Solitary and Periodic Exact Solutions Of the Viscosity-capillarity van der Waals Gas Equations

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Abstract

Periodic and soliton solutions are derived for the (1+1)-dimensional van der Waals gas system in the viscosity-capillarity regularization form. The system is handled via the $e^{-\varphi(\xi)}$-expansion method. The obtained solutions have been articulated by the hyperbolic, trigonometric, exponential and rational functions with arbitrary constants. Mathematical analysis and numerical graphs are provided for some solitons, periodic and kink solitary wave solutions to visualize the dynamics of equations. Obtained results reveal that the method is very influential and effective tool for solving nonlinear partial differential equations in applied mathematics.

1. Introduction

Because of the vital role in describing a wide variety of nonlinear wave phenomena in fluid dynamics and mechanics, considerable efforts have been devoted to find exact solutions to the conservation laws modeled by nonlinear partial differential equations (NPDEs). Exact solutions help to understand the complexity of the phenomena, validate the results of numerical analysis and analyze the stability of these equations. Mixed-type systems of conservation laws have been used to model diverse range of physical phenomena from traffic flow to three-phase flow in porous media. For

example, the systems model the dynamical phase transitions in the propagating phase boundaries in solids and the van der Waals fluid (Benzoni-Gavage (1998)).

The purpose of this contribution is to construct an exact solution for the best known mixed hyperbolic-elliptic system of conservation laws, namely the van der Waals gas equations in the viscosity-capillarity regularization form (Jin (1995))

$$\partial_t u + \partial_x p(v) = \eta\, \partial_x^2 u - \omega\, \eta^2 \partial_x^3 u,\ \partial_t v - \partial_x u = 0, \tag{1}$$

where ∂_x^q denotes the qth partial derivative with respect to x , $u(x,t)$ and $v(x,t)$ are the velocity and the volume respectively, while $p(v)$ represents the pressure of gas. The viscosity constant η and ω are assumed to be positive, where $\omega\, \eta^2$ denotes the coefficient of interfacial capillarity.

The p-system in Equation (1), as is also known, describes the one dimensional longitudinal isothermal motion in elastic bars or fluids. The corresponding eigenvalues are $\pm\sqrt{-p(v)}$. For some material models, the system is of mixed hyperbolic-elliptic type since the constitutive pressure function may not be monotone.

Because of the possibility of shocks in the elliptic region, the well-posedness theory of mixed systems did not develop yet as in the case of hyperbolic systems (Lax (1973)). Theoretically, Berres et al. (2009), Keyfitz (2001), Marchesin and Ploh (2001), Fitt (2009), and Holden et al. (1990) discussed mixed-systems and their applications. Recently, many numeric-analytic schemes have been employed to construct approximate solutions to mixed-type models. Some of these attempts were carried out in (Yildirim and Balci (2011), Kumar et al. (2012), Az-Zo'bi (2014), Az-Zo'bi and Al Dawoud (2014), Al-Khaled (2014), Az-Zo'bi (2015a), Az-Zo'bi (2015b), Az-Zo'bi (2015c), Az-Zo'bi et al. (2015), Az-Zo'bi (2018a), Az-Zo'bi (2018b), Az-Zo'bi et al. (2019)). See also the references included therein.

In this paper, the $e^{-\varphi(\xi)}$-expansion method (Zhao and Li (2008)) is considered to obtain some exact traveling wave solutions of the system in Equation (1). This method has been successfully applied for treating the Fitzhugh-Nagumo equation and Modified Liouville equation (Abdelrahman and Khater (2015)), 1D classical Boussinesq equations (Harun-Or-Roshid and Azizur-Rahman (2014)), the combined KdV-mKdV equation (Rayhanul-Islam et al. (2015), Khater (2015)), and nonlinear evolution equations (Abdus-Salam and Umme Habiba (2017), Abdelrahman et al. (2015)).

The outline of this paper is as follows. Methodology of the $e^{-\varphi(\xi)}$-expansion method is described in Section 2. Exact solutions to the van der Waals gas equations, via the proposed Algorithm, are derived in Section 3. Section 4 provides graphical representation of some obtained solutions. Finally, conclusions are included in Section 5.

2.   The $e^{-\varphi(\xi)}$-expansion method

In the current part, an explanation of the $e^{-\varphi(\xi)}$-expansion scheme will be discussed. For this purpose, consider the following (1+1)-nonlinear evolution equation

$$F(v, \partial_t v, \partial_x v, \partial_{tt} v, \partial_{xt} v, \partial_{xx} v, \ldots) = 0, \tag{2}$$

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where F is a polynomial in $v(x, t)$ and its partial derivatives, in which the highest order derivative and nonlinear terms are involved. In what follows, the main steps of the method are listed.

Step 1. Combine the real variables x and t by the wave-variable $\xi = x \pm \alpha t$, where α is the speed of traveling wave. Equation (2) will be reduced to the formal ordinary differential equation (ODE) in $v(\xi)$ and its total derivatives $v', v'', \ldots$,

$$P\left(v, v', v'', v''', \ldots\right) = 0. \tag{3}$$

Integrate Equation (3) as many times as is applicable and set the constants of integration to be zeros.

Step 2. Assume that the solution of Equation (3) can be expressed in the following form

$$v(\xi) = \sum_{i=0}^{m} A_i e^{-i\varphi(\xi)}, \quad A_m \neq 0, \tag{4}$$

where $A_i's$ are constants to be determined. The positive integer m can be obtained by considering the homogeneous balance between the highest order derivative term and nonlinear term in Equation (3). Moreover, given $deg(v(\xi)) = m$, where $deg(.)$ denotes the degree, implies the following degrees for the other expressions:

$$deg\left(\frac{d^n v}{d\xi^n}\right) = m + n, \qquad deg\left(v^p \left(\frac{d^n v}{d\xi^n}\right)^q\right) = m\,p + q\,(n+m).$$

If m is fraction or negative integer, the following transformations are useful:
1. When $m = \frac{a}{b}, b \neq 0$, is a fraction in lowest term, let $v(\xi) = u(\xi)^{\frac{a}{b}}$.
2. When $m = -k$, k is a positive integer, let $v(\xi) = u(\xi)^{-k}$.

Now, the function $\varphi(\xi)$ in Equation (4) satisfies the following ODE,

$$\varphi'(\xi) = e^{-\varphi(\xi)} + \mu\, e^{\varphi(\xi)} + \lambda, \tag{5}$$

where λ and μ are parameters to be determined. Replacing $e^{\varphi(\xi)}$ by $\psi(\xi)$, we get the Riccati equation whose related method is developed by Ma and Fuchssteine (1996). Later, this method is considered as a special case of the transformed rational function method (Ma and Lee (2009)) and the multiple exponential function method (Ma et al. (2010)).

It is well-known that Equation (5) possess the following classes of solution.

Case I (Rational function solution): When $\lambda^2 - 4\mu = 0$, $\mu = \lambda = 0$, then

$$\varphi(\xi) = \ln(\xi + C). \tag{6}$$

Case II (Rational function solution): When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$, then

$$\varphi(\xi) = \ln\left(-\frac{2(\lambda(\xi + C) + 2)}{\lambda^2(\xi + C)}\right). \tag{7}$$

Case III (Exponential function solution): When $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$, then

$$\varphi(\xi) = -\ln\left(\frac{\lambda}{e^{\lambda(\xi + C)} - 1}\right). \tag{8}$$

Case IV (Hyperbolic function solution): When $\lambda^2 - 4\mu > 0$, $\mu = 0$, then

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right), \tag{9}$$

and,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{10}$$

Case V (Trigonometric function solution): When $\lambda^2 - 4\mu < 0$, $\mu = 0$, then

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right), \tag{11}$$

and,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{4\mu - \lambda^2}\mu\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{12}$$

Step 3. Along with Equation (5), substituting Equation (4) into Equation (3) results a polynomial of $e^{-\varphi(\xi)}$. Collect all the terms of same order and equate each coefficient to zero, a system of algebraic equations would be obtained. With the aid of some symbolic computation software, determine the possible values of parameters α, μ, λ and the $A_i's$.

Step 4. Substitute the obtained values into Equation (4) along with Equations (6)-(12) will complete determining the exact solutions of Equation (2).

An extended technique for the $e^{-\varphi(\xi)}$-expansion scheme was presented by Khater (2015) to obtain exact traveling wave solutions for the generalized Hirota-Satsuma couple KdV system. This method assumes the solution of Equation (2) in the form:

$$v(\xi) = \sum_{i=-m}^{m} A_i e^{-\varphi(\xi)}, \ (A_m \text{ or } A_{-m} \neq 0). \tag{13}$$

The rest steps of this extension run as in the case of $e^{-\varphi(\xi)}$-expansion method.

3. Exact solutions for the van der Waals equations

Exertion of the $e^{-\varphi(\xi)}$-expansion method to construct exact analytic solutions for the (1+1)-dimensional van der Waals gas system Equation (1), with constitutive function $p(v) = v - v^3$, will be achieved in this section.

Utilizing the traveling wave variable $\xi = x + \alpha t$, Equation (1) is carried into following ordinary differential system:

$$\alpha u' + (v - v^3)' = \eta u'' - \omega \eta^2 u''', \alpha v' - u' = 0. \tag{14}$$

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Integrating Equation (14) with respect to ξ once, and equating the integration constants to zero yields,

$$\alpha\, u + \left(v - v^3\right) = \eta\, u' - \omega\, \eta^2 u'', \, \alpha\, v - u = 0. \tag{15}$$

Balancing the highest order derivative and nonlinear term appear in u'' and u^3, as well as for v'' and v^3, implies the formal solutions:

$$u\left(\xi\right) = A_0 + A_1 e^{-\varphi(\xi)}, v\left(\xi\right) = B_0 + B_1 e^{-\varphi(\xi)}. \tag{16}$$

Substituting Equation (16) and its derivative into Equation (15), and equating the coefficients, with the same power of $-\varphi\left(\xi\right)$, to zero results the following set of simultaneous algebraic equations:

$$A_1 - \omega B_1 = 0,$$
$$2\alpha^2 \beta\omega - B_1^2 = 0,$$
$$2\alpha\omega + 12\alpha^2\beta\lambda\omega - 6B_0 B_1 - 3\lambda B_1^2 = 0,$$
$$\mu\left(1 + \alpha\lambda\omega + \alpha^2\beta\lambda^2\omega + 2\alpha^2\beta\mu\omega + \omega^2 - 3B_0^2\right) = 0,$$
$$-1 - 3\alpha\lambda\omega - 7\alpha^2\beta\lambda^2\omega - 8\alpha^2\beta\mu\omega - \omega^2 + 3B_0^2 + 6\lambda B_0 B_1 + 3\mu B_1^2 = 0,$$
$$\lambda + \alpha\lambda^2\omega + \alpha^2\beta\lambda^3\omega + 2\alpha\mu\omega + 8\alpha^2\beta\lambda\mu\omega + \lambda\omega^2 - 3\lambda B_0^2 - 6\mu B_0 B_1 = 0.$$

Solving this system using Mathematica, two clusters of solutions as obtained as following:

$$B_1 = \pm\sqrt{2\,\alpha\,\omega}\,\eta, A_1 = \alpha\, B_1, B_0 = \frac{1}{2\,\omega}\left(\frac{1}{3\,\eta} + \lambda\,\omega\right)B_1, \alpha = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 - 144\,\omega^2}}{12\,\omega},$$

where A_0 is an arbitrary, $\gamma_1 = 1 + 3\,\eta^2\omega^2\gamma_2$ and $\gamma_2 = \lambda^2 - 4\mu$. Therefore, exact solutions following the cases in Section 2, with integration constant C, are listed to be the following.

Case I:

$$u\left(x, t\right) = A_0 + \frac{A_1}{x + \alpha\, t + C}, v\left(x, t\right) = B_0 + \frac{B_1}{x + \alpha\, t + C}. \tag{17}$$

Case II:

$$u\left(x, t\right) = A_0 - A_1\frac{\lambda^2\left(x + \alpha\, t + C\right)}{2\left(\lambda\left(x + \alpha\, t + C\right) + 2\right)}, v\left(x, t\right) = B_0 - B_1\frac{\lambda^2\left(x + \alpha\, t + C\right)}{2\left(\lambda\left(x + \alpha\, t + C\right) + 2\right)}. \tag{18}$$

Case III:

$$u\left(x, t\right) = A_0 + A_1\frac{\lambda}{e^{\lambda(x + \alpha\, t + C)} - 1}, u\left(x, t\right) = B_0 + B_1\frac{\lambda}{e^{\lambda(x + \alpha\, t + C)} - 1}. \tag{19}$$

Case IV:

$$u\left(x, t\right) = A_0 - A_1\frac{2\,\mu}{\sqrt{\gamma_2}\tanh\left(\frac{\sqrt{\gamma_2}}{2}(x + \alpha\, t + C)\right) + \lambda},$$
$$v\left(x, t\right) = B_0 - B_1\frac{2\,\mu}{\sqrt{\gamma_2}\tanh\left(\frac{\sqrt{\gamma_2}}{2}(x + \alpha\, t + C)\right) + \lambda}. \tag{20}$$

Case IV:

$$u\left(x, t\right) = A_0 + A_1\frac{2\,\mu}{\sqrt{-\gamma_2}\tan\left(\frac{\sqrt{-\gamma_2}}{2}(x + \alpha\, t + C)\right) - \lambda},$$
$$v\left(x, t\right) = B_0 + B_1\frac{2\,\mu}{\sqrt{-\gamma_2}\tan\left(\frac{\sqrt{-\gamma_2}}{2}(x + \alpha\, t + C)\right) - \lambda}. \tag{21}$$
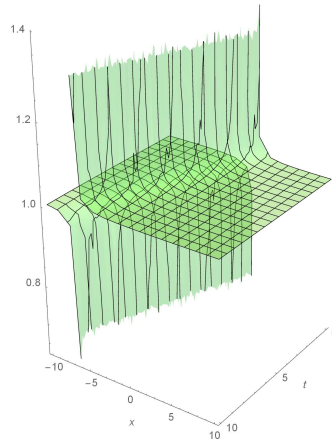
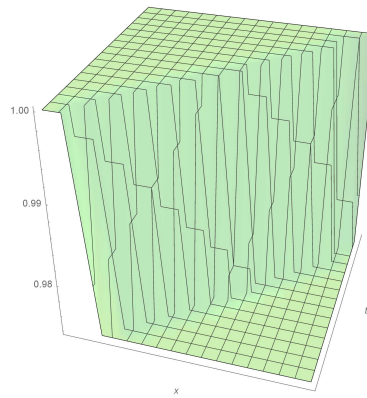**Figure 1.** Singular kink-shaped soliton profile of the rational velocity $u(x,t)$ in Equation (18).

Figure 2. Kink-shaped soliton profile of the exponential velocity $u(x,t)$ in Equation (19).

As an alternative procedure, the proposed scheme can be successfully implemented by introducing the following transformation,

$$v(x,t) = a\,u(x,t) + b, \tag{22}$$

where a and b are constants to be determined, will reduce the p-system in Equation (1) to the NPDE,

$$\left(2 - 3(a + b\,u)^2\right)\partial_t u = \eta\,\partial_{xx}u - \omega\,\eta^2\partial_{xxx}u. \tag{23}$$

Proceeding as before, the same exact solutions in Equations (17)-(21) are obtained.

Also, components of the van der Waals p-system Equation (1) can be merged to get

$$\partial_t^2 v + \left(1 - 3v^2\right)\partial_x^2 v - 6\,v(\partial_x v)^2 = \eta\,\partial_x^2\partial_t v - \omega\,\eta^2\partial_x^4 v. \tag{24}$$

Following the solution steps of the $e^{-\varphi(\xi)}$-expansion algorithm for $v(x,t)$, taking into account that $u(x,t) = \partial_x^{-1}\partial_t v(x,t)$, where $\partial_x^{-1}(.) = \int(.)\,dx$, would present the same results as before.

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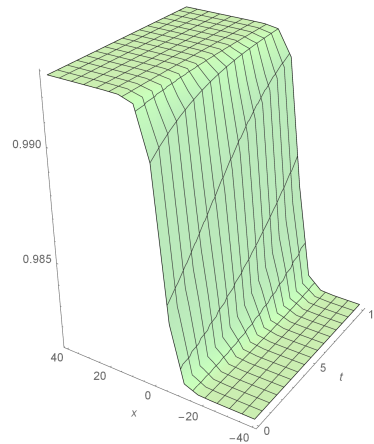
Figure 3. Kink-shaped soliton profile of the hyperbolic velocity $u(x,t)$ in Equation (20).
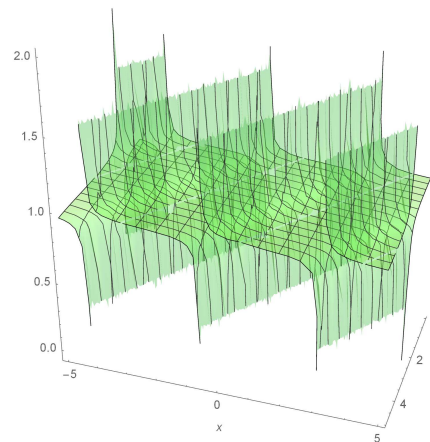

Figure 4. Periodic solution profile of velocity $u(x,t)$ in Equation (21).

4.   Illustration of some exact solutions

As a helping tool, the Mathematica 11 software package is used to illustrate three-dimensional plots for some of investigated solutions. Different profiles of solitons, singular solitons, and periodic solutions to the velocity of gas $u(x,t)$ are shown to visualize the underlying dynamics of the van der Waals system. With $\omega = 0.08$, $\eta = 0.1$, and unity for the other nonzero free parameters, Figure 1 shows the obtained solution in Equation (18). Kink-type solutions derived in Equations (19) and (20) are plotted in Figures 2 and 3 respectively. The trigonometric solution in Equation (21) is represented in Figure 4. Graphs of the volume $v(x,t)$ can be obtained by stretching or compressing, and shifting the obtained figures of the velocity as a result of the linearity in Equation (22).

5.   Conclusion

In this work, some new solitary wave solutions of the 1D van der Waals gas equations are successfully derived via the $e^{-\varphi(\xi)}$-expansion method. We have obtained four types of explicit function

solutions, namely rational, exponential, trigonometric and hyperbolic function solutions. All solutions have been checked by putting them back into the original equations. It can be concluded that the proposed method reduces the size of computational work compared to other existing techniques. This method is practically suitable for wide range of nonlinear evolution equations that arise in mathematical physics and engineering fields.

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