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Analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queueing system with priority services, working breakdown, collision, Bernoulli vacation, immediate feedback, starting failure and repair

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Abstract

This paper considers an $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two types of customers from different classes arrive at the system in different independent compound Poisson processes. The server follows the non-pre-emptive priority rule subject to working breakdown, Bernoulli vacation, starting failure, immediate feedback, collision and repair. After completing each service, the server may go for a vacation or remain idle in the system. The priority customers who find the server busy are queued in the system. If a low-priority customer finds the server busy, he is routed to orbit that attempts to get the service. The system may become defective at any point of time while in operation. However, when the system becomes defective, instead of stopping service completely, the service is continued to the interrupted customer only at a slower rate. Using the supplementary variable technique, the joint distribution of the server state and the number of customers in the queue are derived. Finally, some performance measures are obtained.

Keywords: Batch arrival; Priority queue; Retrial queue; Feedback; Collision; Modified Bernoulli vacation

MSC 2010 Number: $60K25,\ 68M30,\ 90B22$

1. Introduction

The study on queuing models has become an indispensable area due to its wide applicability in real life situations, all the models considered have the property that units proceed to service on a first-come, first-served basis. This is obviously not only manner of service, and there are many alternatives, such as last-come, first-served, the selection in random order and selection by priority. In order to offer different quality of service for different kinds of customers, we often control a queueing system by priority mechanism. This phenomenon is common in practice. For example, in telecommunication transfer protocol, for guaranteeing different layers service for different customers, priority classes control may appear in header of IP package or in ATM cell. Priority control is also widely used in production practice and transportation management.

Retrial queues are characterized by the feature that arriving customers who find the server busy join the retrial group to try their luck again after a random time period. Queues in which customers are allowed to conduct retrials have been extensively used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processor unit.

Several authors have studied a single server retrial queue with orbital search and some of them have studied retrial with collisions. Ayyappan et al $(2009, 2018)$ have studied an $M/M/1$ retrial queueing system with non preemptive priority service and single vacation exhaustive service and retrial queueing system with priority services, working vacations and vacation interruption, emergency vacation, negative arrival and delayed repair, Atencia et al (2005) have studied a single-server retrial queue with general retrial times and non-preemptive priority service. Kim et al. (2016) have studied the $M/G/1$ queue with disasters and working breakdowns. Liou (2013) has discussed a Markovian queue optimisation analysis with an unreliable server subject to working breakdowns and impatient customers.

Yang et al. (2017) have studied the analysis of a finite-capacity system with working breakdowns and retention of impatient customers. Choudhury et al (2012) have studied a batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair, Gomez (1999) has studied a stochastic analysis of a single server retrial queue with general retrial times, Haghighi et al. $(2006, 2013, 2016)$ have studied a parallel priority queueing system with finite buffers, Stochastic Three-stage Hiring Model as a Tandem Queueing Process with Bulk Arrivals and Erlang Phase-Type Selection: $M^K/M^{(k,K)}/1 - M^Y/E_r/1 - \infty$ and delayed network queues. Jain et al (2008) have studied a bulk arrival retrial queue with unreliable server and priority subscribers.

Wang et al. (2010) have studied a batch arrival retrial queue with starting failures, feedback and admission control. Kirupa et al. (2010) have studied a single-server retrial queueing system with two different vacation policies. Krishnakumar et al. (2002) have discussed an $M/G/1$ retrial queue with feedback and starting failures Krishnakumar et al. (2010) have studied a single server feedback retrial queue with collisions, Madan (2011) have studied a non-preemptive priority queueing system with a single server serving two queues $M/G/1$ and $M/D/1$ with optional server vacations based on exhaustive service of the priority units, Chen et al. (2016) have studied a batch arrival retrial G-queue with orbital search and non-persistent customers Varalakshmi et al. (2016) studied an $M/G/1$ retrial queueing system with two phases of service, immediate Bernoulli feedbacks, single vacation and starting failures and Rajadurai et al. (2017) studied a $M/G/1$ feedback retrial queue subject to server breakdown and repair under multiple working vacation policy. Yang et al. (1994) have studied an approximation method for the $M/G/1$ retrial queue with general retrial times.

In this paper we deal with the analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two types of customers from different classes arrive at the system in different independent compound Poisson processes. The server follows the non-pre-emptive priority rule subject to working breakdown, Bernoulli vacation, starting failure, immediate feedback, collision and repair. After completing each service, the server may go for a vacation or remain idle in the system. The priority customers who find the server busy are queued in the system. If a low-priority customer finds the server busy, he is routed to orbit that attempts to get the service. The system may become defective at any point of time while in operation. However, when the system is defective, instead of stopping service completely, the service continues only to the interrupted customer at a slower rate. We assume that the probability of successful commencement of service is δ for a new customer or customer from the orbit. The retrial time, service time, vacation time, repair time are all assumed to follows general (arbitrary) distribution and breakdown time follows exponential distribution. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly. Also some performance measures such as the average number of customer in the priority queue and the non-priority in the orbit and the average waiting time are derived.

The rest of the paper is organized as follows: Mathematical description of the model is presented in Section (2). Definitions, governing equations and the time dependent solution have been obtained in Section (3) and (4). The corresponding steady state results have been derived explicitly in Section (5). Average queue size and the average waiting time are computed in Section (6) and (7). Some particular cases are discussed in Section (8).

2. Mathematical description of our model

(i) High priority and Low-priority units arrive at the system in batches of variable sizes in different compound Poisson processes and they are provided service one by one on a FCFS basis. Let $\lambda_1 c_i\, dt$ and $\lambda_2 c_i\, dt$ $(i = 1, 2, 3, ...)$ be the first order probability that a batch of i customers arrive at the system during a short interval of time $(t, t + dt)$, where $0 \le c_i \le 1$, $\sum_{i=1}^{\infty} c_i = 1$, and $\lambda_1 > 0$, $\lambda_2 > 0$ are the average arrival rates of high priority and low-priority customers and high priority customers only form the queue, if the server is busy. The server must serve all the high priority units present in the system before taking up low-priority units for service. In other words, there is no high priority unit present in the system at the time of starting the service of a low-priority unit. Further, we assume that the server follows a non-pre-emptive priority rule, which means that if one or more high priority units arrive during the service time of a low-priority unit, the current service of a low-priority units is not stopped and a high priority unit will be taken up for service only after the current service of a low-priority unit is completed.

(ii) Low-priority customers are considered as retrial customers. It is assumed that there is no waiting space and on arrival, a customer proceeds to the server with probability p_2 or enters into the orbit with probability q_2. If the server is busy with low-priority customer, the arriving low-priority customer collides with the customer in service resulting in both shifted to the orbit. The retrial time, that is time between successive repeated attempts of each customer in the orbit is assumed to be generally distributed with distribution function $A(x)$, density function $a(x)$. The conditional completion rate for retrials is given by $\eta(x) = \frac{a(x)}{(1-A(x))}$.

(iii) Each customer with high priority and low-priority provided is served by a single server on a first come - first served basis. The service time for both high priority and low-priority units follow general(arbitrary) distribution with distribution functions $B_i(v)$ and the density functions $b_i(v),\ i = 1, 2$.

(iv) Let $\mu_i(x)dx$ be the conditional probability of completion of the high priority and low-priority unit service during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, i = 1, 2$$

and therefore,

$$b_i(v) = \mu_i(v)e^{-\int_0^s \mu_i(x)dx}, i = 1, 2.$$

(v) We further assume that as soon as each service completed, the server has the option to take a vacation of random length with probability θ, in which case the vacation starts immediately or else with probability $(1 - \theta)$ he may decide to remain idle in the system waiting for the new units to arrive.

(vi) The vacation time follows general (arbitrary) distribution with distribution function $V(s)$ and density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)}$$

and therefore,

$$v(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}.$$

(vii) The server may become inactive during busy period. At the time of breakdown the customer who is in service will get fresh service continuously by slower service rate $\mu_w(x)$ and it follows a general distribution. Breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. The repair starts after the current service is completed.

(viii) The Repair time (due to active breakdown) follows general (arbitrary) distribution with distribution function $R_i(t)$ and density function $r_i(t),\ i = 1, 2$. Let $\beta(x)dx$ be the conditional probability of completion of a repair during the interval $(x, x + dx]$, given that the elapsed repair time is x, so that

$$\beta_i(x) = \frac{r_i(x)}{1 - R_i(x)}, i = 1, 2$$

and therefore,

$$r_i(t) = \beta(t)e^{-\int_0^s \beta(x)dx}, i = 1, 2.$$

(ix) The Repair time(due to starting failure) follows general (arbitrary) distribution with distribution function $G_i(t)$ and the density function $g_i(t),\ i = 1, 2.$ Let $\nu(x)dx$ be the conditional probability of completion of a repair during the interval $(x, x + dx]$, given that the elapsed repair time is x, so that

$$\nu(x) = \frac{g_i(x)}{1 - G_i(x)}, i = 1, 2$$

and therefore,

$$r_i(t) = \nu(t)e^{-\int_0^s \nu(x)dx}, i = 1, 2.$$

(x) At the completion of a service for a low priority customer, if he is not satisfied with the service permitted to the head of queue immediately ask for new service as a feedback customer with probability r .

(xi) Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and notations

We define

(i) $P^1_{m,n}(x, t)$ = Probability that at time t, the server is active providing service and there are $m\ (m \geq 0)$ high priority units in the queue and $n\ (n \geq 0)$ low-priority units in the orbit excluding the one high priority unit in service with elapsed service time for this customer is x.

(ii) $P^2_{m,n}(x, t)$ = Probability that at time t, the server is active providing service and there are $m\ (m \geq 0)$ high priority units in the queue and $n\ (n \geq 0)$ low-priority units in the orbit excluding the one low-priority unit in service with elapsed service time for this customer is x.

(iii) $V_{m,n}(x, t)$ = Probability that at time t, the server is on vacation with elapsed vacation time x and there are $m\ (m \geq 0)$ high priority units in the queue and $n\ (n \geq 0)$ low-priority units in the orbit.

(iv) $W^{(i)}_{m,n}(x, t)$ = Probability that at time t, the server is on slower rate service(server inactive due to active breakdown) with elapsed service time is x and there are $m\ (m \geq 0)$ high priority units in the queue and $n\ (n \geq 0)$ low-priority units in the orbit.

(v) $R^{(i)}_{m,n}(x, t)$ = Probability that at time t, the server is on repair(server inactive due to active breakdown) with elapsed repair time is x and there are $m\ (m \geq 0)$ high priority units in the queue and $n\ (n \geq 0)$ low-priority units in the orbit.

(vi) $G^{(i)}_{m,n}(x,t)$ = Probability that at time t, the server is on repair(server inactive due to stating failure) with elapsed repair time is x and there are m $(m \geq 0)$ high priority units in the queue and n $(n \geq 0)$ low-priority units in the orbit.

(vii) $I_{0,0}(t)$ = Probability that at time t, there are no high priority and low-priority customers in the system and the server is idle but available in the system.

4. Equations Governing the System

The Kolmogorov forward equations to govern the model are

$$\frac{d}{dt}I_{(0,0)}(t) = -(\lambda_1+\lambda_2)I_{(0,0)}(t) + (1-\theta)\int_0^\infty P^1_{0,0}(x,t)\mu_1(x)dx$$
$$+ (1-\theta)(1-r)\int_0^\infty P^2_{0,0}(x,t)\mu_2(x)dx + \int_0^\infty R^i_{0,0}(x,t)\beta(x)dx + \int_0^\infty V_{0,0}(x,t)\gamma(x)dx, \quad (1)$$

$$\frac{\partial}{\partial t}I_{(0,n)}(x,t) + \frac{\partial}{\partial x}I_{(0,n)}(x,t) = -(\lambda_1+\lambda_2+\eta(x))I_{(0,n)}(x,t); n \geq 1, \quad (2)$$

$$\frac{\partial}{\partial t}P^1_{m,n}(x,t) + \frac{\partial}{\partial x}P^1_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu_1(x)+\alpha)P^1_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iP^1_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iP^1_{m,n-i}(x,t);\ m,\ n \geq 0, \quad (3)$$

$$\frac{\partial}{\partial t}P^2_{m,n}(x,t) + \frac{\partial}{\partial x}P^2_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu_2(x)+\alpha)P^2_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iP^2_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iP^2_{m,n-i}(x,t);\ m,\ n \geq 0, \quad (4)$$

$$\frac{\partial}{\partial t}W^i_{m,n}(x,t) + \frac{\partial}{\partial x}W^i_{m,n}(x,t) = -(\lambda_1+\lambda_2+\mu_w(x))W^i_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iW^i_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iW^i_{m,n-i}(x,t);\ m,\ n \geq 0, \quad (5)$$

$$\frac{\partial}{\partial t}R^i_{m,n}(x,t) + \frac{\partial}{\partial x}R^i_{m,n}(x,t) = -(\lambda_1+\lambda_2+\beta(x))R^i_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iR^i_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iR^i_{m,n-i}(x,t);\ m,\ n \geq 0, \quad (6)$$

$$\frac{\partial}{\partial t}G^i_{m,n}(x,t) + \frac{\partial}{\partial x}G^i_{m,n}(x,t) = -(\lambda_1+\lambda_2+\nu(x))G^i_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iG^i_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iG^i_{m,n-i}(x,t);\ m,\ n \geq 0, \quad (7)$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t) + \frac{\partial}{\partial x}V_{m,n}(x,t) = -(\lambda_1+\lambda_2+\gamma(x))V_{m,n}(x,t)$$
$$+ \lambda_1\sum_{i=1}^{m}(1-\delta_{m0})C_iV_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}(1-\delta_{0n})C_iV_{m,n-i}(x,t);\ m,\ n \geq 0. \quad (8)$$

The above set of equations are to be solved under the following boundary conditions at $x = 0$.

$$I_{(0,1)}(0,t) = (1-\theta)\int_0^\infty P_{0,1}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{0,1}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{0,1}(x,t)\gamma(x)dx + \int_0^\infty R_{0,1}^i(x,t)\beta(x)dx + \int_0^\infty G_{0,1}^2(x,t)\nu(x)dx, \tag{9}$$

$$I_{(0,2)}(0,t) = (1-\theta)\int_0^\infty P_{0,2}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{0,2}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{0,2}(x,t)\gamma(x)dx + \int_0^\infty R_{0,2}^i(x,t)\beta(x)dx + \int_0^\infty G_{0,2}^2(x,t)\nu(x)dx + \lambda_2 p_2 \int_0^\infty P_{0,0}^2(x,t)dx, \tag{10}$$

$$I_{(0,n)}(0,t) = (1-\theta)\int_0^\infty P_{0,n}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{0,n}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{0,n}(x,t)\gamma(x)dx + \int_0^\infty R_{0,n}^i(x,t)\beta(x)dx + \int_0^\infty G_{0,n}^2(x,t)\nu(x)dx + \lambda_2 p_2 \sum_{i=1}^{n-i} C_{n-i}\int_0^\infty P_{0,i-1}^2(x,t)dx; n \geq 3, \tag{11}$$

$$P_{m,0}^1(0,t) = \delta\lambda_1 C_{m+1} I_{(0,0)}(t) + (1-\theta)\int_0^\infty P_{m+1,0}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{m+1,0}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{m+1,0}(x,t)\gamma(x)dx + \int_0^\infty R_{m+1,0}^i(x,t)\beta(x)dx + \int_0^\infty G_{m+1,0}^i(x,t)\nu(x)dx, \tag{12}$$

$$P_{m,1}^1(0,t) = \delta\lambda_1 C_{m+1} I_{(0,1)}(t) + (1-\theta)\int_0^\infty P_{m+1,1}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{m+1,1}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{m+1,1}(x,t)\gamma(x)dx + \int_0^\infty R_{m+1,1}^i(x,t)\beta(x)dx + \int_0^\infty G_{m+1,1}^i(x,t)\nu(x)dx; m \geq 0, \tag{13}$$

$$P_{m,n}^1(0,t) = \delta\lambda_1 C_{m+1} I_{(0,n)}(t) + (1-\theta)\int_0^\infty P_{m+1,n}^1(x,t)\mu_1(x)dx + (1-\theta)(1-r)\int_0^\infty P_{m+1,n}^2(x,t)\mu_2(x)dx + \int_0^\infty V_{m+1,n}(x,t)\gamma(x)dx + \int_0^\infty R_{m+1,n}^i(x,t)\beta(x)dx + \int_0^\infty G_{m+1,n}^i(x,t)\nu(x)dx + \lambda_2 p_2 \sum_{i=1}^{n-1} C_{n-i}\int_0^\infty P_{m+1,i-1}^2(x,t)dx;\ m \geq 0, n \geq 2, \tag{14}$$

$$P^2_{0,0}(0,t)=\delta\lambda_2 C_1 I_{(0,0)}(t)+\delta\int_0^\infty I_{0,1}(x,t)\eta(x)dx+r\int_0^\infty P^2_{0,0}(x,t)\mu_2(x)dx, \tag{15}$$

$$P^2_{0,n}(0,t)=\delta\lambda_2 C_{n+1} I_{(0,0)}(t)+\delta\int_0^\infty I_{0,n}(x,t)\eta(x)dx+r\int_0^\infty P^2_{0,n}(x,t)\mu(x)dx$$
$$+\delta\lambda_2\sum_{i=1}^{n} C_i\int_0^\infty I_{0,n+1-i}(x,t)dx;\ n\geq 1, \tag{16}$$

$$W^i_{m,n}(0,t)=\alpha\int_0^\infty P^i_{m,n}(x,t)dx, \tag{17}$$

$$R^i_{m,n}(0,t)=\int_0^\infty W^i_{m,n}(x,t)\mu_w(x)dx, \tag{18}$$

$$G^1_{m,n}(0,t)=\overline{\delta}\lambda_1 C_m I_{(0,n)}(t), m\geq 1; n\geq 0, \tag{19}$$

$$G^2_{0,1}(0,t)=\overline{\delta}\lambda_2 C_1 I_{(0,0)}(t)+\overline{\delta}\int_0^\infty I_{0,1}(x,t)\eta(x)dx, \tag{20}$$

$$G^2_{0,n}(0,t)=\overline{\delta}\lambda_2 C_n I_{(0,n)}(t)+\overline{\delta}\int_0^\infty I_{0,n}(x,t)\eta(x)dx$$
$$+\overline{\delta}\lambda_2\sum_{i=1}^{n} C_i\int_0^\infty I_{0,n-i}(x,t)dx;\ n\geq 2, \tag{21}$$

$$V_{m,n}(0,t)=\theta\Big\{\int_0^\infty P^1_{m,n}(x,t)\mu_1(x)dx+\int_0^\infty P^2_{m,n}(x,t)\mu_2(x)dx\Big\}; m,\ n\geq 0. \tag{22}$$

We assume that initially there are no customers in the system so that the server is idle.

$$I_{0,0}(0)=1;\ P^i_{m,n}(0)=W^i_{m,n}(0)=R^i_{m,n}(0)=V_{m,n}(0)=0=G^i_{m,n}(0)=0;\ i=1,\ 2. \tag{23}$$

Next, we define the following probability generating functions:

$$I_0(0,z_2,t)=\sum_{n=1}^{\infty} I_{0,n}(0,t),\quad A(x,z_1,z_2,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(x,t), \tag{24}$$

where

$A=P^i,\ V,\ W^i,\ R^i, G^i;\ i=1,\ 2.$

By taking Laplace transforms from (1) to (22) and solving the equations, we get,

$$\overline{I}_{(0)}(x,z_2,s)=\overline{I}_{(0)}(0,z_2,s)e^{-(s+\lambda_1+\lambda_2)x-\int_0^x \eta(t)dt}, \tag{25}$$

$$\overline{P}^1(x,z_1,z_2,s)=\overline{P}^1(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]+\alpha\right)x-\int_0^x \mu_1(t)dt}, \tag{26}$$

$$\overline{P}^2(x,z_1,z_2,s)=\overline{P}^2(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]+\alpha+\lambda_2 p_2\right)x-\int_0^x \mu_2(t)dt}, \tag{27}$$

$$\overline{W}^i(x,z_1,z_2,s)=\overline{W}^i(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]\right)x-\int_0^x \mu_w(t)dt}, \tag{28}$$

$$\overline{R}^i(x,z_1,z_2,s)=\overline{R}^i(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]\right)x-\int_0^x\beta(t)dt}, \tag{29}$$

$$\overline{G}^i(x,z_1,z_2,s)=\overline{G}^i(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]\right)x-\int_0^x\nu(t)dt}, \tag{30}$$

$$\overline{V}(x,z_1,z_2,s)=\overline{V}(0,z_1,z_2,s)e^{-\left(s+\lambda_1[1-C(z_1)]+\lambda_2[1-C(z_2)]\right)x-\int_0^x\gamma(t)dt}, \tag{31}$$

where

$$\begin{aligned}\overline{I}_{(0)}(0,z_2,s)&=\overline{P}_0^1(0,z_2,s)\Big\{(1-\theta)\overline{B_1}(\psi_1(z,s))+\alpha[\frac{1-\overline{B_1}(\psi_1(z,s))}{\psi_1(z,s)}\Big]\overline{W}^1(A_1(z,s))\\&\overline{R}^1(A_1(z,s))+\theta\overline{V}(A_1(z,s))\overline{B_1}(\psi_1(z,s))\Big\}+\overline{P}_0^2(0,z_2,s)\Big\{\theta\overline{V}(A_1(z,s))(1-r)\\&\overline{B_2}(\psi_2(z,s))+(1-r)(1-\theta)\overline{B_2}(\psi_2(z,s))+\alpha[\frac{1-\overline{B_2}(\psi_2(z,s))}{\psi_2(z,s)}\Big]\overline{W}^2(A_1(z,s))\\&\overline{R}^2(A_1(z,s))+\lambda_2p_2\Big[\frac{1-\overline{B_2}(\psi_2(z,s))}{\psi_2(z,s)}\Big]\Big\}+\overline{\delta}\lambda_2C(z_2)\overline{I}_{0,0}(s)\overline{G}^2(A_1(z,s))+\overline{\delta}\\&\overline{I}_{(0)}(0,z_2,s)\Big\{\overline{M}(s+\lambda_1+\lambda_2)+\lambda_2p_2C(z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}]\Big\}\overline{G}^2(A_1(z,s))\\&-\Big\{(s+\lambda_1+\lambda_2)\overline{I}_{0,0}-1\Big\},\end{aligned} \tag{32}$$

$$\begin{aligned}&\Big\{z_1-(1-\theta)\overline{B_1}(\phi_1(z,s))-\theta\overline{V}(A(z,s))\overline{B_1}(\phi_1(z,s))-\alpha[\frac{1-\overline{B_1}(\phi_1(z,s))}{\phi_1(z,s)}\Big]\\&\overline{W}^1(A(z,s))\overline{R}^1(A(z,s))\Big\}\overline{P}^1(0,z_1,z_2,s)=\lambda_1C(z_1)\overline{I}_{(0)}(0,z_2,s)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}]\\&(\delta+\overline{\delta}\overline{G}^1(A(z,s)))+\overline{P}_0^2(0,z_2,s)\Big\{(1-r)(1-\theta)\overline{B_2}(\phi_2(z,s))+\theta\overline{V}(A(z,s))\\&(1-r)\overline{B_2}(\phi_2(z,s))+\alpha[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\Big]\overline{W}^2(A(z,s))\overline{R}^2(A(z,s))\\&+\lambda_2p_2\Big[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\Big]-(1-r)(1-\theta)\overline{B_2}(\psi_2(z,s))-\theta\overline{V}(A_1(z,s))(1-r)\\&\overline{B_2}(\psi_2(z,s))-\alpha[\frac{1-\overline{B_2}(\psi_2(z,s))}{\psi_2(z,s)}\Big]\overline{W}^2(A_1(z,s))\overline{R}^2(A_1(z,s))-\overline{P}_0^1(0,z_2,s)\Big\{(1-\theta)\\&\overline{B_1}(\psi_1(z,s))+\alpha[\frac{1-\overline{B_1}(\psi_1(z,s))}{\psi_1(z,s)}\Big]\overline{W}^1(A_1(z,s))\overline{R}^1(A_1(z,s))+\theta\overline{V}(A_1(z,s))\\&\overline{B_1}(\psi_1(z,s))\Big\}+\overline{\delta}\Big\{\lambda_2C(z_2)\overline{I}_{(0,0)}(s)+\overline{I}_{(0)}(0,z_2,s)\Big\{\overline{M}(s+\lambda_1+\lambda_2)+\lambda_2p_2C(z_2)\\&[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}]\Big\}\big\{\overline{G}^2(A(z,s)))-\overline{G}^2(A_1(z,s)))\big\}\Big\},\end{aligned} \tag{33}$$

(34)

$$\{z_2 - r\overline{B_2}(\psi_2(z,s))\}\overline{P}_0^2(0,z_2,s) = \delta\lambda_2 C(z_2)\overline{I}_{0,0}(s) + \delta\overline{I}_{(0)}(0,z_2,s)\Big\{\overline{M}(s+\lambda_1+\lambda_2)$$
$$+ \lambda_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\}. \quad (35)$$

We let $A(z,s) = s + \lambda_1[1 - C(z_1)] + \lambda_2[1 - C(z_2)], \phi_1(z,s) = A(z,s) + \alpha$ and $\phi_2(z,s) = A(z,s) + \alpha + \lambda_2 p_2$ $A_1(z,s) = s + \lambda_1 + \lambda_2[1 - C(z_2)], \psi_1(z,s) = A_1(z,s) + \alpha$ and $\psi_2(z,s) = A_1(z,s) + \alpha + \lambda_2 p_2$.

By applying Rouche's theorem, we get,

$$\overline{P}_0^1(0,z_2,s)\Big\{(1-\theta)\overline{B_1}(\psi_1(z,s)) + \alpha\big[\frac{1-\overline{B_1}(\psi_1(z,s))}{\psi_1(z,s)}\Big]\overline{W}^1(A_1(z,s))\overline{R}^1(A_1(z,s)) + \theta$$
$$\overline{V}(A_1(z,s))\overline{B_1}(\psi_1(z,s))\Big\} = \lambda_1 C(g[z_2])\overline{I}_{(0)}(0,z_2,s)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]$$
$$(\delta + \overline{\delta G}^1(A_2(z,s))) + \overline{P}_0^2(0,z_2,s)\Big\{(1-r)(1-\theta)\overline{B_2}(\phi_4(z,s)) + \theta\overline{V}(A_2(z,s))(1-r)$$
$$\overline{B_2}(\phi_4(z,s)) + \alpha\big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]\overline{W}^2(A_2(z,s))\overline{R}^2(A_2(z,s)) + \lambda_2 p_2\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]$$
$$- (1-r)(1-\theta)\overline{B_2}(\psi_2(z,s)) - \theta\overline{V}(A_1(z,s))(1-r)\overline{B_2}(\psi_2(z,s)) - \alpha\big[\frac{1-\overline{B_2}(\psi_2(z,s))}{\psi_2(z,s)}\Big]$$
$$\overline{W}^2(A_1(z,s))\overline{R}^2(A_1(z,s)) + \overline{\delta}\Big\{\lambda_2 C(z_2)\overline{I}_{(0,0)}(s) + \overline{I}_{(0)}(0,z_2,s)\Big\{\overline{M}(s+\lambda_1+\lambda_2)$$
$$+ \lambda_2 p_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\}\big\{\overline{G}^2(A(z,s))) - \overline{G}^2(A_1(z,s)))\big\}\Big\}. \quad (36)$$

Substituting this into the above equations, we get

$$\overline{I}_0(0,z_2,s) = \frac{N_1(z_2)}{d_1(z_2)}, \quad (37)$$
$$\overline{P}_0^2(0,z_2,s) = \frac{N_2(z_2)}{d_1(z_2)}, \quad (38)$$
$$\overline{P^1}(0,z_1,z_2,s) = \frac{N_3(z_2)}{d_2(z_2)}, \quad (39)$$

where

$$N_2(z_2) = \delta\lambda_2 C(z_2)\overline{I}_{(0,0)}(s)\big\{1 - \lambda_1 C(g[z_2])\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big](\delta + \overline{\delta G}^1(A_2(z,s)))\big\} - \delta$$
$$\Big\{(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s) - 1\Big\}\Big\{\overline{M}(s+\lambda_1+\lambda_2) + \lambda_2 p_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\},$$

$$d_1(z_2) = \{z_2 - r\overline{B_2}(\psi_2(z,s))\}\Big\{1 - \lambda_1 C(g[z_2])\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big](\delta + \overline{\delta G}^1(A_2(z,s)))\}$$
$$-\overline{\delta}\Big\{\overline{M}(s+\lambda_1+\lambda_2) + \lambda_2 p_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\}\Big\} - \delta\Big\{\overline{M}(s+\lambda_1+\lambda_2)$$
$$+\lambda_2 p_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\}\Big\{(1-r)(1-\theta)\overline{B_2}(\phi_4(z,s)) + \theta(1-r)$$
$$\overline{B_2}(\phi_4(z,s))\overline{V}(A_2(z,s)) + \lambda_2 p_2 C(z_2)\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big] + \alpha\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]$$
$$\overline{W}^2(A_2(z,s))\overline{R}^2(A_2(z,s))\Big\},$$

$$N_1(z_2,s) = \Big\{\overline{\delta}\lambda_2 C(z_2)\overline{I}_{(0,0)}(s)\overline{G}^2(A_2(z,s))) - \{(s+\lambda_1+\lambda_2)\overline{I}_{0,0} - 1\}\Big\}\{z_2 - r$$
$$\overline{B_2}(\psi_2(z,s))\} + \delta\lambda_2 C(z_2)\overline{I}_{(0,0)}(s)\Big\{(1-r)(1-\theta)\overline{B_2}(\phi_4(z,s)) + \theta(1-r)$$
$$\overline{B_2}(\phi_4(z,s))\overline{V}(A_2(z,s)) + \lambda_2 p_2 C(z_2)\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big] + \alpha\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]$$
$$\overline{W}^2(A_2(z,s))\overline{R}^2(A_2(z,s))\Big\},$$

$$N_3(z_2,s) = \overline{I}_0(0,z_2,s)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\{\lambda_1 C(z_1)(\delta + \overline{\delta G}^1(A(z,s))) - \lambda_1 C(g[z_2])$$
$$(\delta + \overline{\delta G}^1(A_2(z,s)))\Big\} - \overline{\delta}\lambda_2 C(z_2)\overline{I}_{(0,0)}(s)(1-\overline{G}^2(A_2(z,s))) - \overline{I}_0(0,z_2,s)$$
$$(1-\overline{G}^2(A_2(z,s)))\Big\{\overline{M}(s+\lambda_1+\lambda_2) + \lambda_2 p_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\} + \overline{P}_0^2(0,z_2,s)$$
$$\Big\{(1-r)(1-\theta)\{\overline{B_2}(\phi_2(z,s)) - \overline{B_2}(\phi_4(z,s))\} + \theta(1-r)\{\overline{B_2}(\phi_2(z,s))\overline{V}(A(z,s))$$
$$-\overline{B_2}(\phi_4(z,s))\overline{V}(A_2(z,s))\} + \lambda_2 p_2 C(z_2)\{\Big[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\Big] - \Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]\}$$
$$+\alpha\{\Big[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\Big]\overline{W}^2(A(z,s))\overline{R}^2(A(z,s)) - \alpha\Big[\frac{1-\overline{B_2}(\phi_4(z,s))}{\phi_4(z,s)}\Big]$$
$$\overline{W}^2(A_2(z,s))\overline{R}^2(A_2(z,s))\}\Big\},$$

$$d_2(z_2) = \Big\{z_1 - (1-\theta)\overline{B_1}(\phi_1(z,s)) - \alpha\Big[\frac{1-\overline{B_1}(\phi_1(z,s))}{\phi_1(z,s)}\Big]\overline{W}^1(A(z,s))\overline{R}^1(A(z,s))\Big\},$$

$$\overline{G}^1(0,z_1,z_2,s) = \overline{\delta}\lambda_1 C(z_1)\overline{I}_0(0,z_2,s)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big], \tag{40}$$

$$\overline{G}_0^2(0,z_2,s) = \overline{\delta}\lambda_2 C(z_2)\overline{I}_{(0,0)}(s) + \overline{\delta I}_0(0,z_2,s)\Big\{\overline{M}(s+\lambda_1+\lambda_2)$$
$$+\lambda_2 C(z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\big]\Big\}, \tag{41}$$

$$\overline{R}^i(0,z_1,z_2,s) = \alpha\overline{P^1}(0,z_1,z_2,s)[\frac{1-\overline{B_1}(\phi_1(z,s))}{\phi_1(z,s)}\Big]\overline{W}^1(A(z,s))$$
$$+\alpha\overline{P}_0^2(0,z_2,s)[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\Big]\overline{W}^2(A(z,s)), \tag{42}$$

$$\overline{W}^i(0,z_1,z_2,s) = \alpha\overline{P^i}(0,z_1,z_2,s)[\frac{1-\overline{B_i}(\phi_i(z,s))}{\phi_i(z,s)}\Big], \tag{43}$$

$$\overline{V}(0,z_1,z_2,s) = \theta\Big\{\overline{P^1}(0,z_1,z_2,s)\overline{B_1}(\phi_1(z,s)) + (1-r)\overline{P}_0^2(0,z_2,s)\overline{B_2}(\phi_2(z,s))\Big\}. \tag{44}$$

Theorem 4.1.

The inequality $P^1(1,1)+P^2(0,1)+W^i(1,1)=\rho<1,$ is a necessary and sufficient condition for the system to be stable. Under this condition the marginal PGF of the server's state, queue size and orbit size distributions are given by,

$$\overline{I}_{(0)}(z_2,s) = \overline{I}_{(0)}(0,z_2,s)\left[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\right], \tag{45}$$

$$\overline{P}^1(z_1,z_2,s) = \overline{P}^1(0,z_1,z_2,s)\left[\frac{1-\overline{B_1}(\phi_1(z,s))}{\phi_1(z,s)}\right], \tag{46}$$

$$\overline{P}^2(z_1,z_2,s) = \overline{P}_0^2(0,z_2,s)\left[\frac{1-\overline{B_2}(\phi_2(z,s))}{\phi_2(z,s)}\right], \tag{47}$$

$$\overline{W}^i(z_1,z_2,s) = \overline{W}^i(0,z_1,z_2,s)\left[\frac{1-\overline{W}^i(A(z,s))}{A(z,s)}\right], \tag{48}$$

$$\overline{R}^i(z_1,z_2,s) = \overline{R}^i(0,z_1,z_2,s)\left[\frac{1-\overline{R}^i(A(z,s))}{A(z,s)}\right], \tag{49}$$

$$\overline{G}^i(z_1,z_2,s) = \overline{G}^i(0,z_1,z_2,s)\left[\frac{1-\overline{G}^i(A(z,s))}{A(z,s)}\right], \tag{50}$$

$$\overline{V}(z_1,z_2,s) = \overline{V}(0,z_1,z_2,s)\left[\frac{1-\overline{V}(A(z,s))}{A(z,s)}\right]. \tag{51}$$

Proof:

Integrating equations (25) to (31) with respect to x and using the well known result of renewal theory

$$\int_0^\infty [1-H(x)]e^{-sx}dx = \frac{1-\overline{h(s)}}{s}, \tag{52}$$

where $\overline{h(s)}$ is the LST of the distribution function of a random variable $H(x)$, we get the results (44) to (50) respectively. Thus we obtain the complete solution for the probability generating functions for the following states $\overline{I}_{(0)}(z_2,s)$, $\overline{P}^1(z_1,z_2,s)$, $\overline{P}^2(z_1,z_2,s)$, $\overline{W}^i(z_1,z_2,s)$, $\overline{R}^i(z_1,z_2,s)$, $\overline{G}^i(z_1,z_2,s)$ and $\overline{V}(z_1,z_2,s)$. ∎

5. Steady State Analysis: Limiting Behavior

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t),$$

to the above equations, and adding we obtain the steady- state solutions of this model

In order to determine $I_{0,0}$, we use the normalizing condition

$$P^1(1,1) + V(1,1) + P^2(1,1) + W^i(1,1) + R^i(1,1) + G^i(1,1) + I_0(1) + I_{0,0} = 1.$$

For this, let $W_q(z_1, z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then, adding equations from (43) to (49) we obtain,

$$\begin{aligned} W_q(z_1,z_2) =& I_0(z_2) + P^1(z_1,z_2) + V(z_1,z_2) + P^2(z_1,z_2) + W^i(z_1,z_2) \\ &+ R^i(z_1,z_2) + G^i(z_1,z_2), i = 1,2. \end{aligned}$$

$$W_q(z_1,z_2) = \frac{N((z_1,z_2))}{D(z_1,z_2)}, \tag{53}$$

where

$$\begin{aligned} N((z_1,z_2)) =& I_0(0,z_2)\phi_1(z))\phi_2(z))f_1(z_1,z_2) + \overline{\delta}\lambda_2 C(z_2) I_{(0,0)}\phi_1(z))\phi_2(z))(1-\overline{G}^2(A(z))) \\ &+ P^1(0,z_1,z_2)\phi_2(z))f_2(z_1,z_2) + P_0^2(0,z_2)f_3(z_1,z_2), \end{aligned}$$

$$\begin{aligned} f_1(z_1,z_2) =& A(z)\left[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\right] + \overline{\delta}\lambda_1 C(z_1)\left[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\right](1-\overline{G}^1(A(z))) \\ &+ \overline{\delta}(1-\overline{G}^2(A(z)))\left\{\overline{M}(s+\lambda_1+\lambda_2) + \lambda_2 C(z_2)\left[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{(s+\lambda_1+\lambda_2)}\right]\right\}, \end{aligned}$$

$$\begin{aligned} f_2(z_1,z_2) =& A(z)(1-\overline{B_1}(\phi_1(z))) + \alpha(1-\overline{B_1}(\phi_1(z)))(1-\overline{W}^1(A(z))) + \alpha(1-\overline{B_1}(\phi_1(z))) \\ &\overline{W}^1(A(z))(1-\overline{R}^1(A(z))) + \theta\overline{B_1}(\phi_1(z))\phi_1(z)(1-\overline{V}(A(z))), \end{aligned}$$

$$\begin{aligned} f_3(z_1,z_2) =& A(z)(1-\overline{B_2}(\phi_2(z))) + \alpha(1-\overline{B_2}(\phi_2(z)))(1-\overline{W}^2(A(z))) + \alpha(1-\overline{B_2}(\phi_2(z))) \\ &\overline{W}^2(A(z))(1-\overline{R}^2(A(z))) + \theta(1-r)\overline{B_2}(\phi_2(z))(1-\overline{V}(A(z))), \end{aligned}$$

$$D(z_1,z_2) = A(z)\phi_1(z))\phi_2(z)).$$

In order to obtain the idle time probability $I_{0,0}$, we use the normalizing condition,

$$W_q(1,1) + I_{0,0} = 1.$$

From which we can have,

$$I_{0,0} = \frac{d_2^{'}(1)d_1(1)A'(1)\alpha(\alpha+\lambda_2 p_2)}{\left\{\begin{array}{l} d_2^{'}(1)d_1(1)A'(1)\alpha(\alpha+\lambda_2 p_2) + d_2^{'}(1)\alpha(\alpha+\lambda_2 p_2)f_1'(1,1)N_1(1) \\ +\overline{\delta}d_2^{'}(1)d_1(1)\lambda_2\alpha(\alpha+\lambda_2 p_2)E(G_2)A'(1) \\ + d_2^{'}(1)f_3'(1,1)N_2(1) + (\alpha+\lambda_2 p_2)f_2'(1,1)N_3^{'}(1) \end{array}\right\}}. \tag{54}$$

6. The Average Queue Length

The mean number of customers in the priority queue under the steady state is

$$L_{q_1} = \frac{d}{dz_1} W_{q_1}(z_1, 1)|_{z_1=1} \tag{55}$$

and the mean number of customers in the orbit under the steady state is

$$L_{q_2} = \frac{d}{dz_2} W_{q_2}(1, z_2)|_{z_2=1}, \tag{56}$$

thus

$$L_{q_1} = \frac{I_{0,0}\{Nr^{'''}(1)Dr^{''}(1) - Nr^{''}(1)Dr^{'''}(1)\}}{3(Dr^{''}(1))^2}, \tag{57}$$

$$L_{q_2} = \frac{I_{0,0}\{nr^{'''}(1)dr^{''}(1) - nr^{''}(1)dr^{'''}(1)\}}{3(dr^{''}(1))^2}. \tag{58}$$

7. The Average Waiting Time in the Queue

Average waiting time of a customer in the high priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1},$$

Average waiting time of a customer in the low priority orbit is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}.$$

Where L_{q_1} and L_{q_2} have been found in equations (55) and (56).

8. Particular Cases

Case I.

If there are no high priority arrivals, no immediate feedback service to low priority customers, no Bernoulli vacation, no working breakdown and statrting failure, i.e., $\lambda_1 = 0,\ r = 0,\ \overline{B}_1(.) = 0,$ $\theta = 0,\ \alpha = 0,\ \delta = 1$ and $\lambda_2 = \lambda$ then,

$$I_0(z) = \frac{I_{0,0}\{C(z)p_2\overline{B}(\phi(z))\phi(z) + \lambda_2 p_2 C(z)(1 - \overline{B}(\phi(z))) - z\phi(z)\}\{1 - \overline{M}(\lambda)\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda p_2(1 - \overline{B}(\phi(z)))\}\{p_2 C(z) + \overline{M}(\lambda)(1 - C(z)p_2)\}},$$

$$P^2(z) = \frac{I_{0,0}\lambda\{(p_2 C(z) - 1)(1 - \overline{B}(\phi(z)))\overline{M}(\lambda)\}}{z\phi(z) - \{\overline{B}(\phi(z))\phi(z) + \lambda p_2(1 - \overline{B}(\phi(z)))\}\{p_2 C(z) + \overline{M}(\lambda)(1 - C(z)p_2)\}}.$$

In this case if there is no collision, then, the result coincides with Kirupa and Udaya Chandrika (2010).

Case II.

If there are no high priority arrivals, no immediate feedback service to low priority customers, no Bernoulli vacation, no working breakdown, statrting failure and no collision. ie., $\lambda_1 = 0,\ r = 0,$ $\overline{B}_1(.) = 0,\ \theta = 0,\ \alpha = 0,\ \delta = 1,\ p_2 = 1$ and $\lambda_2 = \lambda$ then,

$$I_0(z) = \frac{I_{0,0}\{C(z)\overline{B}(\phi(z)) - z\}\{1 - \overline{M}(\lambda)\}}{z - \overline{B}(\phi(z))\{C(z) + \overline{M}(\lambda)(1 - C(z))\}},$$

$$P^2(z) = \frac{I_{0,0}(1 - \overline{B}(\phi(z)))\overline{M}(\lambda)}{\overline{B}(\phi(z))\{C(z) + \overline{M}(\lambda)(1 - C(z)) - z\}}.$$

This result coincides with Gautam Choudhury, Jau-Chuan Ke (2012).

9. Numerical Results

The above queueing model is analysed numerically with the following assumptions. We consider the service time for both normal and service breakdown period for high priority and low priority services are equal and repair time for both normal and starting failure for high priority and low priority customers are also equal that is $\mu_1 = \mu_2 = \mu,\ \nu_1 = \nu_2 = \nu$ and $\beta_1 = \beta_2 = \beta$, Bernoulli vacation time and repair time are exponentially distributed.

We assume arbitrary values to the parameters such that the stability condition is satisfied. MATLAB software has been used to illustrate the results numerically.

Table 2 shows that for exponential distribution, when high priority arrival rate λ_1 increases then the idle probability $I_{0,0}$ decreases, busy period, mean queue lengths, mean waiting time for customers in the queues all increase for the values of $\alpha = 1\ \lambda_2 = 2,\ \mu = 8,\ \mu_w = 1,\ \theta = 0.6,\ \beta = 1,\ \nu = 1,\ \eta = 1,\ \gamma = 3,\ p_2 = 0.3, E(I) = 1, r = 0.4, \delta = 0.2$ and $E[I(I-1)] = 0$. We choose that λ_1 takes the values $1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.3, 2.4$ and 2.5.

Table 1 shows that for exponential distribution, when normal service rate μ increases then the idle probability $I_{0,0}$ increases, busy period, mean queue lengths, mean waiting time for customers in the queues all decrease for the values of $\alpha = 2\ \lambda_2 = 1,\ \lambda_1 = 2,\ \mu_w = 3,\ \theta = 0.6,\ \beta = 1,\ \nu = 1,\ \eta = 1,\ \gamma = 3,\ p_2 = 0.3, E(I) = 1, r = 0.4, \delta = 0.2$ and $E[I(I-1)] = 0$. We choose that μ takes the values $8.0, 8.2, 8.4, 8.6, 8.8$ and 9.0.

All the trends shown by this tables and the graphs are as expected.

10. Conclusion

In this paper we studied retrial queueing system with priority services, working breakdown, collision, Bernoulli vacation, immediate feedback, starting failure and repair. The server provides

Table 1. Effect of λ_1 on various queue characteristics

Exponential Distribution						
λ_1	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
1.5	0.0939	0.9061	1.2777	13.9775	0.8518	6.9887
1.6	0.0936	0.9064	1.5248	15.5586	0.9530	7.7793
1.7	0.0931	0.9069	1.8241	17.2201	1.0730	8.6100
1.8	0.0925	0.9075	2.1909	18.9644	1.2172	9.4822
1.9	0.0918	0.9082	2.6463	20.7946	1.3928	10.3973
2.0	0.0911	0.9089	3.2193	22.7143	1.6097	11.3571
2.1	0.0902	0.9098	3.9514	24.7272	1.8816	12.3636
2.2	0.0893	0.9107	4.9021	26.8377	2.2282	13.4188
2.3	0.0884	0.9116	6.1600	29.0506	2.6783	14.5253
2.4	0.0874	0.9126	7.8601	31.3711	3.2750	15.6855
2.5	0.0864	0.9136	10.2156	33.8051	4.0863	16.9025

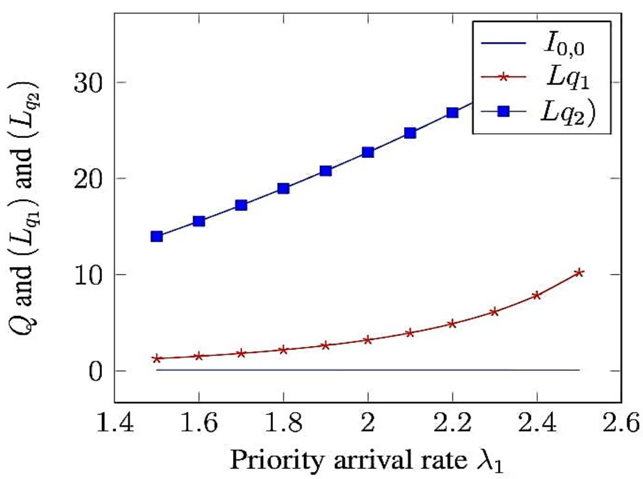

Figure 1: Average queue sizes verses high priority arrival rate λ_1

service for both high-priority customers and low-priority customers under non-preemptive priority rule. We derived the probability generating functions of the number of customers in the high-priority and low-priority customers using the supplementary variable technique. Average queue size, the average waiting time for the high-priority and low-priority customers and numerical results are also obtained. The analytical results are validated numerically may be useful in many real-life situations such as e-mail system, call centers, telecommunication networks, telephone switching system, etc. to design the outputs. The introduction of working breakdown, collision,

Table 2. Effect of μ on various queue characteristics

Exponential Distribution						
μ	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
8.0	0.1271	0.8729	12.4024	12.7922	6.2012	12.7922
8.2	0.1272	0.8728	11.9300	12.2769	5.9650	12.2769
8.4	0.1274	0.8726	11.4894	11.7945	5.7447	11.7945
8.6	0.1274	0.8726	11.0773	11.3419	5.5387	11.3419
8.8	0.1275	0.8725	10.6913	10.9165	5.3456	10.9165
9.0	0.1275	0.8725	10.3289	10.5159	5.1644	10.5159

starting failure and immediate feedback in presence of retrial queueing system with priority services and Bernoulli vacations are the novelty of this investigation. Our model has practical real-time application in computer processing system which processes messages through processor.

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