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An Initial Value Technique using Exponentially Fitted Non Standard Finite Difference Method for Singularly Perturbed Differential-Difference Equations

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Abstract

In this paper, an exponentially fitted non standard finite difference method is proposed to solve singularly perturbed differential-difference equations with boundary layer on left and right sides of the interval. In this method, the original second order differential difference equation is replaced by an asymptotically equivalent singularly perturbed problem and in turn the problem is replaced by an asymptotically equivalent first order problem. This initial value problem is solve by using exponential fitting with non standard finite differences. To validate the applicability of the method, several model examples have been solved by taking different values for the delay parameter δ, advanced parameter η and the perturbation parameter ε. Comparison of the results is shown to justify the method. The effect of the small shifts on the boundary layer solutions has been investigated and presented in figures. The convergence of the scheme has also been investigated.

Keywords: Singularly perturbed differential-difference equations; Boundary Layer; Non standard finite difference

MSC 2010: 65L11, 65Q10

1. Introduction

Singularly Perturbed Differential-difference equations (SPDDEs), also called as a class of functional differential equations, are mathematical models of a number of real phenomenon. Their applications permeate all branches of contemporary sciences such as engineering, physics, economics, biomechanics, and evolutionary biology given in Bellman and Cooke (1963), Mackey (1977) and Kolmanovskii and Myshkis (1992). The study of bistable devices as in Derstine et al. (1982), the description of human pupil-light reflex as in Longtin and Milton (1988), the first exit time problem in the modeling of the activation of neuronal variability as in Stein (1965), the study of a variety of models for physiological processes or diseases as in Longtin and Milton (1988) and Wazewska and Lasota (1976), the study of dynamic systems with time delays which arise in neural networks as in Tuckwell and Ricther (1978) are some examples involving this type of singularly perturbed differential-difference equations.

Lange and Miura (1982, 1985, 1985, 1994, 1994) have made detailed discussion on the solutions of SPDDEs exhibiting rapid oscillations, resonance behaviour, turning point behaviour and boundary and interior layer behaviour. Kadalbajoo and Sharma (2004) have proposed a method consisting the standard upwind finite difference operator on a special type of mesh. Sharma and Kaushik (2006) have discussed a linear problem for which one cannot construct a parameter uniform scheme based on fitted operator approach but for the same problem a parameter uniform numerical scheme based fitted mesh approach can be constructed. Kadalbajoo and Sharma (2004, 2005) have described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. Essam et al. (2013) have proposed a new initial value method for solving a class of nonlinear singularly perturbed boundary value problems with a boundary layer at one end.

Kadalbajoo and Reddy (1987) have derived an initial-value technique, which is simple to use and easy to implement, for a class of nonlinear, singularly perturbed two-point boundary-value problems with a boundary layer on the left end of the underlying interval. Kumara et al. (2016) have presented Galerkin method to solve singularly perturbed differential-difference equations with delay and advanced shifts using fitting factor. A fitting factor in the Galerkin scheme is introduced which takes care of the rapid changes that occur in the boundary layer. Mirzaee and Hoseini (2013) have introduced a method to solve singularly perturbed differential-difference equations of mixed type, in terms of Fibonacci polynomials. Pratima and Sharma (2011) have presented a numerical study of boundary value problems for singularly perturbed linear second-order differential–difference equations with a turning point. Kadalbajoo and Sharma (2002) have proposed a numerical method to solve boundary-value problems for a singularly-perturbed differential-difference equation of mixed type. Reddy and Awoke (2013) have described an appropach for solving singularly perturbed differential difference equations via fitted method.

In this paper, an exponentially fitted non standard finite difference method is proposed to solve singularly perturbed differential-difference equations with boundary layer on left and right sides of the interval. In this method, the original second order differential difference equation is replaced by an asymptotically equivalent singularly perturbed problem and in turn the problem is replaced by an asymptotically equivalent first order problem. This initial value problem is solve by using exponential fitting with non standard finite differences. To validate the applicability of the method, several model examples have been solved by taking different values for the delay parameter δ, advanced parameter η and the perturbation parameter ε. Comparison of the results is shown to justify the method. The effect of the small shifts on the boundary layer solutions has been investigated and presented in figures. The convergence of the scheme has also been investigated.

2.1. Numerical scheme

Consider singularly perturbed differential difference equation of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+d(x)y(x)+c(x)y(x+\eta)=h(x); \quad \forall x\in(0,1), \quad (1)$$

and subject to the boundary conditions

$$y(x)=\varphi(x) \quad on \quad -\delta\le x\le 0, \quad (2)$$

$$y(x)=\gamma(x) \quad on \quad 1\le x\le 1+\eta, \quad (3)$$

where $a(x)$, $b(x)$, $c(x)$, $d(x)$, $h(x)$, $\phi(x)$ and $\gamma(x)$ are bounded and continuously differentiable functions on (0, 1), $0<\varepsilon<<1$ is the singular perturbation parameter; and $0<\delta=o(\varepsilon)$ and $0<\eta=o(\varepsilon)$ are the delay and the advance parameters respectively. In general, the solution of problem: Equations (1)-(3) exhibits the boundary layer behavior of width $O(\varepsilon)$ for small values of ε.

By using Taylor series expansion in the vicinity of the point x, we have,

$$y(x-\delta)\approx y(x)-\delta y'(x), \quad (4)$$

and

$$y(x+\eta)\approx y(x)+\eta y'(x). \quad (5)$$

Using Equations (4) and (5) in Equation (1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x)+p(x)y'(x)+Q(x)y(x)=h(x), \quad (6)$$

with

$$y(0)=\phi(0)=\phi, \quad (7)$$

$$y(1)=\gamma(1)=\gamma, \quad (8)$$

where,

$$p(x)=a(x)+c(x)\eta-b(x)\delta, \quad (9)$$

and

$$Q(x)=b(x)+c(x)+d(x). \tag{10}$$

The transition from Equations (1)-(3) to Equations (6)-(8) is admitted, because of the condition that $0<\delta<<1$ and $0<\eta<<1$ are sufficiently small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Els'golts and Norkin (1973). Thus, the solution of Equations (6)-(8) will provide a good approximation to the solution of Equations (1)-(3).

2.2. Left End Boundary Layer Problems

We assume that

$$Q(x)=b(x)+c(x)+d(x)\le 0,$$

and

$$p(x)=a(x)+c(x)\eta-b(x)\delta\ge M>0,$$

throughout the interval [0, 1], where M is some constant. Further we assume that Equations (6)-(8) has a unique solution $y(x)$ which exhibits a boundary layer of width O(ε) on the left side of the interval, i.e., at x=0. For convenience, we shall write Equation (6) as follows:

$$\varepsilon y''(x)+\left(p(x)y(x)\right)'+q(x)y(x)=h(x), \tag{11}$$

where

$$q(x)=Q(x)-p'(x),$$

with

$$y(0)=\phi(0)=\phi;\quad y(1)=\gamma(1)=\gamma. \tag{12}$$

The initial value method consists of the following steps:

Step 1: Obtain the reduced problem by setting $\varepsilon=0$ in Equation (11) and solve it for the solution to the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of Equation (11), that is,

$$\left(p(x)y_0(x)\right)'+q(x)y_0(x)=h(x), \tag{13}$$

with

$$y_0(1)=\gamma. \tag{14}$$

Step 2: Set up the approximate equation to the Equation (11) as O(ε) as follows:

$$\varepsilon y''(x)+\left(p(x)y(x)\right)'+q(x)y_0(x)=h(x), \tag{15}$$

with

$$y(0)=\phi(0)=\phi; \quad y(1)=\gamma(1)=\gamma, \tag{16}$$

where the term $y(x)$ is replaced by the solution of reduced problem: Equations (13)-(14).

Step 3: Replace the approximated second order problem: Equations (15)-(16) by an asymptotically equivalent first order problem as follows. By integrating Equation (15), we obtain:

$$\varepsilon y'(x)+p(x)y(x)=f(x)+K, \tag{17}$$

where

$$f(x)=\int H(x)dx; \quad H(x)=h(x)-q(x)y_0(x), \tag{18}$$

and K is an integrating constant to be determined.

In order to determine the constant K, we introduce the condition that the reduced equation of Equation (17) should satisfy the boundary condition,

$$y(1)=\gamma,$$

i.e.,

$$p(1)y(1)=f(1)+K.$$

Therefore,

$$K=p(1)\gamma-f(1).$$

Thus, we have replaced the original second order problem: Equations (11)-(12), which is in turn a good approximation to Equations (1)-(3), with an asymptotically equivalent first order problem; Equation (17) with $y(0)=\phi$. We solve this initial value problem to obtain the solutions over the interval $0\le x\le 1$.

In order to solve the initial value problems in our numerical experimentation, we have used the exponential fitting non standard finite difference technique. We consider the exponentially fitted non standard finite difference scheme on Equation (17) given by

$$\varepsilon\sigma(\rho)\tilde{D}_0 y_i+p_i\left(\frac{y_{i-1}+y_i+y_{i+1}}{3}\right)=f_i \quad with\ y_0=\phi, \tag{19}$$

where

$$\tilde{D}_0 y_i\approx\frac{-3y_{i-1}+4y_i-y_{i+1}}{2h}+O(h^2),$$

$$y_i=\frac{y_{i-1}+y_i+y_{i+1}}{3}+O(h^2),$$

$$\rho = \frac{h}{\varepsilon},$$

and σ is a fitting factor which is to be determined in such a way that solves of Equation (19) converges uniformly to the solution of Equation (17). The value of σ is given by

$$\sigma_i(\rho) = \frac{2\rho p_i\left(1+e^{p_i\rho}+e^{-p_i\rho}\right)}{3\left(-4+3e^{p_i\rho}+e^{-p_i\rho}\right)}. \tag{20}$$

Now from Equation (19), we have,

$$\left(\frac{-3\sigma_i\varepsilon}{2h}+\frac{p_i}{3}\right)y_{i-1}-\left(\frac{-4\sigma_i\varepsilon}{2h}-\frac{p_i}{3}\right)y_i+\left(\frac{-\sigma_i\varepsilon}{2h}+\frac{p_i}{3}\right)y_{i+1} = f_i \ ; \quad for \quad i = 1, 2, \ldots, N-1. \tag{21}$$

This is a three term recurrence relation along with the boundary conditions given in the Equation (12), hence, can be solved by the Thomas Algorithm.

2.3. Right End Boundary Layer Problems

We now extend this technique to the right end boundary layer problems on the underlying interval. Consider Equation (6) rewritten as follows, for convenience:

$$\varepsilon y''(x)+\left(p(x)y(x)\right)'+q(x)y(x)=h(x), \tag{22}$$

with

$$y(0)=\phi(0)=\phi \ \text{ and } \ y(1)=\gamma(1)=\gamma, \tag{23}$$

where

$$p(x)=a(x)+c(x)\eta-b(x)\delta,$$
$$q(x)=Q(x)-p'(x),$$

and

$$Q(x)=b(x)+c(x)+d(x).$$

Further, assume that,

$$p(x)=a(x)+c(x)\eta-b(x)\delta \le M<0,$$

throughout the interval [0,1], where M is a constant. This assumption implies that the boundary layer will be in the neighborhood of $x=1$.

Step 1. Obtain the reduced problem by setting $\varepsilon=0$ in Equation (22) and solve it for the solution with the appropriate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of Equation (22), that is,

$$\left(p(x)y_0(x)\right)' + q(x)y_0(x) = h(x), \tag{24}$$

with

$$y_0(0) = \phi. \tag{25}$$

Step 2. Set up the approximate equation to the Equation (22) as O(ε) as follows:

$$\varepsilon y''(x) + \left(p(x)y(x)\right)' + q(x)y_0(x) = h(x), \tag{26}$$

with

$$y(0) = \phi(0) = \phi,$$

and

$$y(1) = \gamma(1) = \gamma,$$

where the term $y(x)$ is replaced by $y_0(x)$, the solution of the reduced problem of Equations (22)-(23).

Step 3. Replace the approximated second order problem Equation (26) by an asymptotically equivalent first order problem as follows. By integrating Equation (26), we obtain:

$$\varepsilon y'(x) + p(x)y(x) = f(x) + K, \tag{27}$$

where

$$f(x) = \int H(x)dx; \quad H(x) = h(x) - q(x)y_0(x), \tag{28}$$

and K is an integrating constant to be determined. In order to determine the constant K, we introduce the condition that the reduced equation of Equation (27) should satisfy the boundary condition,

$$y(0) = \phi,$$

i.e.,

$$p(0)y(0) = f(0) + K.$$

Therefore,

$$K = p(0)\phi - f(0).$$

Thus, we have replaced the original second order problem (11)-(12), which is in turn a good approximation to (1)-(3), with an asymptotically equivalent first order problem Equation (27) with $y(1) = \gamma$.

We solve this initial value problem; Equation (27) with $y(1) = \gamma$ to obtain the solutions over the interval $0 \le x \le 1$. We consider the exponentially fitted non standard finite difference scheme on Equation (27) given by

$$\varepsilon\sigma(\rho)\tilde{D}^{0} y_i + p_i\left(\frac{y_{i-1}+y_i+y_{i+1}}{3}\right) = f_i \quad with \;\; y(1)=\gamma, \tag{29}$$

where

$$\tilde{D}^{0} y_i \approx \frac{y_{i-1}-4y_i+3y_{i+1}}{2h} + O(h^2),$$

$$y_i = \frac{y_{i-1}+y_i+y_{i+1}}{3} + O(h^2),$$

$$\rho = \frac{h}{\varepsilon},$$

and σ is a fitting factor which is to be determined in such a way that solves of Equation (29) converges uniformly to the solution of Equation (27).

The value of σ is given by

$$\sigma_i(\rho) = \frac{2\rho p_i\left(1+e^{p_i\rho}+e^{-p_i\rho}\right)}{3\left(4-e^{p_i\rho}-3e^{-p_i\rho}\right)}. \tag{30}$$

Now, from Equation (29), we have

$$\left(\frac{\sigma_i\varepsilon}{2h}+\frac{p_i}{3}\right)y_{i-1} - \left(\frac{4\sigma_i\varepsilon}{2h}-\frac{p_i}{3}\right)y_i + \left(\frac{3\sigma_i\varepsilon}{2h}+\frac{p_i}{3}\right)y_{i+1} = f_i\,; \quad for\; i = 1, 2, \ldots, N-1. \tag{31}$$

This is a three term recurrence relation along with the boundary conditions given in the Equation (23), hence, can be solved by the Thomas Algorithm.

3. Convergence analysis

Now we consider the convergence analysis of left end boundary layer described in section 2.2 for the problem Equations (1)-(2). Incorporating the boundary conditions we obtain the system of equations in the matrix form as

$$AY + Q + T(h) = 0, \tag{32}$$

in which,

$$A = \left(m_{i,j}\right), \quad 1 \le i, j \le N-1,$$

is a tridiagonal matrix of order N-1, with,

$$a_{i,i-1} = \frac{-3\sigma\varepsilon}{2} + \frac{hp_i}{3},$$

$$a_{i,i} = \frac{4\sigma\varepsilon}{2} + \frac{hp_i}{3},$$

$$a_{i,i+1} = \frac{-\sigma\varepsilon}{2} + \frac{hp_i}{3},$$

$$Q = -hf_i \quad ; \quad T_i(h) = O(h^2),$$

and

$$Y = \left[Y_1, Y_2, ..., Y_{N-1}\right]^T, T(h) = \left[T_1, T_2, ..., T_{N-1}\right]^T, O = \left[0, 0, ..., 0\right]^T,$$

are associated vectors of Equation (32).

Let

$$y = \left[y_1, y_2, ..., y_{N-1}\right]^T \cong Y,$$

which satisfies the equation

$$Ay + Q = 0. \tag{33}$$

Let

$$e_i = y_i - Y_i \text{ , for } i = 1, 2, ..., N-1,$$

be the discretization error so that

$$E = \left[e_1, e_2, ..., e_{N-1}\right]^T = y - Y.$$

Subtracting Equation (32) from Equation (33), we get the error equation

$$AE = T(h). \tag{34}$$

Let $\bar{S}_i$ be the sum of the elements of the ith row of the matrix A, then we have

$$S_i = \frac{3\sigma\varepsilon}{2} + \frac{2hp_i}{3} \text{ , for } i = 1,$$

$$S_i = hp_i \text{ , for } i = 2, 3, ..., N-2,$$

$$S_i = \frac{\sigma\varepsilon}{2} + \frac{2hp_i}{3} \text{ , for } i = N-1.$$

We can choosen h sufficiently small so that the matrix A is irreducible and monotone. It follows that A^{-1} exists. Hence from Equation (34), we have

$$\|E\| \le \left\|(A)^{-1}\right\| \quad \|T\|. \tag{35}$$

Also from theory of matrices, we have

$$\sum_{k=1}^{N-1}(A)_{i,k}^{-1}.\bar{S}_k = 1, \text{ for } i = 1, 2, ..., N-1. \tag{36}$$

Let

$$A_{i,k}^{-1} \text{ be the } (i,k)^{th} \quad \text{element of } \; A^{-1},$$

and

$$C_1 = \max|p(x_i)|.$$

We define

$$\left\|(A)^{-1}\right\| = \max_{1\le i\le N-1}\sum_{k=1}^{N-1}(A)_{i,k}^{-1} \text{ and } \|T(h)\| = \max_{1\le i\le N-1}|T(h_i)|. \tag{37}$$

Hence,

$$(A)_{i,1}^{-1} \le \frac{1}{\bar{S}_1} < \frac{3}{2hC_1}, \tag{38}$$

$$(A)_{i,N-1}^{-1} \le \frac{1}{\bar{S}_{N-1}} < \frac{3}{2hC_1}, \tag{39}$$

Furthermore,

$$\sum_{k=2}^{N-2}(A)_{i,k}^{-1} \le \frac{1}{\min\limits_{2\le k\le N-2} \bar{S}_k} < \frac{1}{h\mathrm{C}_1}\ ;\ i = 2, 3, ..., N-2. \tag{40}$$

By the help of Equations (38) - (40), using Equation (35), we obtain,

$$\|E\| \le O(h). \tag{41}$$

Hence, the proposed method is first order convergent. Similarly the convergence analysis for right end layer can be obtained in the same lines.

4. Numerical Examples

To demonstrate the applicability of the method we have applied it to six problems of the type given by Equations (1) - (3) with left layer. Our numerical solution is compared with the ‘Upwind Method’ which is supposed to be better than classical second oreder method.

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+d(x)y\left(x\right)+c(x)y\left(x+\eta\right)=h(x).$$

The exact solution of such boundary value problems having constant coefficients (i.e. $a(x)=a,$ $\mathrm{b}\left(x\right)=b$ $c\left(x\right)=c,$ $\mathrm{d}\left(x\right)=d$ $h\left(x\right)=h,$ $\phi(x)=\phi$ and $\gamma(x)=\gamma$ are constants) is given by:

$$y\left(x\right)=c_1e^{m_1x}+c_2e^{m_2x}+(h/c),$$

where

$$c_1=\frac{\left[-h+\gamma c_3+\mathrm{e}^{m_2}(h-\phi c_3)\right]}{\left[\left(\mathrm{e}^{m_1}-\mathrm{e}^{m_2}\right)c_3\right]},\quad c_2=\frac{\left[h-\gamma c_3+\mathrm{e}^{m_1}(-h+\phi c_3)\right]}{\left[\left(\mathrm{e}^{m_1}-\mathrm{e}^{m_2}\right)c_3\right]},$$

$$m_1=\frac{\left[-\left(a-b\delta+c\eta\right)+\sqrt{\left(a-b\delta+c\eta\right)^2-4\varepsilon c_3}\right]}{2\varepsilon},$$

$$m_2=\frac{\left[-\left(a-b\delta+c\eta\right)-\sqrt{\left(a-b\delta+c\eta\right)^2-4\varepsilon c_3}\right]}{2\varepsilon},$$

$$c_3=(b+c+d).$$

Example 1.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at the left end with a(x)=1, $\mathrm{b}\left(x\right)=2,$ $\mathrm{c}\left(x\right)=0,$ $\mathrm{d}\left(x\right)=-3,$ $h\left(x\right)=0,$ $\phi\left(x\right)=1,$ $\gamma\left(x\right)=1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{x-1}{p}}\quad with\quad y(0)=1,$$

are presented in Tables 1 and 2. The effect of the small parameters on the boundary layer solutions is shown in the Figures 1 and 2.

Example 2.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at the left end with a(x)=1, $\mathrm{b}\left(x\right)=0,$ $\mathrm{c}\left(x\right)=2,$ $\mathrm{d}\left(x\right)=-3,$ $h\left(x\right)=0,$ $\phi\left(x\right)=1,$ $\gamma\left(x\right)=1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{x-1}{p}}\quad with\quad y(0)=1,$$

are presented in Tables 3 and 4. The effect of the small parameters on the boundary layer solutions is shown in the Figures 3 and 4.

Example 3.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at the left end with a(x)=1, $b(x)=-2$, $c(x)=1$, $d(x)=-5$, $h(x)=0$, $\phi(x)=1$, $\gamma(x)=1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{6(x-1)}{p}} \quad with \quad y(0)=1,$$

are presented in Tables 5 to 8. The effect of the small parameters on the boundary layer solutions is shown in the Figures 5 and 6.

Example 4.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at right end with a(x)=-1, $b(x)=-2$, $c(x)=0$, $d(x)=1$, $h(x)=0$, $\phi(x)=1$, $\gamma(x)=-1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{x}{p}} \quad with \quad y(1)=-1,$$

are presented in Tables 9 and 10. The effect of the small parameters on the boundary layer solutions is shown in the Figures 7 and 8.

Example 5.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at right end with a(x)=-1, $b(x)=0$, $c(x)=-2$, $d(x)=1$, $h(x)=0$, $\phi(x)=1$, $\gamma(x)=-1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{x}{p}} \quad with \quad y(1)=-1,$$

are presented in Tables 11 and 12. The effect of the small parameters on the boundary layer solutions is shown in the Figures 9 and 10.

Example 6.

Numerical results for model problem given by Equations (1)-(3) having boundary layer at right end with a(x)=-1, $b(x)=-2$, $c(x)=-2$, $d(x)=1$, $h(x)=0$, $\phi(x)=1$, $\gamma(x)=-1$ and the resulting initial value problem:

$$\varepsilon y'(x)+py(x)=pe^{\frac{3x}{p}} \quad with \quad y(1)=-1,$$

are presented in Tables 13 to 16. The effect of the small parameters on the boundary layer solutions is shown in the Figures 11 and 12.

5. Conclusions

In general, obtaining the numerical solution of a boundary value problem is difficult than that of the corresponding initial value problem. Hence, we always prefer to convert the second order problem into a first order problem. This technique provides an alternative technique to conventional approaches of converting the second order problems into first order problems. We have implemented the present method on three examples for left layer and three examples for right layer, by taking different values for the delay parameter δ, advanced parameter η and the perturbation parameter ε. To solve these initial value problems, we have used the exponentially fitted method to get a three-term recurrence relation which can be solved by the Thomas Algorithm. We have tabulated the computational results obtained by the proposed technique.

It can be observed from the tables and the figures that the present method approximates the exact solution very well. The effect of delay and advanced parameters on the solutions of the problem has been investigated and plotted in figures. When the solution of the SPDDEs exhibits layer on the left side, it is observed that the effect of delay or advanced parameters on the solution in the boundary layer region is negligible, while in the outer region is considerable. The change in the advanced term affects the solution in the similar manner as the change in delay affects but reversely in (Figures 1-5). Also, when the SPDDEs exhibit layer behavior on the right side, the changes in delay or advanced parameters affect the solution in boundary layer region as well as outer region. The thickness of the layer increases as the size of delay parameter increases while it decreases as the size of advanced parameter increases (Figures 6-10). The present method is independent of perturbation parameter, also is simple and easy technique for solving singularly perturbed differential difference equations. Our method provides an alternative technique for solving singularly perturbed differential difference problems.

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REFERENCES

Bellman, R. and Cooke K. L. (1963). *Differential-Difference Equations*. Academic Press, New York.

Derstine, M. W, Gibbs, and Kaplan, D. L. (1982). Bifurcation gap in a hybrid optical system, Phys. Rev. A, 26 , 3720–3722

Els'golts, L. E. and Norkin, S. B. (1973). *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press

Essam R, El-Zahar, Saber M. M. and EL-Kabeir. (2013). A new method for solving singularly perturbed boundary value problems, Applied Mathematics and Information Sciences, 7, No. 3, 927-938

Farshid Mirzaee and Seyede Fatemeh Hoseini. (2013). Solving singularly perturbed differential-difference equations arising in science and engineering with Fibonacci polynomials, *Results in Physics* 3, 134–141

Kadalbajoo M. K. and Reddy Y. N. (1987). Initial-value technique for a class of non-linear singular perturbation problems, Journal of Optimization Theory and Applications, 53, 395-406

Kadalbajoo, M. K. and Sharma, K. K. (2004). ε -Uniform fitted mesh methods for singular perturbed differential difference equations: mixed type of shifts with layer behavior, Int. J. Comput. Math., 81(1), 49-62

Kadalbajoo, M. K. and Sharma, K. K. (2004). Parameter-Uniform numerical method for a boundary value problem for singularly perturbed nonlinear delay differential equation of neutral type, Int. J. Comput. Math., 81(7), 845-862

Kadalbajoo, M. K. and Sharma, K. K. (2005). Numerical treatment of a mathematical model arising from a model of Neuronal variability, J. Math. Anal. Appl., 307 (2), 606-627

Kadalbajoo, M. K. and Sharma, K.K. (2002). Numerical Analysis of Boundary-Value Problems for Singularly-Perturbed Differential-Difference Equations with Small Shifts of Mixed Type, Journal of Optimization Theory and Applications, 115 (1), 145–163

Kolmanovskii, V. and Myshkis, A. (1992). *Applied Theory of Functional Differential Equations, volume 85 of Mathematics and its Applications (Soviet Series),* Kluwer Academic Publishers Group, Dordrecht

Kumara Swamy, D., Phaneendra K. and Reddy, Y. N. (2016). Solution of Singularly Perturbed Differential-Difference Equations with Mixed Shifts Using Galerkin Method with Exponential Fitting, Chinese Journal of Mathematics, ID 1935853

Lange, C. G. and Miura, R. M. (1985). Singular Perturbation Analysis of Boundary Value Problems for Differential-Difference Equations II. Rapid Oscillations and Resonances, SIAM Journal on Applied Maths., 45, 687-707

Lange, C. G. and Miura, R. M. (1985). Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations III. Turning point problems, SIAM Journal on Applied Mathematics, 45, 708-734

Lange, C. G. and Miura, R. M. (1994). Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations V. Small shifts with layer behaviour, SIAM Journal on Applied Mathematics, 54, 249-272

Lange, C. G. and Miura, R. M. (1994). Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations VI. Small shifts with rapid oscillations, SIAM Journal on Applied Mathematics, 54, 273-283

Lange, C. G. and Miura, R. M. (1982). Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations, SIAM Journal on Applied Mathematics, 54, 502-531

Longtin, A. and Milton, J. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Math. Biosci. 90, 183–199

Mackey, M. C. (1977): Oscillations and chaos in physiological control systems, Science,197, 287–289

Pratima Rai, and Sharma, K. K. (2011). Parameter uniform numerical method for singularlyperturbed differential–difference equations with interior layers, International Journal of Computer Mathematics, Volume 88, Issue 16

Reddy, Y. N. and Awoke A. T. (2013). Solving Singularly Perturbed Differential Difference Equations via Fitted Method; Application and Applied Mathematics: An International Journal, Vol: 8, No:1, 318-332

Sharma, K. K. and Kaushik, A. (2006). A solution of the discrepancy occurs due to using the fitted mesh approach rather than to the fitted operator for solving singularly perturbed differential equations, Appl. Math. Comput., 181, 756-766

Stein, R. B. (1965). A theoretical analysis of neuronal variability, Biophys.J., 5, 173-194

Tuckwell, H. C. and Ricther W. (1978). Neuronal interspike time distributions and the estimation of neurophysiological and neuroanatomical parameters, J. Theor. Biol., 71, 167-183

Wazewska, M-Czyzewska, and Lasota, A. (1976). Mathematical models of the red cell system, Mat. Stos., 6, 25–40.

Table 1: Maximum absolute errors of Example 1 for $\varepsilon=10^{-3}$

δ / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.002853	1.85e+094	0.00067097	0.232912	0.00091355	0.012377
0.3ε	0.002853	1.74e+094	0.00067097	0.232753	0.00091355	0.012373
0.6ε	0.002853	1.63e+094	0.00067097	0.232594	0.00091355	0.012370
0.9ε	0.002853	1.53e+094	0.00067097	0.232436	0.00091355	0.012367

Table 2: Maximum absolute errors of Example 1 for $\varepsilon = 10^{-5}$

δ / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00028558	2.8573e-05	6.7186e-06
0.3ε	0.00028558	2.8573e-05	6.7186e-06
0.6ε	0.00028558	2.8573e-05	6.7186e-06
0.9ε	0.00028558	2.8573e-05	6.7186e-06

Table 3: Maximum absolute errors of Example 2 for $\varepsilon = 10^{-3}$

η / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.002853	1.86e+094	0.00067097	0.232912	0.00091355	0.012377
0.3ε	0.0028513	1.99+094	0.00067013	0.233071	0.00091241	0.012380
0.6ε	0.0028496	2.12e+094	0.00066929	0.233229	0.00091128	0.012383
0.9ε	0.0028479	2.27e+094	0.00066846	0.233388	0.00091014	0.012387

Table 4: Maximum absolute errors of Example 2 for $\varepsilon = 10^{-5}$:

η / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00028558	2.8573e-05	6.7186e-06
0.3ε	0.00028558	2.8573e-05	6.7186e-06
0.6ε	0.00028558	2.8573e-05	6.7186e-06
0.9ε	0.00028558	2.8573e-05	6.7186e-06

Table 5: Maximum absolute errors of Example 3 for $\eta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$:

δ / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.016966	3.54e+094	0.0039948	0.365172	0.005446	0.017050
0.3ε	0.016966	3.22e+094	0.0039948	0.365550	0.005446	0.017064
0.6ε	0.016966	3.54e+094	0.0039948	0.365928	0.005446	0.017078
0.9ε	0.016966	3.79e+094	0.0039948	0.366306	0.005446	0.017092

Table 6: Maximum absolute errors of Example 3 for η =0.5 ε and ε =10^{-5}:

δ / h	10^{-3}	10^{-4}	10^{-5}
0.0 ε	0.0017095	0.00017143	4.0309e-05
0.3 ε	0.0017095	0.00017143	4.0309e-05
0.6 ε	0.0017095	0.00017143	4.0309e-05
0.9 ε	0.0017095	0.00017143	4.0309e-05

Table 7: Maximum absolute errors of Example 3 for δ =0.5 ε and ε =10^{-3}:

η / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0 ε	0.016975	3.54e+094	0.003999	0.365172	0.0054517	0.017050
0.3 ε	0.01697	3.22e+094	0.0039965	0.365550	0.0054483	0.017064
0.6 ε	0.016965	3.54e+094	0.003994	0.365928	0.0054449	0.017078
0.9 ε	0.016959	3.79e+094	0.0039915	0.366306	0.0054415	0.017092

Table 8: Maximum absolute errors of Example 3 for δ =0.5 ε and ε =10^{-5}:

η / h	10^{-3}	10^{-4}	10^{-5}
0.0 ε	0.0017095	0.00017143	4.0309e-05
0.3 ε	0.0017095	0.00017143	4.0309e-05
0.6 ε	0.0017095	0.00017143	4.0309e-05
0.9 ε	0.0017095	0.00017143	4.0309e-05

Table 9: Maximum absolute errors of Example 4 for ε =10^{-3}

δ / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0 ε	0.002853	3.6323e+095	0.00067097	0.0009995	0.00091355	0.00099905
0.3 ε	0.0028547	3.3974e+095	0.00067181	0.0010007	0.00091469	0.00100020
0.6 ε	0.0028564	3.3974e+095	0.00067265	0.0010019	0.00091583	0.00100140
0.9 ε	0.0028582	2.9718e+095	0.0006735	0.0010031	0.00091698	0.00100265

Table 10: Maximum absolute errors of Example 4 for $\varepsilon = 10^{-5}$

δ / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00028558	2.8573e-05	6.1482e-06
0.3ε	0.00028558	2.8573e-05	6.1482e-06
0.6ε	0.00028558	2.8574e-05	6.1483e-06
0.9ε	0.00028559	2.8574e-05	6.1484e-06

Table 11: Maximum absolute errors of Example 5 for $\varepsilon = 10^{-3}$

η / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.002853	3.6322e+095	0.0006709	0.0009995	0.00091355	0.0009990
0.3ε	0.0028513	3.8832e+095	0.0006701	0.0009983	0.00091241	0.0009978
0.6ε	0.0028496	4.15113e+095	0.0006692	0.0009971	0.00091128	0.0009966
0.9ε	0.0028479	4.4377e+095	0.0006684	0.0009959	0.00091014	0.0009954

Table 12: Maximum absolute errors of Example 5 for $\varepsilon = 10^{-5}$

η / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00028558	2.8573e-05	6.1482e-06
0.3ε	0.00028558	2.8573e-05	6.1481e-06
0.6ε	0.00028558	2.8573e-05	6.148e-06
0.9ε	0.00028558	2.8573e-05	6.148e-06

Table 13: Maximum absolute errors of Example 6 for $\eta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$

δ / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.0084942	3.1158e+095	0.0018357	0.0029895	0.0027162	0.0029855
0.3ε	0.0085166	2.9148e+095	0.0019076	0.0029931	0.0027211	0.0029890
0.6ε	0.0085391	2.7266e+095	0.0020628	0.0029967	0.0027563	0.0029926
0.9ε	0.0085615	2.5504e+095	0.0022424	0.0030002	0.0028623	0.0029962

Table 14: Maximum absolute errors of Example 6 for η =0.5ε and ε =10^{-5}

δ / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00085591	8.5713e-05	1.8444e-05
0.3ε	0.00085593	8.5716e-05	1.8444e-05
0.6ε	0.00085596	8.5718e-05	1.8445e-05
0.9ε	0.00085598	8.572e-05	1.848e-05

Table 15: Maximum absolute errors of Example 6 for δ =0.5ε and ε =10^{-3}

η / h	10^{-2}		10^{-3}		10^{-4}	
	Present method	Upwind method	Present method	Upwind method	Present method	Upwind method
0.0ε	0.008569	2.4942e+095	0.0023055	0.0030014	0.0029073	0.0029974
0.3ε	0.0085465	2.6660e+095	0.0021207	0.0029978	0.0027858	0.0029938
0.6ε	0.0085241	2.8507e+095	0.0019555	0.0029943	0.0027233	0.0029902
0.9ε	0.0085016	3.0473e+095	0.001837	0.0029907	0.0027179	0.0029867

Table 16: Maximum absolute errors of Example 6 for δ =0.5ε and ε =10^{-5}

η / h	10^{-3}	10^{-4}	10^{-5}
0.0ε	0.00085599	8.5721e-05	1.8522e-05
0.3ε	0.00085596	8.5719e-05	1.8445e-05
0.6ε	0.00085594	8.5716e-05	1.8444e-05
0.9ε	0.00085592	8.5714e-05	1.8444e-05

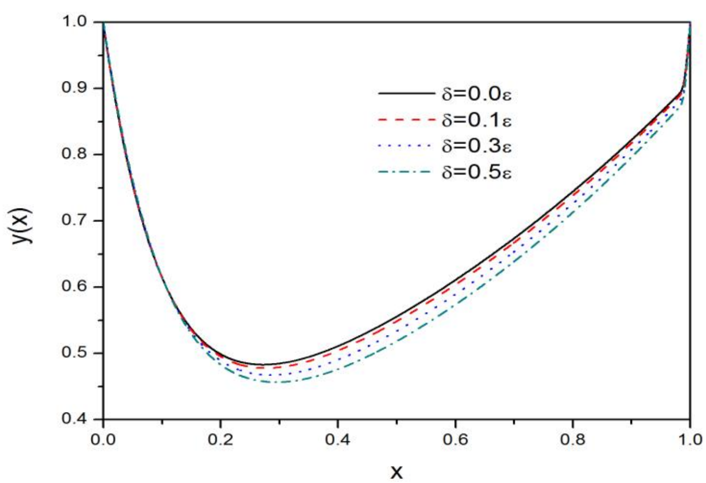

Figure 1: Effect of parameter in the Example 1 for ε =10^{-3}

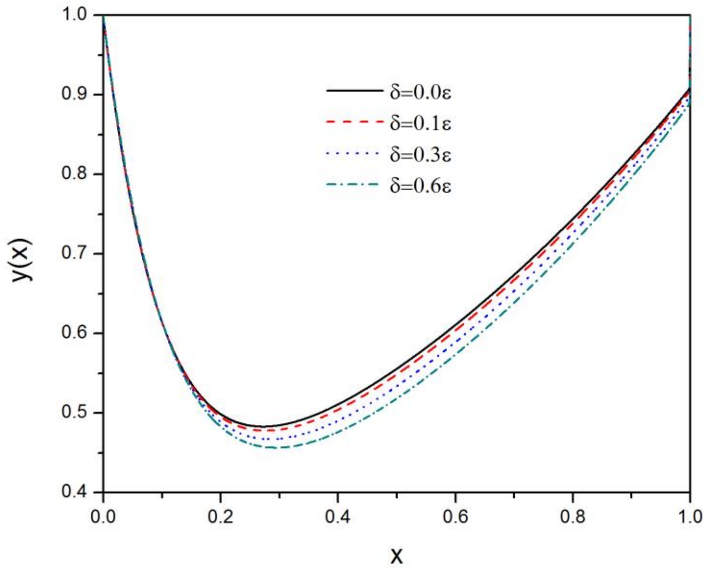

Figure 2: Effect of parameter in the Example 1 for $\varepsilon = 10^{-5}$

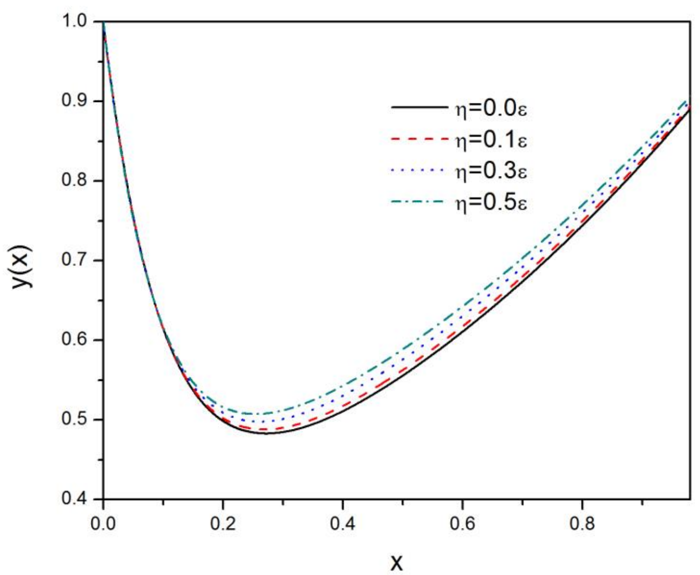

Figure 3: Effect of parameter in the Example 2 for $\varepsilon = 10^{-3}$

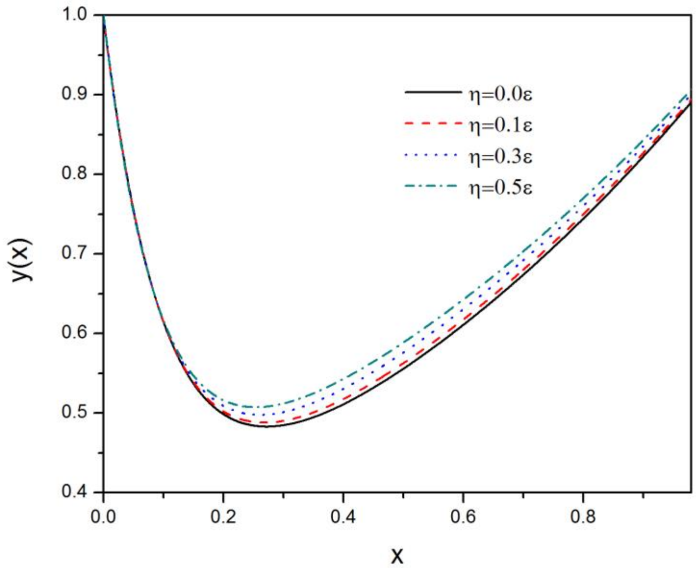

Figure 4: Effect of parameter in the Example 2 for $\varepsilon=10^{-5}$

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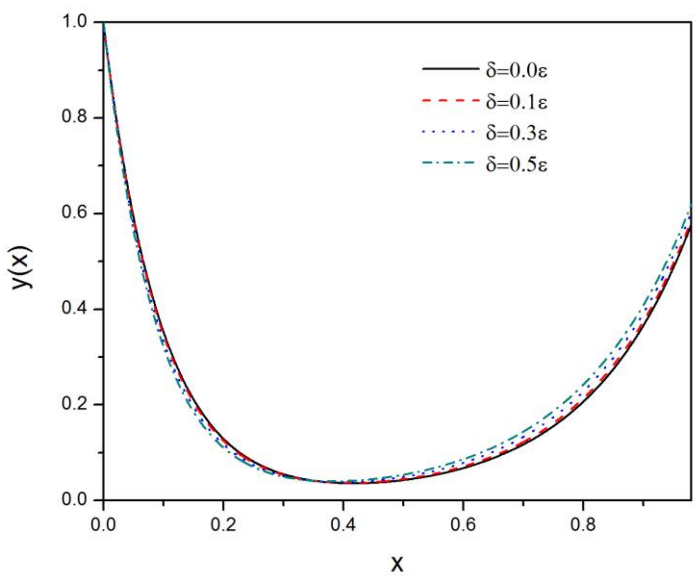

Figure 5: Effect of parameters in the Example 3 for $\eta=0.5\varepsilon$ and $\varepsilon=10^{-3}$:

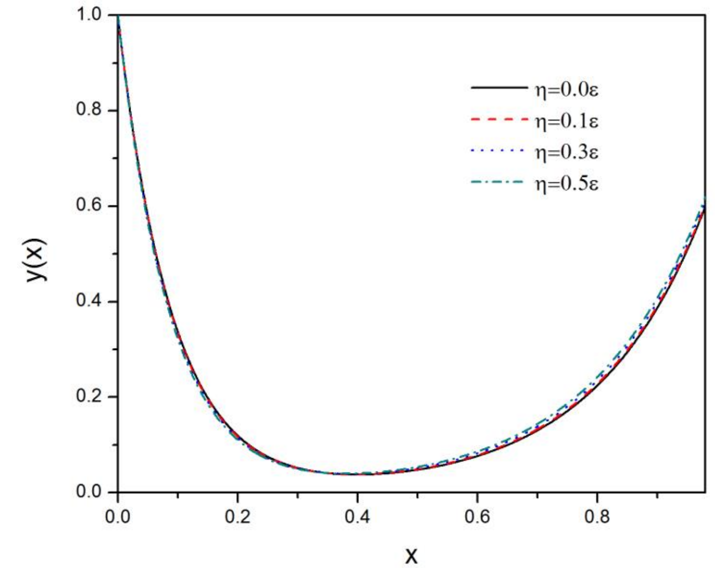

Figure 6: Effect of parameters in the Example 3 for δ =0.5 ε and ε =10^{-3}:

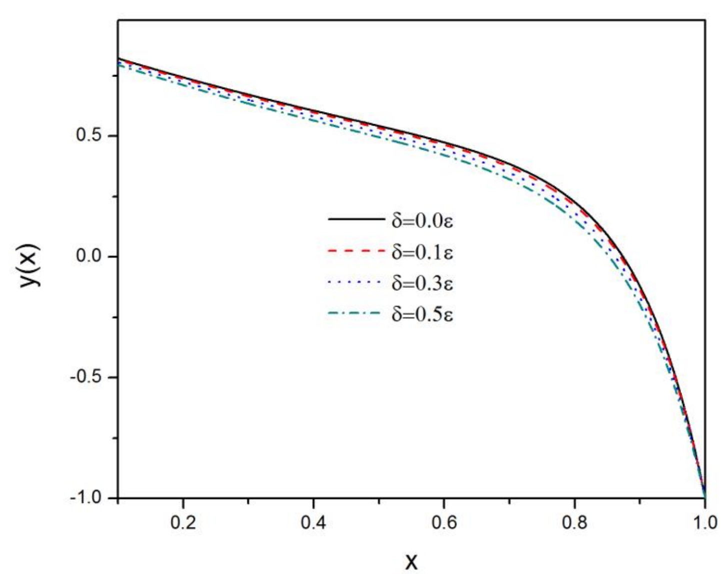

Figure 7: Effect of parameter in the Example 4 for for ε =10^{-3}

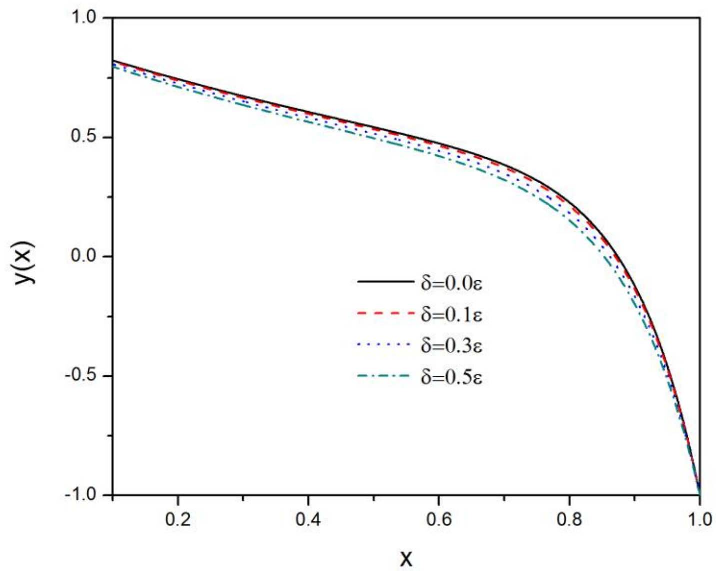

Figure 8: Effect of parameter in the Example 4 for $\varepsilon=10^{-5}$

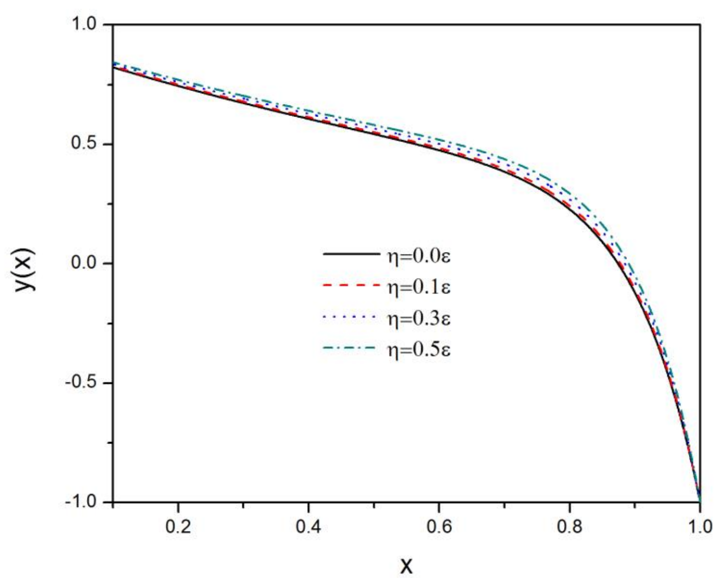

Figure 9: Effect of parameter in the Example 5 for $\varepsilon=10^{-3}$

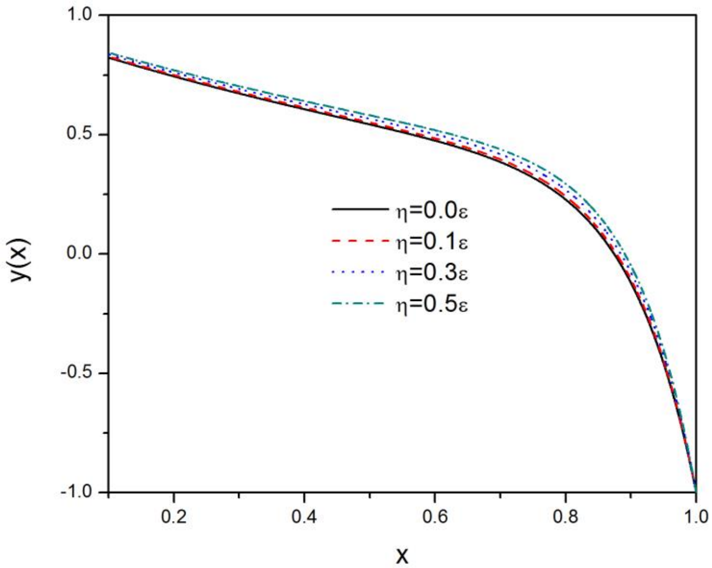

Figure 10: Effect of parameter in the Example 5 for $\varepsilon = 10^{-5}$

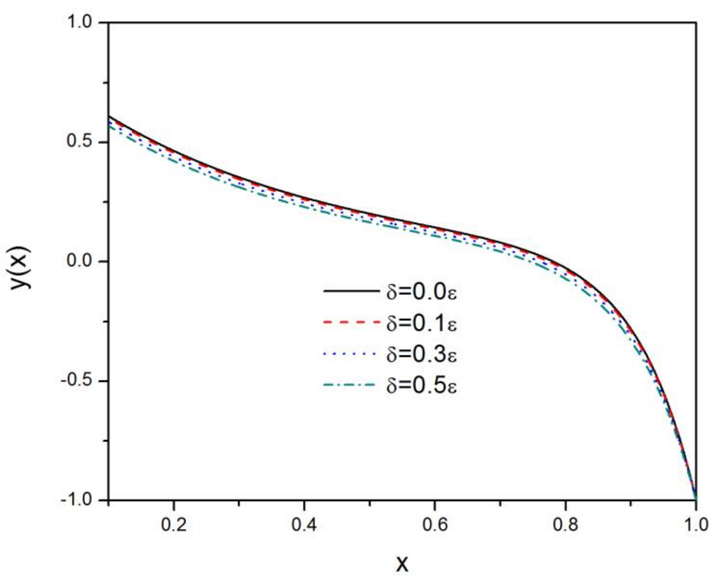

Figure 11: Effect of parameters in the Example 6 for $\eta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$**.**

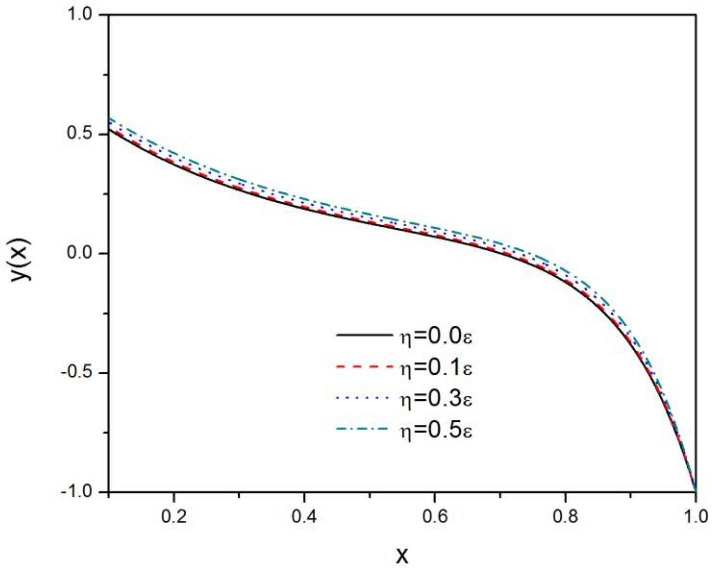

Figure 12: Effect of parameters in the Example 6 for $\delta = 0.5\varepsilon$ and $\varepsilon = 10^{-3}$