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Cubic Interior Ideals in Semigroups

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Abstract

In this paper we apply the cubic set theory to interior ideals of a semigroup. The notion of cubic interior ideals is introduced, and related properties are investigated. Characterizations of (cubic) interior ideals are established, and conditions for a semigroup to be left (right) simple are provided.

Keywords: (Cubic) subsemigroup; (Cubic) left (right) ideal; (Cubic) interior ideal

MSC 2010 No.: 20N25, 03E72

1. Introduction

The notion of fuzzy sets introduced by Zadeh (1965) laid the foundation for the development of fuzzy Mathematics. This theory has a wide range of application in several branches of Mathematics such as logic, set theory, group theory, semigroup theory, real analysis, measure theory, and topology. After a decade, the notion of interval-valued fuzzy sets was introduced by Zadeh (1975) as an extension of fuzzy sets, that is, fuzzy sets with interval-valued membership functions. Based on the (interval-valued) fuzzy sets, Jun et al. (2012) introduced the notion of cubic sets and investigated several properties. This notion is applied to semigroups and BCK/BCI-algebras (see Jun and Khan (2013), Jun et al. (2010), Jun et al. (2011), Jun et al. (2011), Jun and Lee (2010). In the year 2017, Muhiuddin et al. (2017) introduced the notion of stable cubic sets and investigated several properties. Also, Muhiuddin et al. (2014), Muhiuddin and Al-roqi (2014) and Jun et al. (2017) have applied the notion of cubic sets to the soft sets theory. Recently, a number of research papers have been devoted to the study of cubic sets theory applied to different algebraic structures (see e.g., Ahn et al. (2014), Akram et al. (2013), Khan et al. (2015), Jun et al. (2018), Senapati and Shum (2018), Senapati and Shum (2017), Senapati et al. (2015), Yaqoob et al. (2013)).

Motivated by the theory of cubic sets in different algebraic structures, in the present analysis, we introduced the notion of cubic interior ideals in semigroups, and investigated related properties. We provide characterizations of (cubic) interior ideals. We consider the relations between cubic ideals and cubic interior ideals, and we provide examples to show that a cubic interior is not a cubic ideal. We show that the intersection of cubic interior ideals is a cubic interior ideal. We also provide a condition for a cubic interior ideal to be a cubic ideal. We discuss a condition for a semigroup to be left (right) simple.

2. Preliminaries

Let S be a semigroup. For any subsets A and B of S, the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A nonempty subset A of a semigroup S is called

- a *subsemigroup* of S if $AA \subseteq A$,
- a *left (resp. right) ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$),
- an *interior ideal* of S if $SAS \subseteq A$.

We say that a nonempty subset A of S is called a *two-sided ideal* of S if it is both a left and a right ideal of S.

A semigroup S is said to be *left* (resp. *right*) *zero* if $xy = x$ (resp. $xy = y$,) for all $x, y \in S$. A semigroup S is said to be *regular* if for each element $a \in S$, there exists an element x in S such that $a = axa$. A semigroup S is said to be *left* (resp. *right*) *simple* if it contains no proper left (resp. right) ideal. For more details we refer the readers to Kuroki (1981) and Mordeson (2003).

Let I be a closed unit interval, i.e., $I = [0, 1]$. By an *interval number* we mean a closed subinterval $\overline{a} = [a^-, a^+]$ of I, where $0 \le a^- \le a^+ \le 1$. Denote by $D[0, 1]$ the set of all interval numbers. Let us define what are known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $D[0, 1]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $D[0, 1]$. Consider two interval numbers $\overline{a}_1 := \left[a_1^-, a_1^+\right]$ and $\overline{a}_2 := \left[a_2^-, a_2^+\right]$. Then,

$$\begin{aligned}
&\text{rmin}\{\overline{a}_1, \overline{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\text{rmax}\{\overline{a}_1, \overline{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\overline{a}_1 \succeq \overline{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+,
\end{aligned}$$

and similarly we may have $\overline{a}_1 \preceq \overline{a}_2$ and $\overline{a}_1 = \overline{a}_2$. To say $\overline{a}_1 \succ \overline{a}_2$ (resp. $\overline{a}_1 \prec \overline{a}_2$) we mean $\overline{a}_1 \succeq \overline{a}_2$ and $\overline{a}_1 \ne \overline{a}_2$ (resp. $\overline{a}_1 \preceq \overline{a}_2$ and $\overline{a}_1 \ne \overline{a}_2$). Let $\overline{a}_i \in D[0, 1]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \overline{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \overline{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

An *interval-valued fuzzy set* (briefly, *IVF set*) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A := \left\{\left(x, \left[\mu_A^-(x), \mu_A^+(x)\right]\right) \mid x \in X\right\},$$

which is briefly denoted by $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is called the degree of membership of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are refereed to as the lower and upper degrees, respectively, of membership of x to $\tilde{\mu}_A$.

Let X be a nonempty set. A *cubic set* $\mathscr{A}$ in X (see Jun et al. (2010)) is a structure

$$\mathscr{A} = \{\langle x, \tilde{\mu}_A(x), f_A(x)\rangle : x \in X\},$$

which is briefly denoted by $\mathscr{A} = \langle \tilde{\mu}_A, f_A\rangle$, where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and f_A is a fuzzy set in X. In this case, we will use

$$\mathscr{A}(x) = \langle \tilde{\mu}_A(x), f_A(x)\rangle = \left\langle [\mu_A^-(x), \mu_A^+(x)], f_A(x)\right\rangle,$$

for all $x \in X$.

Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A\rangle$ be a cubic set in X. For any $r \in [0,1]$ and $[s,t] \in D[0,1]$, we define $U(\mathscr{A}; [s,t], r)$ as follows:

$$U(\mathscr{A}; [s,t], r) = \{x \in X \mid \tilde{\mu}_A(x) \succeq [s,t],\ f_A(x) \leq r\},$$

and we say it is a *cubic level set* of $\mathscr{A} = \langle \tilde{\mu}_A, f_A\rangle$ (see Jun et al. (2011)). Note that

$$U(\mathscr{A}; [s,t], r) = \mathscr{A}[s,t] \cap L(\mathscr{A}, r),$$

where $\mathscr{A}[s,t] := \{x \in S \mid \tilde{\mu}_A(x) \succeq [s,t]\}$ and $L(\mathscr{A}, r) := \{x \in S \mid f_A(x) \leq r\}$.

For any non-empty subset G of a set X, the *characteristic cubic set* of G in X is defined to be a structure

$$\chi_G = \{\langle x, \tilde{\mu}_{\chi_G}(x), f_{\chi_G}(x)\rangle : x \in X\},$$

which is briefly denoted by $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G}\rangle$, where

$$\tilde{\mu}_{\chi_G}(x) = \begin{cases} [1,1], & \text{if } x \in G, \\ [0,0], & \text{otherwise,} \end{cases} \qquad f_{\chi_G}(x) = \begin{cases} 0, & \text{if } x \in G, \\ 1, & \text{otherwise.} \end{cases}$$

The *whole cubic set* $\mathcal{S}$ in a semigroup S is defined to be a structure

$$\mathcal{S} = \left\{\left\langle x, \tilde{1}_S(x), 0_S(x)\right\rangle : x \in S\right\},$$

with $\tilde{1}_S(x) = [1,1]$ and $0_S(x) = 0$, for all $x \in X$. It will be briefly denoted by $\mathcal{S} = \langle \tilde{1}_S, 0_S\rangle$.

For two cubic sets $\mathscr{A} = \langle \tilde{\mu}_A, f_A\rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B\rangle$ in a semigroup S, we define

$$\mathscr{A} \sqsubseteq \mathscr{B} \iff \tilde{\mu}_A \preceq \tilde{\mu}_B,\ f_A \geq f_B,$$

and the *cubic product* of $\mathscr{A} = \langle \tilde{\mu}_A, f_A\rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B\rangle$ is defined to be a cubic set

$$\mathscr{A} \copyright \mathscr{B} = \{\langle x, (\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B)(x), (f_A \circ f_B)(x)\rangle : x \in S\},$$

which is briefly denoted by $\mathscr{A} \copyright \mathscr{B} = \langle \tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B, f_A \circ f_B\rangle$, where $\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B$ and $f_A \circ f_B$ are defined as follows, respectively:

$$(\tilde{\mu}_A \tilde{\circ} \tilde{\mu}_B)(x) = \begin{cases} \operatorname{rsup}_{x=yz} [\operatorname{rmin}\{\tilde{\mu}_A(y), \tilde{\mu}_B(z)\}], & \text{if } x = yz, \text{ for some } y, z \in S, \\ [0,0], & \text{otherwise,} \end{cases}$$

and

$$(f_A \circ f_B)(x) = \begin{cases} \bigwedge_{x=yz} \max\{f_A(y), f_B(z)\}, & \text{if } x = yz, \text{ for some } y, z \in S, \\ 1, & \text{otherwise,} \end{cases}$$

for all $x \in S$. We also define the cap and union of two cubic sets as follows. Let $\mathscr{A}$ and $\mathscr{B}$ be two cubic sets in X. The *intersection* of $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A} \sqcap \mathscr{B}$, is the cubic set

$$\mathscr{A} \sqcap \mathscr{B} = \langle \tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B, f_A \vee f_B \rangle,$$

where $(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(x) = \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and $(f_A \vee f_B)(x) = \max\{f_A(x), f_B(x)\}$.

The *union* of $\mathscr{A}$ and $\mathscr{B}$, denoted by $\mathscr{A} \sqcup \mathscr{B}$, is the cubic set

$$\mathscr{A} \sqcup \mathscr{B} = \langle \tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B, f_A \wedge f_B \rangle,$$

where $(\tilde{\mu}_A \tilde{\cup} \tilde{\mu}_B)(x) = \text{rmax}\{\tilde{\mu}_A(x), \tilde{\mu}_B(x)\}$ and $(f_A \wedge f_B)(x) = \min\{f_A(x), f_B(x)\}$.

3. Cubic interior ideals

Definition 3.1.

By Jun and Khan (2013), a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is called a *cubic subsemigroup* of S if it satisfies:

$$(\forall x, y \in L) \begin{pmatrix} \tilde{\mu}_A(xy) \succeq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \\ f_A(xy) \leq \max\{f_A(x), f_A(y)\} \end{pmatrix}. \tag{1}$$

Definition 3.2.

By Jun and Khan (2013), a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is called a *left cubic ideal* of S if it satisfies:

$$(\forall a, b \in S)\left(\tilde{\mu}_A(ab) \succeq \tilde{\mu}_A(b),\ f_A(ab) \leq f_A(b)\right). \tag{2}$$

Similarly, we say that a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is a *right cubic ideal* of S if $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ satisfies the following condition:

$$(\forall a, b \in S)\left(\tilde{\mu}_A(ab) \succeq \tilde{\mu}_A(a),\ f_A(ab) \leq f_A(a)\right). \tag{3}$$

By a *two-sided cubic ideal* we mean a left and right cubic ideal.

Definition 3.3.

A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is called a *cubic interior ideal* of S if it satisfies:

$$(\forall x, a, y \in S)\left(\tilde{\mu}_A(xay) \succeq \tilde{\mu}_A(a),\ f_A(xay) \leq f_A(a)\right). \tag{4}$$

Example 3.4.

Consider a semigroup $S = \{a, b, c, d\}$ with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in S as follows:

S	$\tilde{\mu}_A(x)$	$f_A(x)$
a	$[0.4, 0.7]$	0.4
b	$[0, 0]$	1
c	$[0.3, 0.5]$	0.7
d	$[0, 0]$	1

Then, $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S.

We provide characterizations of a cubic interior ideal.

Theorem 3.5.

A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is a cubic interior ideal of S if and only if $\mathscr{A}[s, t]$ and $L(\mathscr{A}, r)$ are interior ideals of S whenever they are nonempty, for all $[s, t] \in D[0, 1]$ and $r \in [0, 1]$.

Proof:

Assume that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic bi-ideal of S. Let $[s, t] \in D[0, 1]$ and $r \in [0, 1]$ be such that $\mathscr{A}[s, t] \neq \emptyset \neq L(\mathscr{A}, r)$. Let $x, z \in S$ and $y \in \mathscr{A}[s, t]$. Then, $\tilde{\mu}_A(y) \succeq [s, t]$, and so $\tilde{\mu}_A(xyz) \succeq \tilde{\mu}_A(y) \succeq [s, t]$ by (4). Hence, $xyz \in \mathscr{A}[s, t]$, which shows that $\mathscr{A}[s, t]$ is an interior ideals of S. Now let $a \in L(\mathscr{A}, r)$. Then, $f_A(a) \leq r$. It follows from (4) that $f_A(xaz) \leq f_A(a) \leq r$ so that $xaz \in L(\mathscr{A}, r)$. Therefore, $L(\mathscr{A}, r)$ is an interior ideals of S.

Conversely, assume that $\mathscr{A}[s, t]$ and $L(\mathscr{A}, r)$ are interior ideals of S, for all $[s, t] \in D[0, 1]$ and $r \in [0, 1]$ with $\mathscr{A}[s, t] \neq \emptyset \neq L(\mathscr{A}, r)$. Suppose that (4) is not valid. Then, there exist $x, a, y \in S$ such that $\tilde{\mu}_A(xay) \not\succeq \tilde{\mu}_A(a)$ or $f_A(xay) \not\leq f_A(a)$. If $\tilde{\mu}_A(xay) \not\succeq \tilde{\mu}_A(a)$, then

$$\tilde{\mu}_A(xay) \prec [s_0, t_0] \preceq \tilde{\mu}_A(a),$$

for some $[s_0, t_0] \in D[0, 1]$. Thus, $a \in \mathscr{A}[s_0, t_0]$ but $xay \notin \mathscr{A}[s_0, t_0]$. This is a contradiction. If $f_A(xay) \not\leq f_A(a)$, then there exists $r_0 \in [0, 1]$ such that $f_A(xay) > r_0 \geq f_A(a)$. Thus, $a \in L(\mathscr{A}, r)$, but $xay \notin L(\mathscr{A}, r)$ which induces a contradiction. Therefore, the condition (4) is valid, and so $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. ■

Corollary 3.6.

If a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S is a cubic interior ideal of S, then every nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is an interior ideal of S.

Proof:

Straightforward. ■

Proposition 3.7.

The intersection of cubic interior ideals is a cubic interior ideal.

Proof:

Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, f_B \rangle$ be cubic interior ideals of a semigroup S. For any $x, a, y \in S$, we have

$$\begin{aligned}(\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(xay) &= \text{rmin}\{\tilde{\mu}_A(xay), \tilde{\mu}_B(xay)\} \\ &\succeq \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_B(a)\} \\ &= (\tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B)(a),\end{aligned}$$

and

$$\begin{aligned}(f_A \vee f_B)(xay) &= \max\{f_A(xay), f_B(xay)\} \\ &\leq \max\{f_A(a), f_B(a)\} \\ &= (f_A \vee f_B)(a).\end{aligned}$$

Therefore, $\mathscr{A} \sqcap \mathscr{B} = \langle \tilde{\mu}_A \tilde{\cap} \tilde{\mu}_B, f_A \vee f_B \rangle$ is a cubic interior ideal of S. ■

Theorem 3.8.

For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in a semigroup S, the following assertions are equivalent:

(1) $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S.
(2) $\mathcal{S} ⓒ \mathscr{A} ⓒ \mathcal{S} \sqsubseteq \mathscr{A}$.

Proof:

Suppose that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. Suppose $x \in S$ and if there exist elements $y, z, a, b \in S$ such that $x = yz$ and $y = ab$, then

$$\begin{aligned}(\tilde{1}_S \tilde{\circ} \tilde{\mu}_A \tilde{\circ} \tilde{1}_S)(x) &= \underset{x=yz}{\text{rsup}} \left[\text{rmin}\left\{(\tilde{1}_S \tilde{\circ} \tilde{\mu}_A)(y), \tilde{1}_S(z)\right\}\right] \\ &= \underset{x=yz}{\text{rsup}} \left[\text{rmin}\left\{\underset{y=ab}{\text{rsup}} \left[\text{rmin}\left\{(\tilde{1}_S(a), \tilde{\mu}_A(b)\right\}\right], \tilde{1}_S(z)\right\}\right] \\ &= \underset{x=yz}{\text{rsup}} \left[\text{rmin}\left\{\underset{y=ab}{\text{rsup}} \left[\text{rmin}\left\{([1,1], \tilde{\mu}_A(b)\right\}\right], [1,1]\right\}\right] \\ &\succeq \tilde{\mu}_A(x),\end{aligned}$$

since $\tilde{\mu}_A(abz) \succeq \tilde{\mu}_A(b)$, and

$$\begin{aligned}(0_S \circ f_A \circ 0_S)(x) &= \bigwedge_{x=yz} \max\{(0_S \circ f_A)(y), 0_S(z)\} \\ &= \bigwedge_{x=yz} \max\left\{\bigwedge_{y=ab} \max\{(0_S(a), f_A(b)\}, 0_S(z)\right\} \\ &= \bigwedge_{x=yz} \max\left\{\bigwedge_{y=ab} \max\{([1,1], f_A(b)\}, [1,1]\right\} \\ &\leq f_A(x),\end{aligned}$$

since $f_A(abz) \leq f_A(b)$. In the other case, we have $(\tilde{1}_S \,\tilde{\circ}\, \tilde{\mu}_A \,\tilde{\circ}\, \tilde{1}_S)(x) = [0,0] \preceq \tilde{\mu}_A(x)$ and $(0_S \circ f_A \circ 0_S)(x) = 1 \geq f_A(x)$. Therefore, $\mathcal{S} ⓒ \mathscr{A} ⓒ \mathcal{S} \sqsubseteq \mathscr{A}$.

Conversely, assume that $\mathcal{S} ⓒ \mathscr{A} ⓒ \mathcal{S} \sqsubseteq \mathscr{A}$. Let x, a and y be any elements of S. Then,

$$\begin{aligned}\tilde{\mu}_A(xay) &\succeq (\tilde{1}_S \,\tilde{\circ}\, \tilde{\mu}_A \,\tilde{\circ}\, \tilde{1}_S)(xay) \\ &= \operatorname*{rsup}_{xay=pq} \left[\operatorname{rmin}\left\{(\tilde{1}_S \,\tilde{\circ}\, \tilde{\mu}_A)(p), \tilde{1}_S(q)\right\}\right] \\ &\succeq \operatorname{rmin}\left\{(\tilde{1}_S \,\tilde{\circ}\, \tilde{\mu}_A)(xa), \tilde{1}_S(y)\right\} \\ &= \operatorname{rmin}\left\{(\tilde{1}_S \,\tilde{\circ}\, \tilde{\mu}_A)(xa), [1,1]\right\} \\ &= \operatorname*{rsup}_{xa=pq} \left[\operatorname{rmin}\{\tilde{1}_S(p), \tilde{\mu}_A(q)\right\} \\ &\succeq \operatorname{rmin}\left\{\tilde{1}_S(x), \tilde{\mu}_A(a)\right\} \\ &= \operatorname{rmin}\{[1,1], \tilde{\mu}_A(a)\} \\ &= \tilde{\mu}_A(a),\end{aligned}$$

and

$$\begin{aligned}f_A(xay) &\leq (0_S \circ f_A \circ 0_S)(xay) \\ &= \bigwedge_{xay=pq} \max\{(0_S \circ f_A)(p), 0_S(q)\} \\ &\leq \max\{(0_S \circ f_A)(xa), 0_S(y)\} \\ &= \max\{(0_S \circ f_A)(xa), 0\} \\ &= \bigwedge_{xa=pq} \max\{0_S(p), f_A(q)\} \\ &\leq \max\{0_S(x), f_A(a)\} \\ &= \max\{0, f_A(a)\} \\ &= f_A(a).\end{aligned}$$

Consequently, $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. ■

Theorem 3.9.

For any nonempty subset G of a semigroup S, the following assertions are equivalent:

(1) G is an interior ideal of S.

(2) The characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a cubic interior ideal of S.

Proof:

Assume that G is an interior ideal of S. For any $x, y, a \in S$, if $a \in G$, then $xay \in SGS \subseteq G$. Hence,

$$\tilde{\mu}_G(xay) = [1, 1] = \tilde{\mu}_G(a) \text{ and } f_G(xay) = 0 = f_G(a).$$

If $a \notin G$, then $\tilde{\mu}_G(xay) \succeq [0, 0] = \tilde{\mu}_G(a)$ and $f_G(xay) \leq 1 = f_G(a)$. Hence,

$$\tilde{\mu}_G(xay) \succeq \tilde{\mu}_G(a) \text{ and } f_G(xay) \leq f_G(a),$$

for all $x, a, y \in S$. Therefore, $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ is a cubic interior ideal of S.

Conversely, suppose that the characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ is a cubic interior ideal of S. Let $w \in SGS$. Then, $w = xay$, for some $x, y \in S$ and $a \in G$. It follows from (4) that

$$\tilde{\mu}_G(w) = \tilde{\mu}_G(xay) \succeq \tilde{\mu}_G(a) = [1, 1],$$
$$f_G(w) = f_G(xay) \leq f_G(a) = 0.$$

Thus, $\tilde{\mu}_G(w) = [1, 1]$ and $f_G(w) = 0$, which imply that $w \in G$. Hence, $SGS \subseteq G$, and therefore, G is an interior ideal of S. ■

As it can be easily seen that every two-sided cubic ideal of a semigroup S is a cubic interior ideal of S. The following example shows that the converse of this property may not hold in general.

Example 3.10.

(1) The cubic interior ideal $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in Example 3.4 is not a left cubic ideal of S, since

$$\tilde{\mu}_A(dc) = \tilde{\mu}_A(b) = [0, 0] \not\succeq [0.3, 0.5] = \tilde{\mu}_A(c),$$

and/or

$$f_A(dc) = f_A(b) = 1 > 0.7 = f_A(c).$$

Hence, $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is not a two-sided cubic ideal of S.

(2) Consider a semigroup $S = \{0, a, b, c\}$ with the Cayley table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Define a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ in S as follows:

S	$\tilde{\mu}_A(x)$	$f_A(x)$
0	$[0.5, 0.8]$	0.2
a	$[0.3, 0.6]$	0.6
b	$[0.5, 0.8]$	0.4
c	$[0.2, 0.4]$	0.6

It is easy to verify that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a cubic interior ideal of S. But it is not a left cubic ideal of S, since $\tilde{\mu}_A(cb) = \tilde{\mu}_A(a) = [0.3, 0.6] \not\succeq [0.5, 0.8] = \tilde{\mu}_A(b)$ and/or $f_A(cb) = f_A(a) = 0.6 > 0.4 = f_A(b)$. Therefore, it is not a two-sided cubic ideal of S.

We provide a condition for a cubic interior ideal to be a cubic ideal.

Theorem 3.11.

In a regular semigroup S, every cubic interior ideal is a two-sided cubic ideal.

Proof:

Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a cubic interior ideal of a regular semigroup S. Let a and b be any elements of S. Then, since S is regular, there exist elements $x, y \in S$ such that $a = axa$ and $b = byb$. Hence, we have

$$\tilde{\mu}_A(ab) = \tilde{\mu}_A((axa)b) = \tilde{\mu}_A((ax)ab) \succeq \tilde{\mu}_A(a),$$
$$f_A(ab) = f_A((axa)b) = f_A((ax)ab) \leq f_A(a),$$

and

$$\tilde{\mu}_A(ab) = \tilde{\mu}_A(a(byb)) = \tilde{\mu}_A(ab(yb)) \succeq \tilde{\mu}_A(b),$$
$$f_A(ab) = f_A(a(byb)) = f_A(ab(yb)) \leq f_A(b).$$

Therefore, $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is a two-sided cubic ideal of S. ■

Lemma 3.12 (Jun and Khan (2013)).

A non-empty subset G of a semigroup S is a left (resp. right) ideal of S if and only if the characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a left (resp. right) cubic ideal of S.

Lemma 3.13.

For a semigroup S, the following conditions are equivalent:

(1) S is left (resp. right) simple.
(2) Every left (resp. right) cubic ideal of S is constant.

Proof:

Assume that S is left simple. Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a left cubic ideal of S and let a and b be any elements of S. Note that there exist elements $x, y \in S$ such that $a = yb$ and $b = xa$. It follows from

(2) that

$$\tilde{\mu}_A(a) = \tilde{\mu}_A(yb) \succeq \tilde{\mu}_A(b) = \tilde{\mu}_A(xa) \succeq \tilde{\mu}_A(a),$$
$$f_A(a) = f_A(yb) \leq f_A(b) = f_A(xa) \leq f_A(a).$$

and so that $\tilde{\mu}_A(a) = \tilde{\mu}_A(b)$ and $f_A(a) = f_A(b)$. Since a and b are arbitrarily, we know that $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is constant.

Conversely, suppose that (2) holds. Let G be a left ideal of S. Using Lemma 3.12, we know that the characteristic cubic set $\chi_G = \langle \tilde{\mu}_{\chi_G}, f_{\chi_G} \rangle$ of G in S is a left cubic ideal of S, and so it is constant by (2). Let x be any element of S. Since G is nonempty, $\tilde{\mu}_{\chi_G}(x) = [1, 1]$ and $f_{\chi_G}(x) = 0$, and so $x \in G$. This implies that $S \subseteq G$, and thus $G = S$. Hence, S is left simple. Similarly, we can obtain the right case. ■

Theorem 3.14.

For a semigroup S, the following conditions are equivalent:

(1) Every two-sided cubic ideal of S is constant.
(2) Every cubic interior ideal of S is constant.

Proof:

Assume that every two-sided cubic ideal of S is constant. Then, S is simple by Lemma 3.13. Let $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ be a cubic interior ideal of S and $a, b \in S$. Then, there exist $x, y \in S$ such that $a = xby$. It follows from (4) that

$$\tilde{\mu}_A(a) = \tilde{\mu}_A(xby) \succeq \tilde{\mu}_A(b) \text{ and } f_A(a) = f_A(xby) \leq f_A(b).$$

Similarly, we can prove that $\tilde{\mu}_A(b) \succeq \tilde{\mu}_A(a)$ and $f_A(b) \leq f_A(a)$. Therefore, $\tilde{\mu}_A(b) = \tilde{\mu}_A(a)$ and $f_A(b) = f_A(a)$, and so $\mathscr{A} = \langle \tilde{\mu}_A, f_A \rangle$ is constant, since a and b are arbitrarily. It is clear that (2) implies (1) because every two-sided cubic ideal is a cubic interior ideal. ■

4. Conclusion

Using the notion of cubic set theory, we introduced the notion of cubic interior ideals in semigroups, and investigated several properties of them. We provide characterizations of (cubic) interior ideals. Further, we consider the relations between cubic ideals and cubic interior ideals, and we provide examples to show that a cubic interior is not a cubic ideal. Furthermore, we show that the intersection of cubic interior ideals is a cubic interior ideal. In addition, we provide a condition for a cubic interior ideal to be a cubic ideal. Finally, we discuss a condition for a semigroup to be left (right) simple.

We hope that this work will provide a deep impact on the upcoming research in this field and other cubic structures studies to open up new horizons of interest and innovations. Indeed, this work may serve as a foundation for further study of cubic set theory. To extend these results, one can further study the cubic interior ideals of different algebras such as MTL-algerbas, subtraction algebras, B-algebras, MV-algebras, d-algebras, Q-algebras etc. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, medical diagnosis and data analysis, etc.

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REFERENCES

Ahn, S.S., Kim, Y.H. and Ko, J.M. (2014). Cubic subalgebras and filters of CI-algebras, Honam Math. J., Vol. 36, pp. 43–54.

Akram, M., Yaqoob, N. and Gulistan, M. (2013). Cubic KU-subalgebras, Int. J. Pure Appl. Math., Vol. 89, pp. 659–665.

Khan, M., Jun, Y.B., Gulistan, M. and Yaqoob, N. (2015). The generalized version of JunâĂŹs cubic sets in semigroups, J. Intell. Fuzzy Syst., Vol. 28, pp. 947–960.

Jun, Y. B. and Khan, A. (2013). Cubic Ideals in Semigroups, Honam Mathematical J., Vol. 35, No. 4, pp. 607–623.

Jun, Y. B., Kim, C. S. and Kang, M. S. (2010). Cubic subalgebras and ideals of BCK/BCI-algebras, Far East. J. Math. Sci., Vol. 44, pp. 239–250.

Jun, Y. B., Kim, C. S. and Kang, J. G. (2011). Cubic q-ideals of BCI-algebras, Ann. Fuzzy Math. Inform., Vol. 1, pp. 25–34.

Jun, Y. B., Kim, C. S. and Yang, K. O. (2012). Cubic sets, Ann. Fuzzy Math. Inform., Vol. 4, pp. 83–98.

Jun, Y. B., and Lee, K. J. (2010). Closed cubic ideals and cubic $\circ$-subalgebras in BCK/BCI-algebras, Appl. Math. Sci., Vol. 4, pp. 3395–3402.

Jun, Y. B., Lee, K. J. and Kang, M. S. (2011). Cubic structures applied to ideals of BCI-algebras, Comput. Math. Appl., Vol. 62, pp. 3334–3342.

Jun, Y. B., Muhiuddin, G., Ozturk, M. A. and Roh, E. H. (2017). Cubic soft ideals in BCK/BCI-algebras, Journal of Computational Analysis and Applications, Vol. 22, No. 5, pp. 929-940.

Jun, Y. B., Song, S. Z. and Kim, S. J. (2018). Cubic interval-valued intuitionistic fuzzy sets and their application in BCK/BCI-algebras, Axioms 2018, 7, 7. doi:10.3390/axioms7010007

Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems, Vol. 5, pp. 203–215.

Mordeson, J.N., Malik, D.S. and Kuroki, N. (2003). *Fuzzy semigroups,* Springer-Verlag, Berlin-Heidelberg-New York.

Muhiuddin,G., Ahn, S. S., S. K., Chang and Jun, Y.B. (2017). Stable Cubic Sets, Journal of Computational Analysis and Applications, Vol. 23, No. 5, pp. 802–819.

Muhiuddin, G., Feng, F. and Jun, Y.B. (2014). Subalgebras of BCK/BCI-Algebras Based on Cubic Soft Sets, The Scientific World Journal, Volume 2014, Article ID 458638, 9 pages.

Muhiuddin, G. and Al-roqi, A. M. (2014). Cubic soft sets with applications in BCK/BCI-algebras, Annals of Fuzzy Mathematics and Informatics, Vol. 8, No. 2, pp. 291–304.

Senapati, T. and Shum, K.P. (2018). Cubic commutative ideals of BCK-algebras, Missouri Journal of Mathematical Sciences, Vol. 30, No. 1, pp. 5–19.

Senapati, T. and Shum, K.P. (2017). Cubic implicative ideals of BCK-algebras, Missouri J. Math. Sci., Vol. 29, No. 2, pp. 125–138.

Senapati, T., Kim, C.S., Bhowmik, M. and Pal, M. (2015). Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy Inf. Eng., Vol. 7, No. 2, pp. 129–149.

Yaqoob, N., Mostafa, S.M. and Ansari, M.A. (2013). On cubic KU-ideals of KU-algebras. ISRN Algebra 2013, doi:10.1155/2013/935905

Zadeh, L.A. (1965). Fuzzy sets, Inform. Control, Vol. 8, pp. 338–353.

Zadeh, L.A. (1975). The concept of a linguistic variable and its application to approximate reasoning I, Inform. Sci., Vol. 8, pp. 199–249.