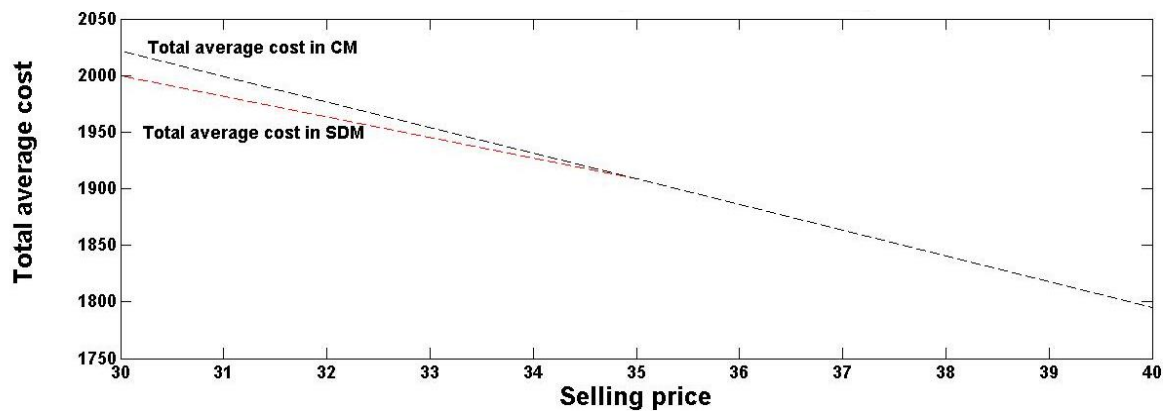| $\tilde{C}_1$	SDM			CM		
	t_1	t_2	$\widetilde{TAC_S}$	t_1	t_2	$\widetilde{TAC_C}$
(25,30,35)	47.4065	319.9242	2000.0275	47.4063	319.9240	2022.0266
(30,35,40)	47.3825	319.9002	1908.4200	47.3822	319.9000	1908.4200
(35,40,45)	47.3554	319.8730	1794.8100	47.3551	319.8726	1794.8100

Figure 6. Variation of total avg. cost w.r.t selling price

Table 5. Sensitivity analysis on shortage cost parameter $(\tilde{C}_2)$

$\tilde{C}_1$	SDM			CM		
	t_1	t_2	$\widetilde{TAC_S}$	t_1	t_2	$\widetilde{TAC_C}$
(8,9,10)	43.2755	312.4957	1714.1375	43.2668	312.4816	1716.1300
(9,10,11)	47.3824	319.8842	2027.7075	47.3741	319.8706	2029.6000
(10,11,12)	51.3900	327.1100	2364.0175	51.3821	327.0970	2365.8200

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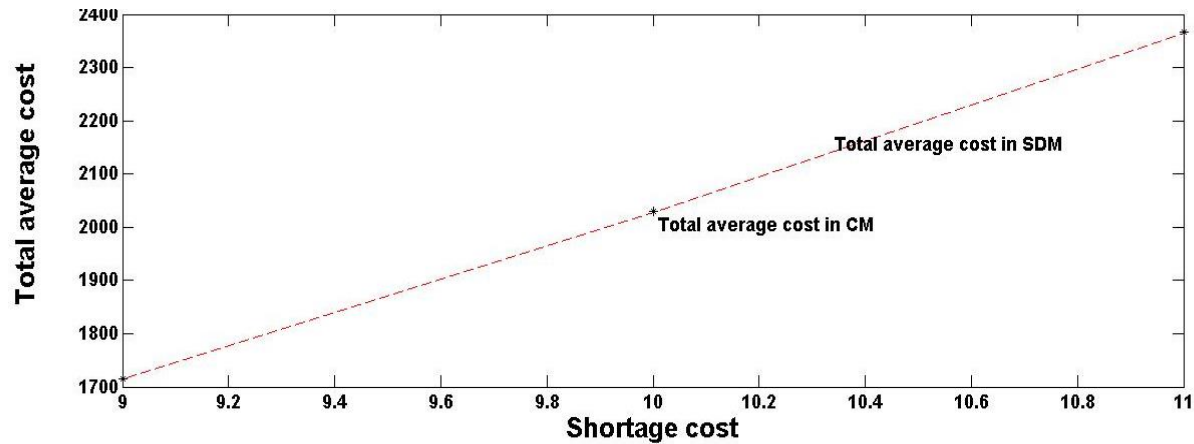

Figure 7. Variation of total avg. cost w.r.t shortage cost parameter

Table 6. Sensitivity analysis on lost sale cost parameter $(\tilde{C}_3)$

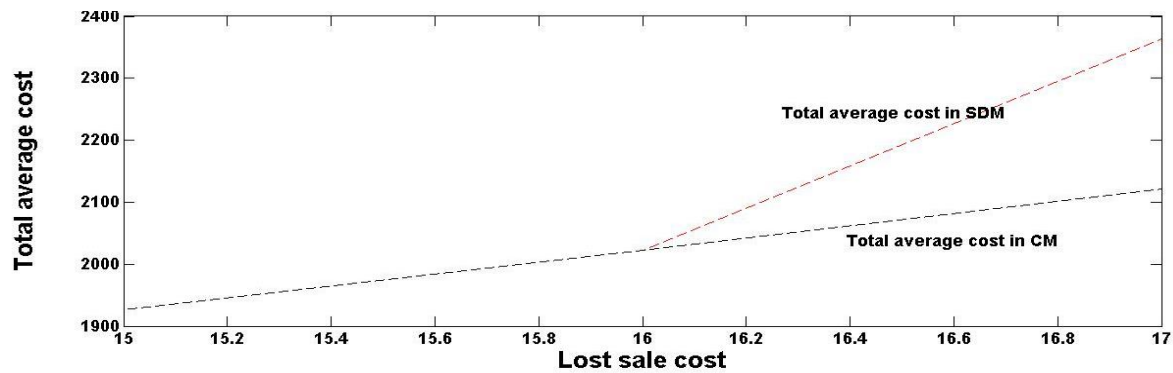| $\tilde{C}_3$	SDM			CM		
	t_1	t_2	$\widehat{TAC_S}$	t_1	t_2	$\widehat{TAC_C}$
(14,15,16)	46.1838	317.7225	1925.9475	46.1831	317.7213	1926.1200
(15,16,17)	47.4049	319.9212	2022.5375	47.4042	319.9200	2022.7066
(16,17,18)	48.6171	322.1052	2364.0175	48.6164	322.1040	2121.3400

Figure 8. Variation of total avg. cost w.r.t lost sale cost parameter

Table7. Sensitivity analysis on deterioration parameter $(\tilde{\theta})$

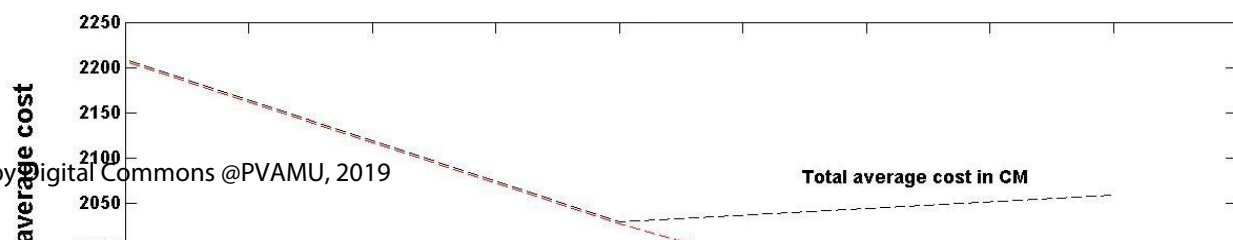| $\tilde{\theta}$	SDM			CM		
	t_1	t_2	$\widehat{TAC_S}$	t_1	t_2	$\widehat{TAC_C}$
(0.0056,0.0058,0.006)	49.0711	326.3462	2206.2625	49.0808	326.3720	2208.57
(0.0058,0.006,0.0062)	47.4335	319.9982	2027.8575	47.4423	320.0226	2029.80
(0.006,0.0062,0.0064)	45.9018	313.9437	1872.9675	45.9097	313.9663	2058.56

Figure 9. Variation of total avg. cost w.r.t deterioration rate

Table 8. Sensitivity analysis on holding cost parameter $\tilde{h}_o$

$\tilde{h}_o$	SDM			CM		
	t_1	t_2	$\widetilde{TAC}_S$	t_1	t_2	$\widetilde{TAC}_C$
(0.05,0.055,0.06)	47.4411	332.0655	1977.9581	47.4411	332.1770	1977.4775
(0.055,0.06,0.065)	47.4072	320.2042	2020.8375	47.4072	320.2973	2022.4400
(0.06,0.065,0.07)	47.3732	309.4727	2058.8925	47.3732	309.5516	2058.5600
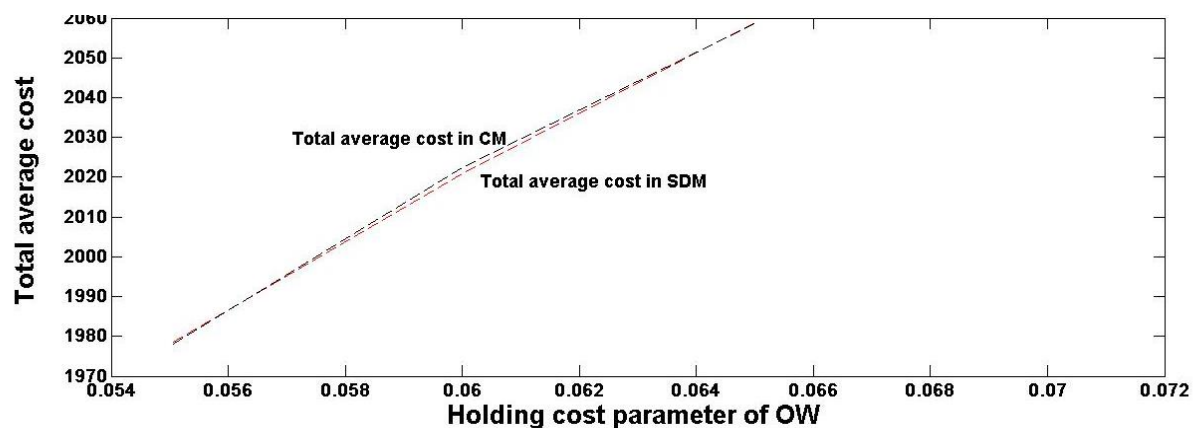
Figure 10. Variation of total avg. cost w.r.t holding cost parameter of OW

Above observations can be sum up as follows:

i.   In Table 1 and Fig. 3, it is observed that increase of the value of $\tilde{\alpha}$, the values of t_1 and t_2 also increase in both the cases of SDM and CM. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ increase.

ii.   In Table 2 and Fig. 4, if we increase the value of $\tilde{\beta}$, then the values of t_1 and t_2 decrease in both the cases slowly. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ decrease rapidly.

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iii. In Table 3 and Fig. 5, by increasing the value of back-logging rate $\tilde{k}$, the values of t_1 and t_2 decrease slowly in both the cases. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ decrease rapidly.

iv. In Table 4 and Fig. 6, it is observed that, if the value $\tilde{C}_1$ increases, the values of t_1 and t_2 decrease slowly in both the cases. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ decrease very rapidly.

v. Table 5 and Fig. 7, show that when the value of shortage cost $\tilde{C}_2$ increases, then the values of t_1 and t_2 increase rapidly with increase of the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$.

vi. In Table 6 and Fig. 8, it is observed that, if the value of lost-sale-cost $\tilde{C}_3$ increases, the values of t_1 and t_2 increase rapidly in both the cases. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ in both the cases at its peak.

vii. In Table 7 and Fig. 9, it is observed that, if the value of deterioration rate $\tilde{\theta}$ increases, the values of t_1 and t_2 in both the cases decrease rapidly. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ decrease in both the cases.

viii. Table 8 and Fig. 10, show that, if the value of holding cost of OW $\tilde{h}_o$ increases, the value of t_1 and t_2 in both the cases decrease slowly. With this effect, the total average cost $\widetilde{TAC}_S$ and $\widetilde{TAC}_C$ in both the cases increase.

ix. Defuzzification by Centroid method gives more profit as compared to Signed distance method.

x. Deterioration rate is more sensitive in both the cases.

8. Conclusion

Most of the researchers worked in two-warehouse inventory modelling by assuming both deterioration rate and demand constant. But in real situations, these quantities are not exactly constant. In the present paper, a fuzzy two-warehouse inventory model for deteriorating items with price-dependent demand and shortages under partially backlogged, where the demand rate is a function of selling-price, has been proposed. The developed model is discussed for both crisp as well as in fuzzy environment. Since demand, selling-price, deterioration rate, holding-cost, shortage-cost and lost-sale are uncertain, these parameters have been considered as triangular fuzzy numbers. The main objective of the study is to determine the optimum result of fuzzy model in which the fuzzy numbers are defuzzified by Signed Distance Method (SDM) and Centroid Method (CM). In this paper, we observed that an uncertainty nature of the system parameters, the total average cost decreases in fuzzy model as compared to crisp model. Also, we have observed that in Centroid Method gives more accurate result as compared to Signed Distance Method. Sensitivity analysis indicates that the total cost function is more sensitive to the changes in deterioration rate. After analyze the result, the decision maker can plan for the optimal value for total cost and for other related parameters. The model can be used for the products like potato, onion, fruits in the countries like India, Pakistan, Sri Lanka, Bangladesh as the demand of the food grains increases with time for a fixed time horizon. The present model can be extended by considering the demand function to be time and price dependent, or stock dependent under time dependent deterioration rate.

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