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Analysis of an M/M/1 Queue With Working Vacation and Vacation Interruption

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Abstract

In this paper, an $M/M/1$ queue with working vacation and vacation interruption is investigated. The server is supposed to interrupt the vacation and return back to the normal working period, if there are at least N customers waiting in the system at a service completion instant during the working vacation period. Otherwise, the server continues the vacation until the system is non-empty after a vacation ends or there are at least N customers after a service ends. In terms of the quasi birth and death process and matrix-geometric solution method, we obtain the distributions for the stationary queue length. Moreover, we demonstrate stochastic decomposition structures of the queue length and waiting time, and obtain the distributions of the additional queue length and additional delay for the case $N = 2$. Finally, numerical examples are presented.

Keywords: Working vacation; Vacation interruption; Stochastic decomposition; $M/M/1$ queue

MSC 2010 No.: 60G10, 60G27

1. Introduction

During the last three decades, researchers have extensively analyzed the vacation queuing models and successfully utilized in numerous applied problems. In the classical vacation queuing models, the server completely ceases service during a vacation and such a policy may lead to the dissatisfaction of the customers and ultimately to the loss of costumer base. However, there are many situations where the server does not remain completely inactive during a vacation. But provides service to the customers at a lower rate. This idea was first introduced by Servi and Finn (2002).

Servi and Finn (2002) introduced a class of semi vacation policy, where the server does not completely stops working during a vacation, but it will render service at a lower rate to the queuing system. This type of vacation is called a working vacation (WV). Servi and Finn (2002) analyzed an $M/M/1$ queue with multiple working vacation policy and derived the probability generating function for the number of customers in the system and LST for waiting time distribution, and utilized the results to analyze the system performance of gateway router in fiber communication networks. Subsequently, working vacation queues have received considerable attention in literature, Baba (2005), Wu and Takagi (2006), Liu et al. (2007). Recently, Majid and Manoharan (2017*a*, 2017*b*) derived the steady state solution of $M/M/c$ queue with WV by utilizing PGF method.

Generally, in a WV policy, the server returns back to regular service period only if there are customers waiting in the system at the end of a vacation. Definitely such assumption seems to be much more limited in real world situations. To overcome such restrictions, the concept of vacation interruption in an $M/M/1$ queue with WV was introduced by Li and Tain (2007). In this vacation policy, if the server at the instant of service completion during the vacation period finds that there are customers waiting in the system, the sever ends his vacation and returns back to regular service period, otherwise the server continues the WV until the queuing system is non empty after a service or a WV ends. Due to its strong application in the stochastic service models, various productive theoretical results are presented in this area. Li et al. (2008), Baba (2010), Zhang and Hou (2011), Gao and Liu (2013), and Lee and Kim (2015) are those who have contributed in this area. Majid and Manoharan (2017*c*) extended the work of Li and Tain (2007) with single working vacation. Goswami (2014) analyzed the $M/M/1$ queue with impatient customers with multiple working vacations and Bernoulli-schedule vacation interruption. Laxmi amd Jyothsna (2015) studied the impatient customers in an $M/M/1$ queue with single and multiple working vacation policy under Bernoulli schedule vacation interruption. Recently, Majid and Manoharan (2019) analyzed impatient customers in an $M/M/1$ Queue with vacation interruptions under Bernoulli schedule vacation.

Although, in Li and Tian (2007) service discipline, only the first arrival during a vacation period gets lower service rate. When the server switches from the lower service to the normal service rate during his vacation, the switching cost is incurred. The system has to face more additional cost if the service is mostly interrupted. Therefore, in practical application, the vacation interruption policy introduced by Li and Tian (2007) has some limitations. In this paper, a modified vacation interruption policy is presented to reduce the switching cost of the system. Under the modified vacation interruption policy, the server ends his vacation and resumes regular service rate as soon as at least N customers accumulate in the system upon the completion of a service in the vacation period. Otherwise, the server continues the vacation until the system is non-empty after a vacation ends or there are at least N customers waiting after a service ends.

Neuts (1981) has developed a new approach called matrix-geometric method which expands and enhances the earlier transform methods and presented efficient and stable algorithms involving only real arithmetic. This technique is applicable to both continuous and discrete-time Markov processes. Matrix-geometric method is a effective technique to study and examine complex queuing systems. This method manages the block matrices of the states of the system and transitions within the states despite of engaging with individual states or transition probabilities. This technique of solving queuing model makes the expressions in matrix form simpler than the corresponding expressions given in terms of eigenvalues. Also the basic matrices have direct probabilistic evaluation, while the eigenvalues do not. This method establishes a geometric relationship between vectors of the stationary probability distribution and enables one to find a closed-form derivations for the calculation of different performance measures such as the mean queue length, mean waiting time and busy cycle. One of the practical advantages of this method is that these elementary matrix operations can easily be programmed for a high-speed computer. The use of PH-distributions in the representation of system elements and the matrix-geometric method in their analysis has significantly expanded the scope of queuing systems.

The rest of the paper is structured as follows. In Section 2, we discuss the model as a quasi birth and death process and obtain the stationary distribution of the queue length. In Section 3, we investigate this model for the case $N = 2$ and derive the stochastic decomposition structures of the number of customers in the system and waiting time, and obtain the distributions of the additional queue length and additional delay. In Section 4, numerical illustrations are presented followed by conclusions in Section 5.

2. Model Description

We consider a multiple working vacation policy in an $M/M/1$ queuing model under vacation interruption, where the server provides service to the customers at a reduced rate rather than stopping the service completely during his vacation period. The customers arrive according to a Poisson process with parameter λ. The server serves the customers at an exponential rate μ during a regular busy period and service discipline is first come first served (FCFS). The server begins a working vacation as soon as the system becomes empty. The arriving customers during working vacation period are served at a rate lower than the regular service rate. The service times during the working vacation and vacation times are also assumed to be exponentially distributed with rates η and θ, respectively. The server is supposed to interrupt the vacation and return back to the normal busy period, if there are at least N customers waiting in the system at a service completion instant during a working vacation period. Otherwise, he continues the vacation until the system is non-empty after a vacation ends or there are at least N customers after a service ends. Furthermore, if the server does not find any customer waiting in the system after completing a working vacation he will take another working vacation, else he will resume his regular busy period instantly. The inter-arrival times, the service times, and the working vacation times all are taken to be mutually independent.

Let $Q(t)$ be the number of customers in the system at time t and $J(t)$ be the status of the server, which is defined as follows,

$$J(t) = \begin{cases} 0, & \text{when the server stays in a WV period at time } t, \\ 1, & \text{when the server stays in non vacation period at time } t, \end{cases}$$

then the stochastic process $\{(Q(t), J(t)), t \geq 0\}$ is a quasi birth-and-death (QBD) process with the state space

$$S = \{(0,0)\} \bigcup \{(k,j), k \geq 1, j = 0,1\},$$

where state $(k, 0)$ represents that the system is in WV period and there are $k(k \geq 0)$ customers in the system; state $(k, 1)$ represents that the system is in normal working level and there are $k(k \geq 1)$ customers in the system.

Using the lexicographical order for the states, the infinitesimal generator for the QBD process is

$$Q = \begin{pmatrix} A_{00} & A_{01} & & & & & & & \\ B_{10} & A & C & & & & & & \\ & B_2 & A & C & & & & & \\ & & B_3 & A & C & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & B_N & A & C & & \\ & & & & & B & A & C & \\ & & & & & & B & A & C \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

where

$$A_{00} = -\lambda, \qquad A_{01} = (\lambda, 0), \qquad B_{10} = (\eta, \mu)^T,$$

$$A = \begin{pmatrix} -(\lambda+\eta+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{pmatrix}, \qquad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \eta \\ 0 & \mu \end{pmatrix}, \qquad B_i = \begin{pmatrix} \eta & 0 \\ 0 & \mu \end{pmatrix}, \qquad i = 2, 3, \ldots.$$

Lemma 2.1.

If the system workload $\frac{\lambda}{\mu} < 1$, the following matrix quadratic equation

$$R^2 B + RA + C = 0$$

has a minimal non-negative solution

$$R = \begin{pmatrix} r & \frac{r(\lambda+\theta)}{\mu} \\ 0 & \rho \end{pmatrix}, \tag{2}$$

where

$$r = \frac{\lambda}{\lambda+\theta+\eta}, \qquad 0 < r < 1. \tag{3}$$

Proof:

Since A, B and C in are all upper triangular matrices, therefore we can consider that the solution matrix R has the same structure as

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}.$$

Substituting the matrix value R into $R^2B + RA + C = 0$, we obtain

$$-(\lambda+\theta+\eta)r_{11}+\lambda=0,$$
$$\mu r_{22}^2-(\lambda+\mu)r_{22}+\lambda=0,$$
$$\eta r_{11}^2+\mu r_{12}(r_{11}+r_{22})+\theta r_{11}-(\lambda+\mu)r_{12}=0.$$

Solving the above equations, we can obtain *R*. ■

Lemma 2.2.

The QBD process $(Q(t), J(t))$ is positive recurrent if and only if $\rho < 1$.

Proof:

Based on Theorem 3.1.1 in Neuts (1981), the QBD process $\{(Q(t), J(t)), t \geq 0\}$ is positive recurrent (ergodic) iff the spectral radius *SP(R)* of the rate matrix *R* satisfies $SP(R) < 1$ and the linear system of equations $xB[R] = 0$ have positive solution, where x is $2N+1$-dimensional row vector and $B[R]$ is a $2N+1$-order matrix given by

$$B[R]=\begin{pmatrix} A_{00} & A_{01} & & & & & \\ B_{10} & A & C & & & & \\ & B_2 & A & C & & & \\ & & B_3 & A & C & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B_N & A & C \\ & & & & & B & RB+A \end{pmatrix}.$$

$B[R]$ is an irreducible and aperiodic generator with finite state. Therefore, $xB[R] = 0$ has a positive solution. Thus, the process $(Q(t), J(t))$ is positive recurrent if and only if

$$SP(R) = max(r, \rho) < 1.$$

Note that $0 < r < 1$, the above relation means that $\rho < 1$. ■

2.1. Stationary Distribution of Queue Length

If $\rho < 1$, let (Q, J) be the stationary limit of the QBD process $\{(Q(t), J(t)), t \geq 0\}$ and define

$$\Pi_k = (\pi_{k0}, \pi_{k1}), \quad k \geq 1,$$

$$x = (\pi_{00}, \Pi_1, \Pi_2, ..., \Pi_N),$$

$$\pi_{kj} = P\{Q = k, J = j\} = \lim_{t\to\infty}\{(Q(t) = k, J(t) = j\}, \quad (k,j) \in S.$$

In order to derive the stationary distribution of (Q, J), define a series of numbers as

$$\begin{cases} \beta_0 = 1, \\ \beta_k = 1 - \frac{\eta r}{\lambda} + \frac{\eta}{\lambda}\beta_{k-1} + \frac{\theta}{\lambda}\sum\limits_{v=0}^{k-1}\beta_N, \quad 1 \leq k \leq N-1. \end{cases} \tag{4}$$

Theorem 2.3.

If $\rho < 1$, the joint probability distribution of (Q, J) is

$$\pi_{k0} = \begin{cases} K\beta_{N-1-k}, & 0 \leq k \leq N-1, \\ K, & k = N, \end{cases} \tag{5}$$

$$\pi_{k1} = \begin{cases} K\left(\frac{\lambda}{\mu}\beta_{N-1} - \frac{\eta}{\mu}\beta_{N-2}\right), & k = 1, \\ K\Bigg\{(\frac{\lambda}{\mu})^{k-1}\left(\frac{\lambda}{\mu}\beta_{N-1} - \frac{\eta}{\mu}\beta_{N-2}\right) + \frac{r(\lambda+\theta)}{\lambda}\sum\limits_{j=1}^{k-1}(\frac{\lambda}{\mu})^j & \\ \quad + \frac{\theta}{\lambda}\sum\limits_{j=1}^{k-1}(\frac{\lambda}{\mu})^j \sum\limits_{v=k+1-j}^{N-1}\beta_{N-1-v}\Bigg\}, & 2 \leq k \leq N \end{cases}$$

and

$$\begin{cases} \pi_{k0} = \pi_{N0} r^{k-N}, & k > N, \\ \pi_{k1} = \pi_{N1}\rho^{k-N} + \frac{r(\lambda+\theta)}{\mu} + \sum\limits_{v=0}^{k-N-1} r^v \rho^{k-N-1-v}, & k > N, \end{cases} \tag{6}$$

where the constant K can be determined by the normalization condition

$$\sum_{k=0}^{\infty}\pi_{k,0} + \sum_{k=1}^{\infty}\pi_{k,1} = 1.$$

Proof:

The linear system of equations $xB[R] = 0$ can be written as

$$-\lambda\pi_{00} + \eta\pi_{10} + \mu\pi_{11} = 0, \tag{7}$$

$$\theta\pi_{10} - (\lambda+\mu)\pi_{11} + \mu\pi_{21} = 0, \tag{8}$$

$$\lambda\pi_{k-1,0} - (\lambda+\theta+\eta)\pi_{k0} + \eta\pi_{k+1,0} = 0, \quad 1 \leq k \leq N-1, \tag{9}$$

$$\lambda\pi_{k-1,1} + \theta\pi_{k0} - (\lambda+\mu)\pi_{k1} + \mu\pi_{k+1,1} = 0, \quad 2 \leq k \leq N-1, \tag{10}$$

$$\lambda\pi_{N-1,0} - \frac{\lambda}{r}\pi_{N0} = 0, \tag{11}$$

$$\lambda\pi_{N-1,1} + (\lambda+\theta)\pi_{N0} - \mu\pi_{N1} = 0. \tag{12}$$

From (11), we have

$$\pi_{N,0} = r\pi_{N-1,0}. \tag{13}$$

Substituting (13) into (9), we recursively obtain

$$\pi_{k,0} = \pi_{N-1,0}\,\beta_{N-1-k}, \qquad 0 \leq k \leq N-1.$$

Denoting $\pi_{N-1,0} = K$, then (5) is derived.

From (12), we have

$$\mu\pi_{N1} - \lambda\pi_{N-1,1} = (\lambda+\theta)\pi_{N0}. \tag{14}$$

Using (14) and (10), we recursively obtain

$$\mu\pi_{k1} = \lambda\pi_{k-1,1} + (\lambda+\theta)\pi_{N0} + \theta\sum_{v=k}^{N-1}\pi_{v0}, \quad 2 \le k \le N-1.$$

After manipulating, we recursively achieved

$$\pi_{k1} = \left(\frac{\lambda}{\mu}\right)^{k-1}\pi_{1,1} + \frac{(\lambda+\theta)}{\lambda}\pi_{N0}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j} + \frac{\theta}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}\sum_{v=k+1-j}^{N-1}\pi_{v0}, \quad 2 \le k \le N. \tag{15}$$

Applying (5) in (7), we get

$$\pi_{11} = K\left[\frac{\lambda}{\mu}\beta_{N-1} - \frac{\eta}{\mu}\beta_{N-2}\right]. \tag{16}$$

Substituting (16) and (5) into (15), we have

$$\pi_{k1} = K\Bigg\{\left(\frac{\lambda}{\mu}\right)^{k-1}\left(\frac{\lambda}{\mu}\beta_{N-1} - \frac{\eta}{\mu}\beta_{N-2}\right) + \frac{r(\lambda+\theta)}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j} + \frac{\theta}{\lambda}\sum_{j=1}^{k-1}\left(\frac{\lambda}{\mu}\right)^{j}\sum_{v=k+1-j}^{N-1}\beta_{N-1-v}\Bigg\}, \quad 2 \le k \le N.$$

Furthermore, using the matrix geometric solution method (Neuts (1981)), we obtain

$$\Pi_k = \Pi_N R^{k-N} = (\pi_{N,0}, \pi_{N,1})R^{k-N}, \quad k > N.$$

From (2), we get

$$R^k = \begin{pmatrix} r^k & \frac{r(\lambda+\theta)}{\mu}\sum_{j=0}^{k-1} r^j\rho^{k-1-j} \\ 0 & \rho^k \end{pmatrix}.$$

Hence, the theorem is proved. ■

Remark 2.4.

The results are the same as those of Li and Tian (2007) when $N = 1$. Therefore, the model is a generalization of Li and Tian (2007).

3. A Special Case of $N = 2$

It is difficult to obtain the stochastic decomposition structures of the mean queue length and mean waiting time at a steady state of this model as the distribution expressions of these indices are very complicated and hard to operate. Hence, we analyze the stochastic decomposition structure of the steady state indices in a special case $N = 2$.

Suppose that $N = 2$, then the system of equations (7) to (12) take the form

$$
\begin{aligned}
-\lambda\pi_{00} + \eta\pi_{10} + \mu\pi_{11} &= 0, \\
\theta\pi_{10} - (\lambda+\mu)\pi_{11} + \mu\pi_{21} &= 0, \\
\lambda\pi_{0,0} - (\lambda+\theta+\eta)\pi_{10} + \eta\pi_{2,0} &= 0, \\
\lambda\pi_{1,0} - \frac{\lambda}{r}\pi_{20} &= 0, \\
\lambda\pi_{1,1} + (\lambda+\theta)\pi_{20} - \mu\pi_{21} &= 0.
\end{aligned}
$$

Assume that $\pi_{0,0} = K$, then we obtain

$$
\begin{aligned}
\Pi_1 &= K\left(\frac{\lambda}{\lambda+\theta+(1-r)\eta}, \frac{\rho(\lambda+\theta+r\eta)}{\lambda+\theta+(1-r)\eta}\right), \\
\Pi_2 &= K\left(\frac{\rho\lambda}{\lambda+\theta+(1-r)\eta}, \frac{\rho(\rho(\lambda+\theta+r\eta)+r(\lambda+\theta))}{\lambda+\theta+(1-r)\eta}\right).
\end{aligned}
$$

Using the matrix geometric solution method (Neuts (1981)), we obtain

$$(\pi_{k0}, \pi_{k1}) = (\pi_{20}, \pi_{21})R^{k-2},$$

thus, we have

$$\pi_{k0} = K\frac{\lambda}{\lambda+\theta+(1-r)\eta}r^{k-1}, \qquad k \geq 2,$$

$$\pi_{k1} = K\left[\frac{\rho(\lambda+\theta)}{\lambda+\theta+(1-r)\eta}\sum_{j=2}^{k-1} r^j\rho^{k-1-j} + \rho^{k-1}\frac{\rho(\lambda+\theta+r\eta)+r(\lambda+\theta)}{\lambda+\theta+(1-r)\eta}\right], \quad k \geq 2, \tag{17}$$

where *K* can be determined by the normalization condition as

$$
\begin{aligned}
K = &\left\{1 + \frac{\lambda}{(1-r)(\lambda+\theta+(1-r)\eta)} + \frac{\rho r(\lambda+\theta)}{(1-r)(1-\rho)(\lambda+\theta+(1-r)\eta)}\right. \\
&\left. + \frac{\rho(\lambda+\theta-r\mu_v)}{(1-\rho)(\lambda+\theta+(1-r)\eta)}\right\}^{-1}.
\end{aligned} \tag{18}
$$

From (17), the probabilities that the system is in a working vacation period and in a regular busy period are as follows, respectively

$$
\begin{aligned}
P(J=0) &= \sum_{k=0}^{\infty}\pi_{k0} = K\left[1 + \frac{\lambda}{(1-r)(\lambda+\theta+(1-r)\eta)}\right], \\
P(J=1) &= \sum_{k=1}^{\infty}\pi_{k1} = K\left[\frac{\rho r(\lambda+\theta)}{(1-r)(1-\rho)(\lambda+\theta+(1-r)\eta)} + \frac{\rho(\lambda+\theta-r\mu_v)}{(1-\rho)(\lambda+\theta+(1-r)\eta)}\right].
\end{aligned}
$$

Theorem 3.1.

If $\rho < 1$ and $\mu > \eta$, the stationary queue length L in system can be decomposed into a sum of two independent random variables: $Q = Q_0 + Q_d$, where Q_0 is the stationary queue length of the classical $M/M/1$ queue without vacation and follows a geometric distribution with parameter 1-ρ and the additional queue length Q_d has a modified geometric distribution

$$P\{Q_d\text{=k}\} = \begin{cases} K^*\phi_1, & K=0, \\ K^*\phi_2, & K=1, \\ K^*\phi_3(1-r)r^{k-1}, & K\geq 2, \end{cases} \tag{19}$$

where $\phi_1 = 1-r, \quad \phi_2 = \dfrac{(\mu-\eta)(1-r)\rho}{\lambda+\theta+(1-r)\eta}, \quad \phi_3 = \dfrac{(\mu-\eta)\rho}{\lambda+\theta+(1-r)\eta},$

$$K^* = \left[(1-r)(1-\rho) + \frac{\lambda(1-\rho)}{(\lambda+\theta+(1-r)\eta)} + \frac{\rho r(\lambda+\theta)}{(\lambda+\theta+(1-r)\eta)} + \frac{\rho(1-r)(\lambda+\theta-r\eta)}{(\lambda+\theta+(1-r)\eta)}\right]^{-1}.$$

Proof:

Using (17), the PGF of Q can be expressed as follows

$$\begin{aligned}
Q(z) &= \sum_{k=0}^{\infty}\pi_{k0}z^k + \sum_{k=1}^{\infty}\pi_{k1}z^k \\
&= K\left\{1 + \frac{\lambda}{\lambda+\theta+\eta(1-r)}\frac{z}{1-rz} + \frac{\rho(\lambda+\theta-\eta r)z}{\lambda+\theta+\eta(1-r)} + \frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{r^2z^3}{(1-rz)(1-\rho z)}\right. \\
&\quad \left. + \frac{\rho(\lambda+\theta-\eta r)+\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{z^2}{(1-\rho z)}\right\} \\
&= \frac{1-\rho}{1-\rho z}K^*\left\{(1-r)(1-\rho z) + \frac{\lambda}{\lambda+\theta+\eta(1-r)}\left((1-r)z + (r-\rho)\frac{(1-r)z^2}{1-rz}\right)\right. \\
&\quad \left. + \frac{\rho(\lambda+\theta-\eta r)z}{\lambda+\theta+\eta(1-r)} + \frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}r(1-r)z^2 + \frac{\rho(\lambda+\theta)}{\lambda+\theta+\eta(1-r)}\frac{r^2(1-r)z^3}{(1-rz)}\right\} \\
&= \frac{1-\rho}{1-\rho z}K^*\left[\phi_1 + \phi_2 z + \phi_3\frac{r(1-r)}{1-rz}z^2\right] \\
&= \frac{1-\rho}{1-\rho z}Q_d(z).
\end{aligned}$$

It is easy to verify that $\phi_1 + \phi_2 + r\phi_3 = (K^*)^{-1}$, therefore, $Q_d(z)$ is a PGF. Expanding $Q_d(z)$ in power series of z, we get the distribution of additional number of customers Q_d. With the stochastic decomposition structure in Theorem 3.1, we can easily get the means

$$E(Q_d) = K^*\left[\phi_2 + \frac{r(2-r)}{1-r}\phi_3\right], \qquad E(Q) = \frac{\rho}{1-\rho} + E(Q_d). \qquad \blacksquare$$

Theorem 3.2.

If $\rho < 1$ and $\mu > \eta$, the waiting time W of an arrival can be decomposed into the sum of two independent variables: $W = W_0 + W_d$, where W_0 is the waiting time of an arrival in a corresponding classical $M/M/1$ queue and is exponentially distributed with parameter $\mu(1-\rho)$ and W_d is the additional delay with the LST given by

$$W_d^*(s) = K^* \left\{ \psi_1 + \psi_1 \frac{\alpha}{\alpha + s} \right\}, \tag{20}$$

where

$$\alpha = \frac{\lambda(1-r)}{r}, \qquad \psi_1 = \phi_1 + \phi_2 - \phi_3 \frac{1-r^2}{r}, \qquad \psi_2 = \frac{\phi_3}{r}.$$

Proof:

The classical relationship between the PGF of Q and the LST of waiting time W is

$$Q(z) = W^*(\lambda(1-z)).$$

From Theorem 3.1, we get

$$Q(z) = \frac{1-\rho}{1-\rho z} K^* \left[\phi_1 + \phi_2 z + \phi_3 \frac{r(1-r)}{1-rz} z^2 \right]. \tag{21}$$

Taking $z = 1 - \frac{s}{\lambda}$ in (21) and denoting $\frac{\lambda(1-r)}{r} = \alpha$, we get

$$\begin{aligned} W^*(s) &= \frac{\mu(1-\rho)}{\mu(1-\rho)+s} K^* \left\{ \phi_1 + \phi_2 (1 - \frac{s}{\lambda}) + \phi_3 \frac{(1-r)}{\lambda} \left[\frac{\frac{\lambda}{r}}{\alpha + s} - (2\lambda + \alpha) + s \right] \right\} \\ &= \frac{\mu(1-\rho)}{\mu(1-\rho)+s} K^* \left[\psi_1 + \psi_2 \frac{\alpha}{\alpha + s} \right] \\ &= \frac{\mu(1-\rho)}{\mu(1-\rho)+s} W_d^*(s). \end{aligned}$$

It is easy to verify that $\psi_1 + \psi_2 = \phi_1 + \phi_2 + r\phi_3 = (K^*)^{-1}$. Therefore, $W_d^*(s)$ is a LST. ■

The result of Theorem 3.2 indicates that additional delay W_d equals zero with probability $K^*\psi_1$ and follows an exponential distribution with parameter α. It is easy to obtain

$$E(W_d) = K^* \psi_2 \frac{1}{\alpha} = \frac{1}{\lambda} E(Q_d), \qquad E(W) = \frac{1}{\mu(1-\rho)} + E(W_d).$$

4. Numerical Results

In this section, we illustrate the influence of the system parameters on the performance measures by presenting some numerical examples. Figures 1 and 2 depict the expected queue length $E(Q)$ against the vacation service rate η for different values of θ and ρ respectively. In Figure 3, we present the state probability of the server for the change of η and different vacation rate θ . Figure 4 gives the comparison of our model ($M/M/1/MWV + VI$) with $M/M/1/MWV$ (Liu et al.(2007)) in terms of mean queue length. Figure 5 shows how the mean waiting time $E(W)$ changes with the mean vacation time and presents the comparison of the mean waiting time in our model with two different vacation policies i.e the multiple vacation (MV) and the multiple working vacation (MWV). The main findings in this study are itemized as follows.

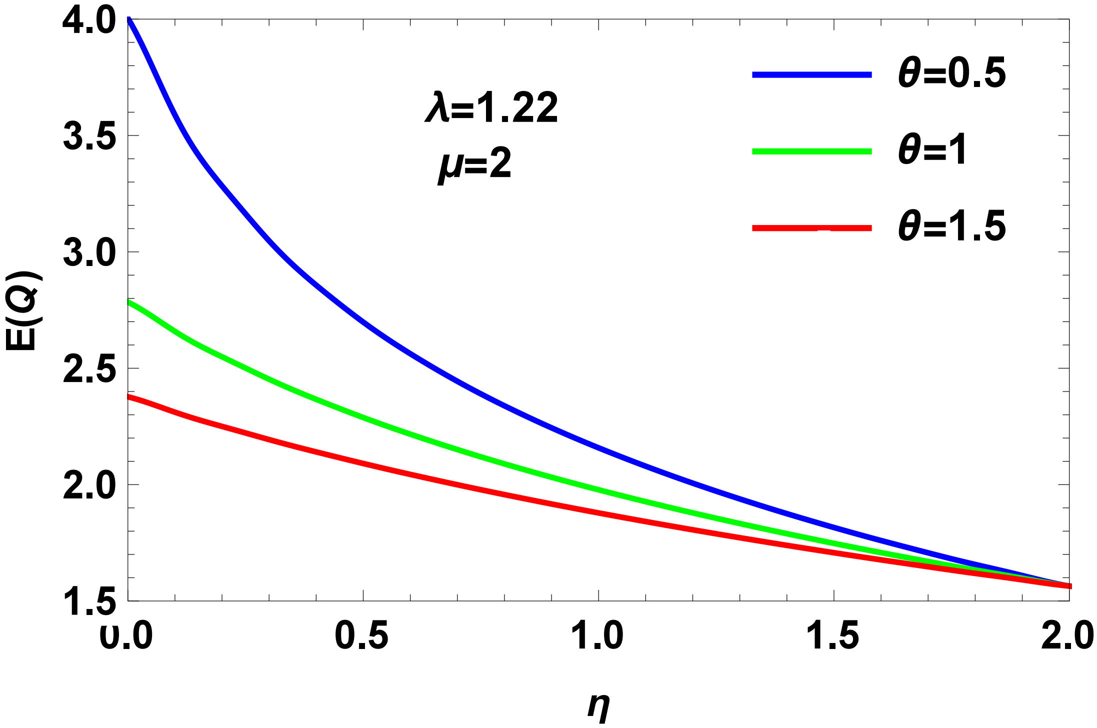

Figure 1. The effect of η on $E(Q)$ for different values of θ.

- As explained in Figures 1 and 2, with the increase in vacation service rate η, the mean queue length $E(Q)$ apparently decreases. Meanwhile, when the vacation service rate η tends to $\mu = 2$, $E(Q)$ will approach to a constant value and the model reduces to the corresponding queue without vacation, regardless of the length of vacation times. Furthermore, when we increase the values of θ and ρ, the expected number of customers in the queue decreases and increases respectively.
- From Figure 3, the probability that the server stays in working vacation $P(J = 0)$, evidently increases and the probability that the server remains in normal working level $P(J = 1)$ decreases with an increase in vacation service rate η . Hence, the utilization level of the system idle time becomes larger. Moreover, the state probability of the server is also affected by the vacation rate θ . For example, when $\theta = 1.5$, $P(J = 0)$ are evidently lesser than those for $\theta = 0.5$.
- From Figure 4, $M/M/1/MWV$ (Liu et al. (2007)) yields higher mean queue length $E(Q)$ when compared to our model $M/M/1/MWV + VI$ for fixed η and hence in the former case more customers wait in the queue. Therefore, the vacation interruption policy is appreciably more desirable in terms of $E(Q)$. Thus, we can provide a better service, if we consider vacation interruptions under working vacation policy so that we can make use of server productively and consequently decrease the waiting time of customers.
- As illustrated in Figure 5, increase in θ^{-1} leads to an increase in $E(W)$ and when θ^{-1} advances towards 0, $E(W)$ will approach to a constant value i.e. our model becomes a classical $M/M/1$ queue. Moreover, $MWV + VI$ policy performs better than the MV policy and the MWV policy, as the server will return back to a regular busy level more frequently if the mean vacation time is longer. Consequently, more customers are served at a higher rate.

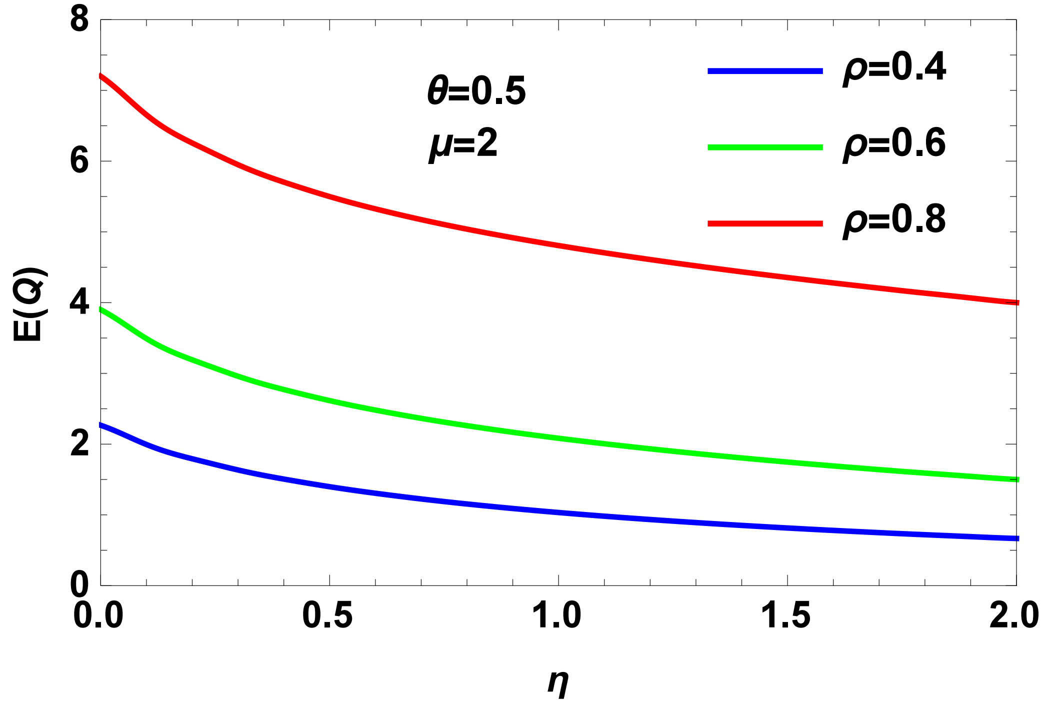

Figure 2. The effect of η on $E(Q)$ for different values of ρ.

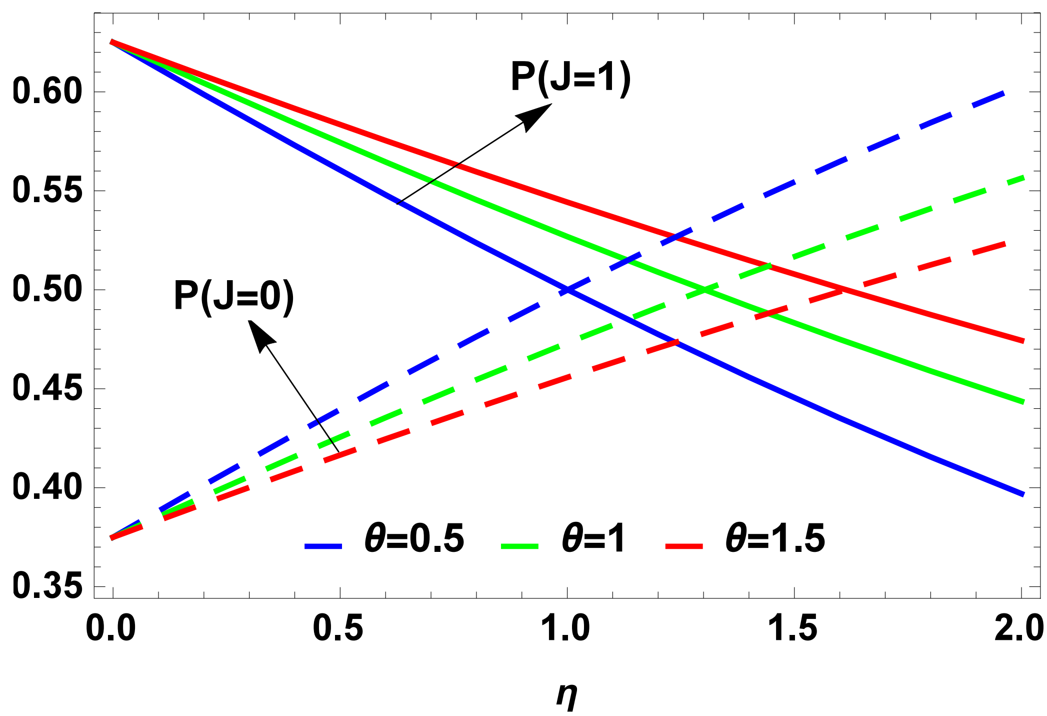

Figure 3. The state probability of the server with the change of η

5. Conclusion

In this paper, we have carried out the analysis of $M/M/1$ queue with working vacation and vacation interruption. Using the QBD process and matrix-geometric solution technique, we obtained

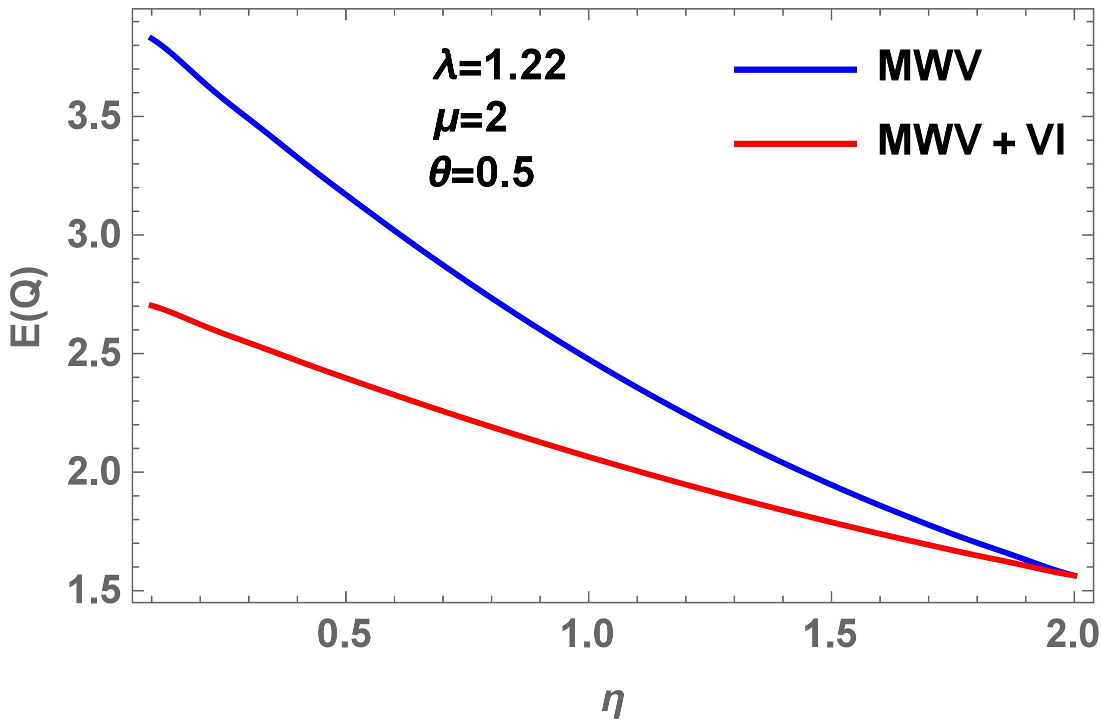

Figure 4. The comparison of models without and with vacation interruption.

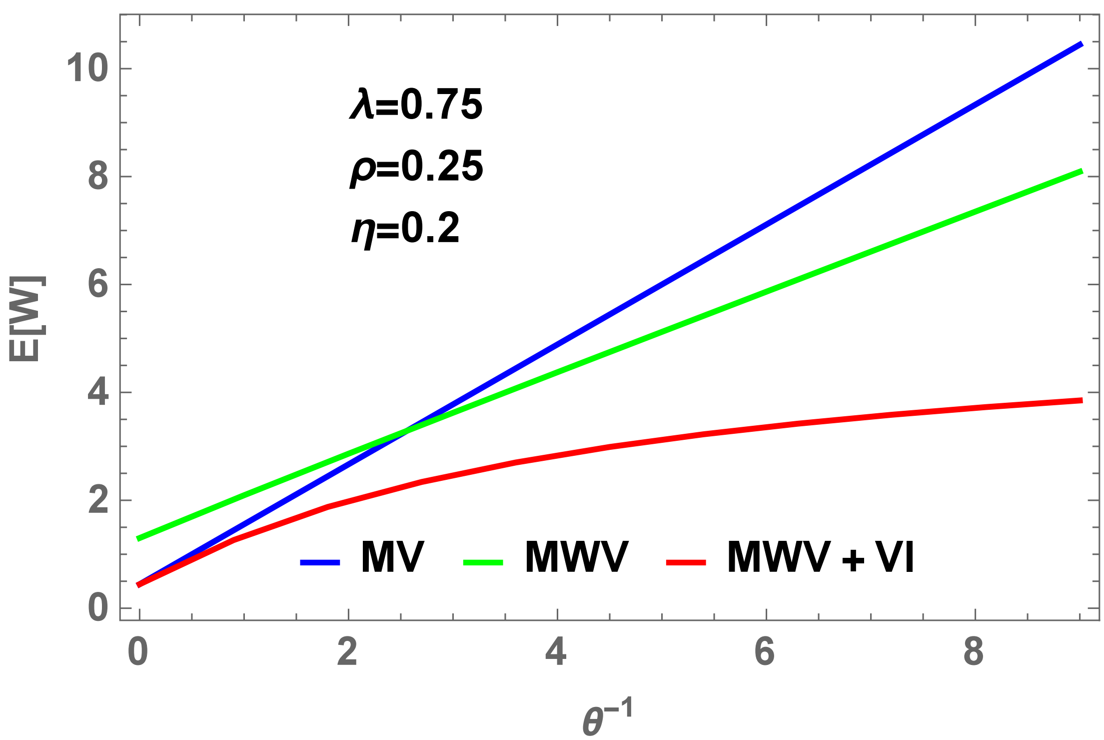

Figure 5. Comparisons among different models.

the stationary probability distribution of the number of customers in the system. Furthermore, we also derived the conditional stochastic decomposition structures and some performance measures for $N = 2$. We find that $MWV + VI$ policy outperforms both the MV policy and MWV policy. Therefore, the vacation interruption policy is appreciably more desirable in terms of mean queue length and mean waiting time. Hence, we can achieve a better service, if we consider vacation interruptions under working vacation policy so that we can make use of server effectively and consequently decrease the waiting time of customers. For future research, one can consider a $M/M/1$ queue with working vacations and vacation interruption under N policy.

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