Generalized Differential Transform Method for Solving Some Fractional Integro-Differential Equations

S. Shahmorad
University of Tabriz

A. A. Khajehnasiri
North Tehran Branch, Islamic Azad University

Generalized Differential Transform Method for Solving Some Fractional Integro-Differential Equations

[1]**S. Shahmorad and** [2]**A. A. Khajehnasiri**

[1]Department of Applied Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, Iran
shahmord@tabrizu.ac.ir

[2] Department of Mathematics
North Tehran Branch, Islamic Azad University
Tehran, Iran
a.khajehnasiri@gmail.com

Abstract

In this paper, we use a generalized form of two-dimensional Differential Transform (2D-DT) to solve a new class of fractional integro-differential equations. We express some useful properties of the new transform as a proposition and prove a convergence theorem. Then we illustrate the method with numerical examples.

1. Introduction

Since solving fractional integro-differential equations, especially the fractional order partial integro-differential equations are a new subject in physical and mathematical problems, there are only a few techniques for solving these types of equations. The most commonly used methods in

211

this area are the Adomian decomposition method (AD) (Momani (2006); Momani (2006)), collocation (Rawashdeh (2006)), differential transform and fractional differential transform methods (Arikoglu (2009); Nazari (2010); Biazar (2010); Garg (2011)).

The DTM was firstly proposed by J. K. Zhouin (Zhou (1986)) and then applied by A. Fatma for solving differential equations and systems of differential equations (Fatma (2004)). Arikoglu (Arikoglu (2007)) used this method for solving fractional differential equations. The FDTM was used and developed for solving various types of linear and nonlinear differential, integral and integro-differential equations of fractional and integer orders (see for example (Nazari (2010)) and the references therein). About two-dimensional cases, there are a few works, for example, Shahmorad et al. applied differential transform method for solving nonlinear Volterra integro-differential equations (Tari (2009); Tari (2011)), Aghazadeh in (Aghazadeh (2013)) have developed the Block-pulse operational matrix to evaluate the approximate solution of the nonlinear 2D Volterra integro-differential equation, Khajehnasiri used 2DTFs for solving 2D Volterra-Fredholm integro-differential equations (Khajehnasiri (2016)). Maleknejad has extended hybrid functions method to solving two-dimensional nonlinear integral equations (Maleknejad (2018)). Hesameddini also developed the shifted Legendre polynomials operational matrix method for solving the two-dimensional integral equations of fractional order (Hesameddini (2018)). In this research work, we are interested in solving equations of the form (1) by a faster and remarkably simple method, i.e. the generalized DTM.

Consider a fractional order partial Volterra integro-differential equation in the general form

$$a_1(x,y)\frac{\partial^\gamma u(x,y)}{\partial x^\gamma} + a_2(x,y)\frac{\partial^\mu u(x,y)}{\partial y^\mu} + a_3(x,y)\frac{\partial^{\gamma+\mu} u(x,y)}{\partial x^\gamma \partial y^\mu} + a_4(x,y)u(x,y)$$
$$+ \lambda \int_{x_0}^x \int_{y_0}^y K(x,y,t,z,u(t,z))dtdz = f(x,y),$$
$$n-1 < \gamma \le n, m-1 < \mu \le m, \tag{1}$$

with some supplementary conditions which will be determined duo to order of equation, where $h_i(x), g_j(x), a_1(x,y), \cdots, a_4(x,y), f(x,y)$ and $k(x,y,t,z,u)$ are given continuous functions on $\Omega_1 = \{x \in \mathbb{R}, x_0 \le x \le X\}, \Omega_2 = \{(x,y) \in \mathbb{R}^2, x_0 \le x \le X, y_0 \le y \le Y\}$ and

$$\Omega_3 = \{(x,y,z,t,u) \in \mathbb{R}^5, x_0 \le x \le X, y_0 \le y \le Y, x_0 \le t \le x, y_0 \le z \le y, -\infty < u < \infty\}.$$

Existence and uniqueness of solution for the special cases of the problem (1) may be found in literature, but for the general case, it is an open problem. The fractional derivatives are taken in Caputo sense.

Equations of the form (1) may arise in the mathematical modeling of dynamic fractional order viscoelastic problem (Adolfsson (2006)) and dynamical processes occurring in the system exhibiting anomalous diffusive behavior (Bandrowski (2010)). A special case of Equation (1) is the generalized KdV equation (Kurulay (2010)).

Shahmorad and Khajehnasiri: Generalized Differential Transform for Solving Fractional Integro-Differential

AAM: Intern. J., Vol. 14, Issue 1 (June 2019) 213

2. Fractional calculus

There are many definitions of a fractional derivative of order $\alpha > 0$ (Ebadian (2015); Rahmani (2015); Podlubny (1999)). In this context, we follow the essential definition of the Caputo fractional order integration and differentiation given as below, which are used in this work.

Definition 2.1.

A real function $u(x)$, $x > 0$, in the space $c_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $u(x) = x^p u_1(x)$, where $u_1(x) \in c[0, \infty]$ and it is said to be in the space, if $u^{(m)} \in c_\mu$, $m \in N$.

Definition 2.2.

The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $u \in c_\mu$, $\mu \geq -1$ is defined as

$$I_{x_0}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x - t)^{\alpha-1} u(t) dt, \quad \alpha > 0, \quad x > 0. \tag{2}$$

Definition 2.3.

The fractional derivative of $u(x)$ in the Caputo sense is defined as

$$D_*^\alpha u(x) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dx^m} \int_{x_0}^x (x - t)^{m-\alpha-1} u(t) dt, \quad x > x_0, \tag{3}$$

for $m - 1 < \alpha < m$, $m \in N$, $x > 0$, $u \in c_{-1}^n$.

Definition 2.4.

For m to be the smallest integer that greater than α, the Caputo time fractional derivative of order $\alpha > 0$ is defined as

$$D_{*t}^\alpha u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} \frac{\partial^m u(x,t)}{\partial t^m}, & m - 1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N, \end{cases} \tag{4}$$

3. 2D-DT and its generalization

In this section we recall 2D-DT and generalize it for solving fractional integro-differential equations of the form (1).

Definition 3.1.

The 2D-DT of the analytical and continuously differentiable function $u(x, y)$ around the point (x_0, y_0) is defined by

$$U(k, h) = \frac{1}{k!h!} \frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x, y) \mid_{(x,y)=(x_0,y_0)}, \tag{5}$$

where $u(x, y)$ and $U(k, h)$ denote the original function and its differential transform respectively.

Definition 3.2.

The inverse differential transform of $U(k, h)$ is defined by

$$u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)(x - x_0)^k (y - y_0)^h. \tag{6}$$

From (5) and (6), we obtain

$$u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x, y) \mid_{(x,y)=(x_0,y_0)} (x - x_0)^k (y - y_0)^h, \tag{7}$$

which is the bivariate Taylor expansion of the function $u(x, y)$ around the point (x_0, y_0).

Note: Throughout this paper, the lower and upper case letters are used to represent the original function and its transform respectively.

Definition 3.3.

The generalized 2D-DT of the function $u(x, y)$ is defined by

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} [(D_{x_0}^\alpha)^k (D_{y_0}^\beta)^h u(x, y)]_{(x_0,y_0)}, \tag{8}$$

where $(D_{x_0}^\alpha)^k = D_{x_0}^\alpha . D_{x_0}^\alpha \cdots D_{x_0}^\alpha$ (k-times), and $D_{x_0}^\alpha$ denotes the fractional derivative in the Caputo sense (Podlubny (1999)). Based on the generalized 2D-DT, the function $u(x, y)$ can be represented as

$$u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha} (y - y_0)^{h\beta}, \quad 0 < \alpha, \beta \leq 1. \tag{9}$$

Then, for $N \in \mathbb{N}$, we use the truncation

$$u_{N,N}(x, y) \simeq \sum_{k=0}^{N} \sum_{h=0}^{N} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha} (y - y_0)^{h\beta}, \tag{10}$$

of (9) to get an approximation for $u(x, y)$.

In the following proposition, we summarize fundamental properties of the generalized 2D-DT.

Shahmorad and Khajehnasiri: Generalized Differential Transform for Solving Fractional Integro-Differential

AAM: Intern. J., Vol. 14, Issue 1 (June 2019)   215

Proposition 3.4.

Let $U_{\alpha,\beta}(m,n)$, $V_{\alpha,\beta}(m,n)$ and $W_{\alpha,\beta}(m,n)$ be generalized 2D-DT of the functions $u(x,y)$, $v(x,y)$ and $w(x,y)$ respectively. Then,

(a) If $u(x,y) = av(x,y) \pm bw(x,y)$ for $a, b \in R$, then

$$U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h) \pm bW_{\alpha,\beta}(k,h).$$

(b) If $u(x,y) = v(x,y)w(x,y)$, then

$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s).$$

(c) If $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then

$$U_{\alpha,\beta}(k,h) = \delta(k-n,h-m) = \delta(k-n)\delta(h-m),$$

where

$$\delta(k-n) = \begin{cases} 1, & k=n, \\ 0, & otherwise, \end{cases} \qquad \delta(h-m) = \begin{cases} 1, & h=m, \\ 0, & otherwise. \end{cases}$$

(d) If $u(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t,z)dtdz$, then

$$U_{\alpha,\beta}(k,h) = \frac{V_{\alpha,\beta}(k-\frac{1}{\alpha}, h-\frac{1}{\beta})}{\alpha k \beta h}, \qquad k \geq \frac{1}{\alpha}, h \geq \frac{1}{\beta}.$$

(e) If $u(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t,z)w(t,z)dtdz$, then

$$U_{\alpha,\beta}(k,h) = \frac{1}{\alpha k \beta h} \sum_{r=0}^{k-\frac{1}{\alpha}} \sum_{s=0}^{h-\frac{1}{\beta}} V_{\alpha,\beta}(r,h-s-\frac{1}{\beta})W_{\alpha,\beta}(k-r-\frac{1}{\alpha},s), \ k \geq \frac{1}{\alpha}, h \geq \frac{1}{\beta}.$$

(f) If $u(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t)w(t,z)dtdz$, then

$$U_{\alpha,\beta}(k,h) = \frac{1}{2\alpha k \beta h} V_{\alpha}(k-\frac{1}{2\alpha})W_{\alpha,\beta}(k-\frac{1}{2\alpha},h-\frac{1}{\beta}).$$

(g) If $u(x,y) = v(x,y)\int_{x_0}^{x} \int_{y_0}^{y} w(t,z)dtdz$, then

$$U_{\alpha,\beta}(k,h) = \sum_{k_1=\frac{1}{\alpha}}^{k} \sum_{h_1=\frac{1}{\beta}}^{h} \frac{1}{\alpha k_1 \beta h_1} W_{\alpha,\beta}(k_1-\frac{1}{\alpha}, h_1-\frac{1}{\beta})V_{\alpha,\beta}(k-k_1,h_1).$$

Proof:

We prove parts **(d)** and **(e)**, the other parts are proved similarly.
(d) Using the expansion (9), we have

$$u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha,\beta}(k, h)(t - x_0)^{k\alpha}(z - y_0)^{h\beta} dt dz,$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha,\beta}(k, h) \int_{x_0}^{x} \int_{y_0}^{y} (t - x_0)^{k\alpha}(z - y_0)^{h\beta} dt dz,$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha,\beta}(k, h) \frac{(t - x_0)^{k\alpha+1}}{k\alpha + 1} \Big|_{x_0}^{x} \int_{y_0}^{y} (z - y_0)^{h\beta} dz$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha,\beta}(k, h) \frac{(x - x_0)^{k\alpha+1}}{k\alpha + 1} \frac{(y - y_0)^{h\beta+1}}{h\beta + 1}.$$

Starting the indices from $k = \frac{1}{\alpha}$ and $h = \frac{1}{\beta}$, we obtain

$$u(x, y) = \sum_{k=\frac{1}{\alpha}}^{\infty} \sum_{h=\frac{1}{\beta}}^{\infty} V_{\alpha,\beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta}) \frac{(x - x_0)^{k\alpha}}{k\alpha} \frac{(y - y_0)^{h\beta}}{h\beta}. \tag{11}$$

Then, by equating the coefficients on both sides on can get

$$U_{\alpha,\beta}(k, h) = \frac{V_{\alpha,\beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta})}{\alpha k \beta h}, \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta}. \tag{12}$$

(**e**) Let $c(t, z) = v(t, z)w(t, z)$ and $C_{\alpha,\beta}(k, h)$ be its transform. Then, by using part (d), we have

$$U_{\alpha,\beta}(k, h) = \frac{C_{\alpha,\beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta})}{\alpha k \beta h}, \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta},$$

and using part **(b)** yields

$$U_{\alpha,\beta}(k, h) = \frac{1}{\alpha k \beta h} \sum_{r=0}^{k-\frac{1}{\alpha}} \sum_{s=0}^{h-\frac{1}{\beta}} V_{\alpha,\beta}(r, h - s - \frac{1}{\beta}) W_{\alpha,\beta}(k - r - \frac{1}{\alpha}, s), \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta}. \quad \blacksquare$$

Theorem 3.5 (Momani (2007)).

If $u(x, y) = D_{x_0}^{\gamma} v(x, y)$, $m - 1 < \gamma \leq m$, then

$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \frac{\gamma}{\alpha}, h). \tag{13}$$

Shahmorad and Khajehnasiri: Generalized Differential Transform for Solving Fractional Integro-Differential

AAM: Intern. J., Vol. 14, Issue 1 (June 2019)   217

Theorem 3.6 (Momani (2007)).

If $u(x,y) = D_{x_0}^\gamma D_{y_0}^\mu v(x,y)$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} \frac{\Gamma(\beta h + \mu + 1)}{\Gamma(\beta h + 1)} V_{\alpha,\beta}(k + \frac{\gamma}{\alpha}, h + \frac{\mu}{\beta}), \tag{14}$$

where $\gamma, \mu \in Q^+$ and $0 < \alpha, \beta \leq 1$.

4. Error bound and convergence

In this section, we obtain an error bound for the approximate solution, then from which we conclude convergence of the method. We define the error function as

$$e_{N,N}(x,y) = u(x,y) - u_{N,N}(x,y), \tag{15}$$

where $u(x,y)$ and $u_{N,N}(x,y)$ are defined by (9) and (10).

Theorem 4.1.

Let

$$\mid D_x^{(N+1)\alpha} u(x,y) \mid \leqslant M_1, \tag{16}$$

and

$$\mid D_y^{(N+1)\beta} u(x,y) \mid \leqslant M_2, \tag{17}$$

for some nonnegative constants M_1 and M_2. Then,

$$\mid e_{N,N}(x,y) \mid \leqslant M_1 \frac{\mid (x-x_0) \mid^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)} + M_2 \frac{\mid (y-y_0) \mid^{(N+1)\beta}}{\Gamma((N+1)\beta + 1)}.$$

Proof:

By using (8), (9) and (10) we have

$$\mid u(x,y) - u_{N,N}(x,y) \mid = \mid \sum_{k=0}^\infty \sum_{h=0}^\infty U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}$$

$$- \sum_{k=0}^N \sum_{h=0}^N U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \mid$$

$$\leq \frac{\mid (x-x_0) \mid^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)} \mid D_x^{(N+1)\alpha} u(x,y)\mid_{(x,y)=(\eta_1,\eta_2)} \mid$$

$$+ \frac{\mid (y-y_0) \mid^{(N+1)\beta}}{\Gamma((N+1)\beta + 1)} \mid D_y^{(N+1)\beta} u(x,y)\mid_{(x,y)=(\eta_1,\eta_2)} \mid,$$

for some η_1 and η_2 with $min\{x_0, x\} \leq \eta_1 \leq max\{x_0, x\}$ and $min\{y_0, y\} \leq \eta_2 \leq max\{y_0, y\}$. Then, using (16) and (17) completes the proof. ∎

Corollary 4.2 (Convergence).

By the hypothesis of Theorem 4.1, we have $e_{N,N}(x,y) \to 0$, as $N \to \infty$.

5.   Application of the method

In this section, we apply proposition 3.4 to obtain a recurrence relation for $U_{\alpha,\beta}(k,h)$. To this end, we need some starting values of U that can be obtained from supplementary conditions. Since the initial conditions are implemented by the integer-order derivatives, the transformations of the initial conditions for $k = 0, 1, \cdots, (\gamma\alpha - 1)$ and $h = 0, 1, \cdots, (\mu\beta - 1)$ are defined by

$$U_{\alpha,\beta}[x_0, h] = \begin{cases} \frac{1}{\Gamma(\beta h + 1)} D^{\beta h} u(x_0, y) \mid_{y=y_0}, & if \ \ \beta h \in \mathcal{Z}^+, \\ 0, & \beta h \notin \mathcal{Z}^+, \end{cases} \tag{18}$$

and

$$U_{\alpha,\beta}[k, y_0] = \begin{cases} \frac{1}{\Gamma(\alpha k + 1)} D^{\alpha k} u(x, y_0) \mid_{x=x_0}, & if \ \ \alpha k \in \mathcal{Z}^+, \\ 0, & \alpha k \notin \mathcal{Z}^+, \end{cases} \tag{19}$$

where γ and μ are the orders of the corresponding fractional equations (Nazari (2010)).

Example 5.1.

Consider the nonlinear partial Volterra integro-differential equation

$$\frac{\partial^{1/2} u(x,y)}{\partial x^{1/2}} + \frac{\partial^{1/3} u(x,y)}{\partial y^{1/3}} + \frac{\partial^{\frac{5}{6}} u(x,y)}{\partial y^{1/3} \partial x^{1/2}} + u(x,y)$$

$$+ \int_0^x \int_0^y u^3(t,z) dt dz = g(x,y), \quad x, y \in [0,1], \tag{20}$$

with

$$g(x,y) = \frac{3}{2} \frac{y^{2/3}}{\Gamma(2/3)} + \frac{2\sqrt{x}}{\sqrt{\pi}} + x + y + \frac{1}{4} xy^4 + \frac{1}{2} x^2 y^3 + \frac{1}{2} x^3 y^2 + \frac{1}{4} x^4 y,$$

subject to the initial condition

$$u(x,0) = x. \tag{21}$$

By choosing $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and applying the generalized 2D-DT to the both sides of Equation (20)

Shahmorad and Khajehnasiri: Generalized Differential Transform for Solving Fractional Integro-Differential

AAM: Intern. J., Vol. 14, Issue 1 (June 2019)   219

and using Proposition 3.4, Theorems 3.5 and 3.6, we get the recurrence relation

$$
U_{\frac{1}{2},\frac{1}{3}}(k+1,h+1) = \frac{\Gamma(\frac{k}{2}+1)\Gamma(\frac{h}{3}+1)}{\Gamma(\frac{k}{2}+\frac{3}{2})\Gamma(\frac{h}{3}+\frac{4}{3})}\left(-\frac{\Gamma(\frac{k}{2}+\frac{3}{2})}{\Gamma(\frac{k}{2}+1)}U_{\frac{1}{2},\frac{1}{3}}(k+1,h)\right.
$$

$$
-\frac{\Gamma(\frac{h}{3}+\frac{4}{3})}{\Gamma(\frac{h}{3}+1)}U_{\frac{1}{2},\frac{1}{3}}(k,h+1) - U_{\frac{1}{2},\frac{1}{3}}(k,h)
$$

$$
-\sum_{k_1=1}^{k}\sum_{h_1=1}^{h}\sum_{l_2=0}^{k_1-1}\sum_{r_2=0}^{h_1-1}\sum_{l_1=0}^{l_2}\sum_{r_1=0}^{r_2}\frac{1}{k_1 h_1}U_{\frac{1}{2},\frac{1}{3}}(l_1,r_1)
$$

$$
U_{\frac{1}{2},\frac{1}{3}}(l_2-l_1,r_2-r_1)U_{\frac{1}{2},\frac{1}{3}}(k_1-l_2-1,h_1-r_2-1)
$$

$$
+\frac{3}{2\Gamma(\frac{2}{3})}\delta(h-\frac{2}{3})\delta(k) + \frac{2}{\sqrt{\pi}}\delta(k-\frac{1}{2})\delta(h)
$$

$$
+\delta(k-1)\delta(h) + \delta(h-1)\delta(k) + \frac{1}{4}\delta(k-1)\delta(h-4)
$$

$$
+\frac{1}{2}\delta(k-2)\delta(h-3) + \frac{1}{2}\delta(k-3)\delta(h-2) + \frac{1}{4}\delta(k-4)\delta(h-1)).
$$

The 2D-DT of the initial conditions is given by

$$
U_{\frac{1}{2},\frac{1}{3}}(k,0) = \delta(k-1).
$$

Then, for $N=2$, $N=3$ and $N=5$, the approximate solutions are listed as follows:

$$
u_{2,2} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{3}} - 1.1161725y^{\frac{2}{3}}) + x(1 + 0.9999995y^{\frac{1}{3}} - 2.9783642y^{\frac{2}{3}}).
$$

$$
u_{3,3} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{3}} - 1.1161725y^{\frac{2}{3}} + 2.1359985y) + x(1 + 0.9999995y^{\frac{1}{3}}
$$
$$
-2.9783642y^{\frac{2}{3}} + 5.5708657y) + x^{\frac{3}{2}}(0.75225274y^{\frac{1}{3}} - 3.7368505y^{\frac{2}{3}} + 9.8046058y).
$$

$$
u_{5,5} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{3}} - 1.1161725y^{\frac{2}{3}} + 2.1359985y - 2.9223719y^{\frac{4}{3}} + 2.3126065y^{\frac{5}{3}})
$$
$$
+x(1 + 0.9999995y^{\frac{1}{3}} - 2.9783642y^{\frac{2}{3}} + 5.5708657y - 9.1617499y^{\frac{4}{3}} + 11.889491y^{\frac{5}{3}})
$$
$$
+x^{\frac{3}{2}}(0.75225274y^{\frac{1}{3}} - 3.7368505y^{\frac{2}{3}} + 9.8046058y - 19.317391y^{\frac{4}{3}} + 31.122589y^{\frac{5}{3}})
$$
$$
-x^{2}(0.61984651y^{\frac{1}{3}} - 2.3706321y^{\frac{2}{3}} + 11.140679y - 28.713412y^{\frac{4}{3}} + 56.248187y^{\frac{5}{3}})
$$
$$
-x^{\frac{5}{2}}(0.97482722y^{\frac{1}{3}} - 0.08934489y^{\frac{2}{3}} + 8.2117925y - 30.881239y^{\frac{4}{3}} + 75.567816y^{\frac{5}{3}}).
$$

Example 5.2.

Consider a linear partial Volterra integro-differential equation of the form

$$
\frac{\partial^{1/3}u(x,y)}{\partial y^{1/3}} + x\frac{\partial u(x,y)}{\partial x} + \int_{0}^{x}\int_{0}^{y}t^2 z^2 u(t,z)dtdz = 3x^2 + \frac{1}{18}x^3 y^6, \quad x,y \in [0,1], \quad (22)
$$

subject to the initial condition

$$
u(x,0) = x^3. \tag{23}
$$

Let $\alpha = 1$ and $\beta = \frac{1}{3}$. Then, by a similar way as in Example 1, we get the recurrence relation

$$U_{1,\frac{1}{3}}(k, h+1) = \frac{\Gamma(\frac{h}{3}+1)}{\Gamma(\frac{h}{3}+\frac{4}{3})}\Big(-\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-1, h-s)(k-r+1)U_{1,\frac{1}{3}}(k-r+1, s)$$

$$-\frac{3}{kh}\sum_{r=0}^{k-3}\sum_{s=0}^{h-1}\delta(k-2)\delta(h-s-5)U_{1,\frac{1}{3}}(k-r-2, s) + 3\delta(k-3)\delta(h)$$

$$+\frac{1}{18}\delta(k-3)\delta(h-6)\Big).$$

for the Equation (22) and

$$U_{1,\frac{1}{3}}(k, 0) = \delta(k-3).$$

for the initial condition (23).

Hence, for $N = 2, N = 3$ and $N = 5$, the approximate solutions are given as follows:

$$u_{2,2} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y)$$
$$+x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}} + 0.00012500y).$$
$$u_{3,3} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y + 0.00462929y^{\frac{4}{3}})$$
$$+x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}} + 0.00012500y + 0.62499999 \times 10^{-5}y^{\frac{4}{3}}).$$
$$u_{5,5} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y + 0.00462929y^{\frac{4}{3}}$$
$$+0.00077160y^{\frac{5}{3}} + 0.00012860y^2) + x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}}$$
$$+0.00012500y + 0.62499999 \times 10^{-5}y^{\frac{4}{3}} + 0.31249998 \times 10^{-6}y^{\frac{5}{3}}$$
$$+0.15624999 \times 10^{-7}y^2).$$

6.   Conclusion

In this paper, we analyzed the applicability of the generalized differential transform method for solving partial Volterra integro-differential equations of fractional order. We showed simplicity and reliability of the method by handling examples of linear and nonlinear partial Volterra integro-differential equations of fractional order. Also, we proved convergence of the method.

Acknowledgment

Shahmorad and Khajehnasiri: Generalized Differential Transform for Solving Fractional Integro-Differential

AAM: Intern. J., Vol. 14, Issue 1 (June 2019)   221

REFERENCES

Adolfsson, K. Enelund, M. Larsson, S. and Racheva, M. (2006). Discretization of Integro-Differential Equations Modeling Dynamic Fractional Order Viscoelasticity, Springer-Verlag Berlin Heidelberg, Vol. 3743, pp. 76-83.

Arikoglu, A. and Ozkol, I . (2007). Solution of fractional differential equations by using differential transform method, Chaos, Solitons and Fractals, Vol. 34, pp. 1473-148.

Aghazadeh, N. and Khajehnasiri, A. A. (2013). Solving nonlinear two-dimensional Volterra integro-differential equations by block-pulse functions, Mathematical Sciences, Vol. 7, pp. 1-6.

Arikoglu, A. and Ozkol, I. (2009). Solution of fractional integro-differential equations by using fractional differential transform method, Chaos, Solitons and Fractals, Vol. 30, pp. 521-529.

Bandrowski, B. Karczewska, A. and Rozmej, P. (2010). Numerical Solutions to Integral Equations Equivalent to Diffrential Equations with Fractional Time, Int. J. Appl. Math. Comput. Sci, Vol. 20, pp. 261-269.

Biazar, J. and Eslami, M. (2010). Differential Transform Method for Nonlinear Parabolic-hyperbolic Partial Differential Equations, Appl. Appl. Math, Vol. 5, pp. 396-406.

Ebadian, A. Rahmani Fazli, H. and Khajehnasiri, A. A. (2015). Solution of nonlinear fractional diffusion-wave equation by traingular functions, SeMA Journal, Vol. 72, pp. 37-46.

Fatma, A. (2004). Solution of the system of differential equations by differential transform, Applied Mathematics and Computation, Vol. 147, pp. 547-567.

Garg, M. Manohar, P. and Kalla, S.L. (2011). Generalized Differential Transform Method to Space-Time Fractional Telegraph Equation, International Journal of Differential Equations, Vol. 54, pp. 9-19.

Hesameddini, E. and Shahbazi, M. (2018). Two-dimensional shifted Legendre polynomials operational matrix method for solving the two-dimensional integral equations of fractional order, Applied Mathematics and Computation, Vol. 322, pp. 40-54.

Khajehnasiri, A. A. (2016). Numerical Solution of Nonlinear 2D VolterraâFredholm Integro-Differential Equations by Two-Dimensional Triangular Function, International Journal of Applied and Computational Mathematics, Vol. 2, pp. 575-591.

Kurulay, M. and Bayram, M. (2010). Approximate analytical solution for the fractional modified KdV by differential transform method, Commun Nonlinear Sci Numer Simulat, Vol. 15, pp. 1777-1782.

Momani, S. and Noor, M.A. (2006). Numerical methods for fourth order fractional integro-differential equations, Applied Mathematics and Computation, Vol. 182, pp. 754-760.

Momani, S. and Qaralleh, A. (2006). An efficient method for solving systems of fractional integro-differential equations, Applied Mathematics and Computation, Vol. 52, pp. 452-470.

Maleknejad, K. and Shahabi, M. (2018). Application of hybrid functions operational matrices in the numerical solution of two-dimensional nonlinear integral equations, Applied Numerical Mathematics, Vol. 22, pp. 1-27.

Momani, S. Odibat, Z. and Erturk, V. (2007). Generalized differential transform method for solving a space and time-fractional diffusion-wave equation, Physics Letters A, Vol. 370, pp. 379-387.

Nazari, D. and Shahmorad, S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions, Journal of Computational and Applied Mathematics, Vol. 234, pp. 883-891.

Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, San Diego.

Rahmani Fazli, H. Hassani, F. Ebadian, A. and Khajehnasiri, A. A. (2015). National economies in state-space of fractional-order financial system, Afrika Matematika, Vol. 10, pp. 1-12.

Rawashdeh, E.A. (2006). Numerical solution of fractional integro-differential equations by collocation method, Applied Mathematics and Computation, Vol. 176, pp. 1-6.

Tari, A. Rahimi, M.Y. Shahmorad, S. and Talati, F. (2009). Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method, Journal of Computational and Applied Mathematics, Vol. 228, pp. 70-76.

Tari, A. and Shahmorad, S. (2011). Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations, Computers and Mathematics with Applications, Vol. 61, pp. 2621-2629.

Zhou, J.K. (1986). *Differential Transfomation and its Application for Electrical Circuits*, Huazhong University Press, Wuhan, China, (in Chinese).