Applications and Applied Mathematics: An International Journal (AAM)

Volume 13 | Issue 2 Article 37

12-2018

Restricted Three-Body Problem Under the Effect of Albedo When Smaller Primary is a Finite Straight Segment

Shipra Chauhan
University of Delhi

Dinesh Kumar
University of Delhi

Bhavneet Kaur
University of Delhi

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 1200 – 1215

Restricted Three-Body Problem Under the Effect of Albedo When Smaller Primary is a Finite Straight Segment

[1]Shipra Chauhan, [2]*Dinesh Kumar and [3]Bhavneet Kaur

[1]Department of Mathematics
University of Delhi
Delhi, India
chauhanshipra10@gmail.com

[2]Department of Mathematics
Deshbandhu College
University of Delhi
Delhi, India
dineshk8392@gmail.com

[3]Department of Mathematics
Lady Shri Ram College for Women
University of Delhi
Delhi, India
bhavneet.lsr@gmail.com

*Corresponding Author

Received: March 31, 2018; Accepted: September 7, 2018

Abstract

This paper addresses the dynamics of the infinitesimal body in the restricted three-body problem under the effect of Albedo when the smaller primary is a finite straight segment and bigger one is a source of radiation. The measure of diffusive reflection of solar radiation out of the total solar radiation received by a body is Albedo which is measured on a scale from 0 to 1. The equations of motion of the infinitesimal body are derived and it is found that there exist five libration points, out of which three are collinear and the rest are non-collinear with the primaries. All the collinear libration points are found to be unstable while the non-collinear libration points are stable for a critical value of the mass parameter. The perturbation of libration points and its stability due to the effect of Albedo and straight segment in the present problem are investigated. Further, it is found that the effect of Jacobian constant, length of the straight segment and Albedo parameter has a substantial influence on the possible regions of motion of the infinitesimal body. It is observed that when the value of the Jacobian constant decreases, the region of possible motion increases. When the length of the straight segment increases, the region of possible motion increases. Further, it is observed that as we increase the Albedo parameter, the region of possible motion decreases.

Keywords: Restricted Three-body Problem; Albedo; Libration Points; Finite Straight Segment; Stability

MSC 2010 No.: 37N05, 70F07, 70F15

1. Introduction

The restricted three-body problem is a special case of general three-body problem in which the mass of one of the bodies is considered negligibly small, so that two other bodies (called primaries) can be described by the two-body problem and the body with negligible mass moves in the given field of two bodies such that it does not influence the motion of the primaries but influenced by them. This problem possesses five libration points out of which three are collinear and two non-collinear. The collinear libration points are unstable for all values of mass parameter μ, whereas the non-collinear libration points are stable for $0 < \mu < \mu_c$, where $\mu_c = 0.0385209$ (Szebehely (1967)). There are many examples of the restricted three-body problem in space dynamics. One of them is the Earth-Moon-Satellite system. A lot of work has been done on the restricted three-body problem. Riaguas et al. (2001) have discussed non-linear stability of the libration points corresponding to the motion of a particle orbiting around a finite straight segment. They have determined the orbital stability of the equilibria for all values of the parameter of the problem. Jain and Sinha (2014) have studied stability and regions of motion in the restricted three-body problem by taking both the primaries as finite straight segments.

The existence and stability of libration points in the restricted three-body problem by taking bigger primary as a triaxial rigid body and source of radiation have been studied by Sharma et al. (2001). Mittal et al. (2009) have discussed periodic orbits generated by libration points of the restricted three-body problem by taking bigger primary a source of radiation and the smaller primary an oblate body. They have also studied the effect of radiation pressure on the periodic orbits by taking some fixed values of the mass and oblate parameter.

Albedo is the amount of solar energy reflected back from the planet into the space. It is a measure of the reflectivity of the planet's surface (Harris and Lyle (1969), Rocco (2010)), given by

$$\text{Albedo} = \frac{\text{radiation reflected back to space}}{\text{incident radiation}}.$$

The effect of Albedo on the motion of a satellite has been studied by Anselmo et al. (1983), Grotte and Holzinger (2017), Harris and Lyle (1969), Pontus (2005) and Rocco (2010). The existence and linear stability of the libration points for all the values of radiation pressure of both luminous bodies, and all values of mass ratio have been studied by Simmons et al. (1985). Idrisi (2017) has studied the effect of Albedo on the libration points and their stability in the restricted three-body problem when the smaller primary is an ellipsoid. Further Idrisi and Ullah (2018) have studied the existence and stability of non-collinear libration points in the elliptic restricted three-body problem taking into account the oblateness of smaller primary.

Many authors have studied the motion of the spacecraft under the effect of Albedo in the restricted three-body problem, but they didn't take primary as a finite straight segment. Our model has a lot of practical applications in astrophysics, since there are natural bodies which are in the form of straight segment like asteroids. In the present paper, we are interested to study the motion of the third body under the effect of Albedo when smaller primary is a finite straight segment. In our solar system the Sun is a source of radiation so this work is applicable to the study of Sun-Asteroid-Spacecraft system.

The description of the present paper is as follows: In Section 2, the equations of motion of our model are presented. In Section 3, the existence of collinear and non-collinear libration points is shown. The stability of collinear and non-collinear libration points is discussed in Section 4. Section 5 is devoted to the zero velocity surfaces in which we have discussed the possible and forbidden regions of motion of the third body. This paper ends with Section 6, where the conclusion and discussion for the problem are presented.

2. Equations of Motion

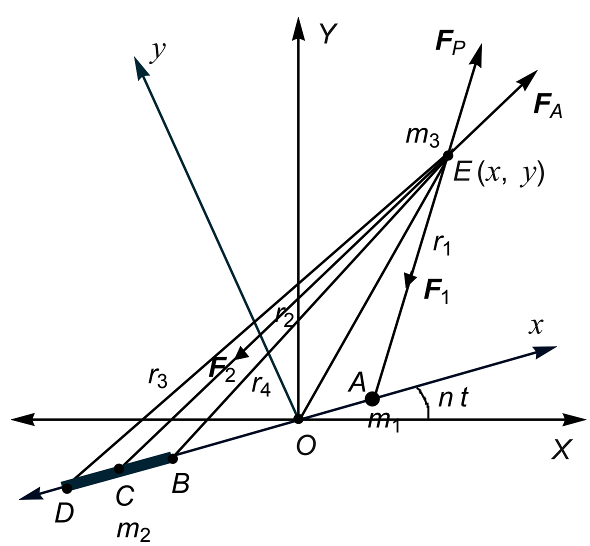

Figure 1. Geometry of the restricted three-body problem when m_2 is a finite straight segment.

Let m_1 be a point mass and a source of radiation and m_2 a finite straight segment of length $2l'$ ($m_1 > m_2$, in the sense of mass), move in the circular orbit around their common center of mass O with angular velocity n. The mass of the infinitesimal body is taken as m_3 which is negligibly small in comparison with the bodies having masses m_1 and m_2, moves in the plane of motion of m_1 and m_2. The line joining the origin to m_1 is taken as $X-$axis. The line passing through origin and perpendicular to OX in the plane of motion of the primaries is taken as $Y-$axis. The distance of mass m_3 from A, C, D and B is r'_1, r'_2, r'_3 and r'_4 respectively. Let us consider a synodic system of coordinates $Oxyz$ initially coincides with the inertial system $OXYZ$, rotating with angular velocity n about $Z-$axis. It is to note that $z-$axis coincides with $Z-$axis. Let $(x'_1, 0), (x'_2, 0)$ and (x', y') be the dimensional coordinates of A, C and E in synodic system respectively (Figure 1).

The various forces acting on m_3 are:

(1) The gravitational force $\boldsymbol{F_1}$ due to m_1;
(2) The solar radiation pressure $\boldsymbol{F_p}$ due to m_1;
(3) The gravitational force $\boldsymbol{F_2}$ due to m_2;
(4) The Albedo force $\boldsymbol{F_A}$ (solar radiation reflected by m_2 in space) due to m_2.

The total forces acting on m_3 due to m_1 and m_2 are

$$F_1 - F_p = F_1(1-\alpha) \text{ and } F_2 - F_A = F_2(1-\beta),$$

respectively, where

$$\alpha = \frac{F_P}{F_1} = \frac{\pounds_1}{2\pi G m_1 c\sigma},\ \beta = \frac{F_A}{F_2} = \frac{\pounds_2}{2\pi G m_2 c\sigma},\ 0 \leq \alpha < 1,\ 0 \leq \beta < \alpha,$$

$\pounds_1$ and $\pounds_2$ are the luminosities of m_1 and m_2 respectively, c is the speed of light, σ is the mass per unit area of m_3 and G is the gravitational constant. The relation between α and β is $\beta\mu = \alpha(1-\mu)k$, where $k = \pounds_2/\pounds_1$ is a constant, $0 \leq k < 1$.

We consider that the sum of the masses of the primaries is one unit and the distance between the primaries is also taken as one unit. The unit of time is so chosen such that G becomes unity. Let us assume that $\mu = \frac{m_2}{m_1+m_2}$, thus $m_2 = \mu$ and $m_1 = 1-\mu$. The coordinates of the points A, C and E in the dimensionless variables are $(\mu, 0), (\mu-1, 0)$ and (x, y) respectively in the synodic system.

The equations of motion of m_3 using the terminology of Szebehely (1967) in the dimensionless synodic coordinate system are given by

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \end{aligned} \tag{1}$$

where

$$\begin{aligned}
&\Omega = \frac{1}{2}n^2\left(x^2+y^2\right) + \frac{(1-\mu)(1-\alpha)}{r_1} + \frac{\mu(1-\beta)}{2l}\log\left(\frac{r_3+r_4+2l}{r_3+r_4-2l}\right),\\
&2l = \frac{2l'}{R},\ r_1 = \frac{r'_1}{R},\ r_3 = \frac{r'_3}{R},\ r_4 = \frac{r'_4}{R},\\
&R = AC(\text{distance between the primaries}) = 1 \text{ unit},\\
&r_1^2 = (x-\mu)^2 + y^2,\ r_3^2 = \{x-(\mu-1-l)\}^2 + y^2,\\
&r_4^2 = \{x-(\mu-1+l)\}^2 + y^2.
\end{aligned}$$

2.1. Mean motion of the primaries

Let the distances of m_1 and m_2 from the center of mass O be a and b respectively. The gravitational force between m_1 and m_2 is

$$F = G\frac{m_1 m_2}{(a+b)^2 - l^2}.$$

Since m_1 and m_2 are moving in circular orbits around their center of mass O, therefore

$$m_1 a n^2 = m_2 b n^2 = G\frac{m_1 m_2}{(a+b)^2 - l^2}, \tag{2}$$

On solving Equation (2), we get

$$(a+b)n^2 = G\frac{(m_1 + m_2)}{(a+b)^2 - l^2}.$$

Using dimensionless variables, we get $n^2 = (1 + l^2),\ l << 1$. Here, we are taking terms containing l up to second order.

3. Libration Points

The libration points are the solutions of the equations

$$\Omega_x = n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{2\mu(x-\mu+1)(1-\beta)}{r_3 r_4 (r_3 + r_4)} = 0, \tag{3}$$

$$\Omega_y = y\left[n^2 - \frac{(1-\mu)(1-\alpha)}{r_1^3} - \frac{2\mu(1-\beta)(r_3+r_4)}{r_3 r_4\left\{(r_3+r_4)^2 - 4l^2\right\}}\right] = 0. \tag{4}$$

3.1. Collinear Libration Points

The collinear libration points are the solutions of Equation (3) when $y = 0$, that is,

$$f(x) = n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{2\mu(x-\mu+1)(1-\beta)}{r_3 r_4 (r_3 + r_4)} = 0, \tag{5}$$

where $r_1 = |x-\mu|$, $r_3 = |x - (\mu - 1 - l)|$ and $r_4 = |x - (\mu - 1 + l)|$. It is perceived that

- $f(x)$ is strictly increasing in $(-\infty,\ \mu - 1 - l)$, $(\mu - 1 + l,\ \mu)$ and $(\mu,\ \infty)$.
- $f(x) \to -\infty$ as $x \to -\infty$ or $(\mu - 1 - l)^+$ or μ^+.
- $f(x) \to \infty$ as $x \to \infty$ or $(\mu - 1 + l)^-$ or μ^-.

There exists one and only one value of x in $(-\infty, \mu - 1 - l)$, $(\mu - 1 + l, \mu)$ and (μ, ∞) for each interval, where $f(x) = 0$. Further, $f(\mu - 2) < 0$, $f(0) \geq 0$ and $f(\mu + 1) > 0$. From these inequalities, it is observed that Equation (5) has three real roots, that lie in $(\mu - 2,\ \mu - 1 - l)$, $(\mu - 1 + l,\ \mu)$ and $(\mu,\ \mu + 1)$. Therefore, there are only three collinear libration points $L_1(x_1, 0)$, $L_2(x_2, 0)$ and $L_3(x_3, 0)$.

Following the procedure of Szebehely (1967), when $x \in (\mu-2, \mu-1-l)$, $r_1 = \mu - x_1$, $r_3 = \mu - 1 - x_1 - l$, $r_4 = \mu - 1 - x_1 + l$ and $x_1 = \mu - 1 - \xi_1$. On substituting these values in Equation (5), we get

$$\begin{aligned}&(1+l^2)\xi_1^5 + (3-\mu)(1+l^2)\xi_1^4 + \left\{(3-2\mu) + 2(1-\mu)l^2\right\}\xi_1^3 \\ &- \left\{2l^2 + \mu + \alpha(1+k)(-1+\mu)\right\}\xi_1^2 - \left\{2 - 2\alpha(1-\mu)k + (3-2\mu)l^2\right\}\xi_1 \\ &- \left\{1 - \alpha(1-\mu)k + (1-\mu)\alpha l^2\right\} = 0. \qquad (6)\end{aligned}$$

By Descartes' sign rule, there exists one and only one positive real root of Equation (6). Then the abscissa of L_1 is obtained by $x_1 = \mu - 1 - \xi_1$, where ξ_1 is a positive real root of Equation (6).

Following the same procedure when $x \in (\mu - 1 + l, \mu)$, we get

$$\begin{aligned}&(1+l^2)\xi_2^5 - (3-\mu)(1+l^2)\xi_2^4 + \left\{(3-2\mu) + 2(1-\mu)l^2\right\}\xi_2^3 \\ &+ \left\{2l^2 - \mu - \alpha(1-\mu)(1-k)\right\}\xi_2^2 + \left\{2\mu - 2\alpha(1-\mu)k - (3-2\mu)l^2\right\}\xi_2 \\ &+ \left\{-\mu + \alpha(1-\mu)k + (1-\mu)\alpha l^2\right\} = 0. \qquad (7)\end{aligned}$$

The abscissas of L_2 is obtained by $x_2 = \mu - 1 + \xi_2$, where ξ_2 is a positive real root of Equation (7).

Similarly when $x \in (\mu, \mu + 1)$, we get

$$\begin{aligned}&(1+l^2)\xi_3^5 + (2+\mu)(1+l^2)\xi_3^4 + \left\{(1+2\mu) + 2\mu l^2\right\}\xi_3^3 + (1-\mu)\left\{\alpha(1+k) - 1\right\}\xi_3^2 \\ &-2(1-\alpha)(1-\mu)\xi_3 - \left\{(1-l^2)(1-\mu)(1-\alpha)\right\} = 0. \qquad (8)\end{aligned}$$

The abscissa of L_3 is obtained by $x_3 = \mu + \xi_3$, where ξ_3 is a positive real root of Equation (8).

In this case, the collinear libration points agree with classical case of the restricted three-body problem (Szebehely (1967)) if l, α and k are zero. While these libration points are affected due to the finite straight segment as well as Albedo if l, α and k are non-zero. The effect of finite straight segment and Albedo on these libration points is shown in Figure 2.

Table 1. Locations of the libration points, when $\alpha = 0.1$, $k = 0.05$, $l = 0.0001$

μ	L_1	L_2	L_3	$L_{4,5}$
0.1	(-1.24581, 0)	(-0.599370, 0)	(1.008220, 0)	$(-0.381667, \pm 0.838120)$
0.15	(-1.25859, 0)	(-0.508935, 0)	(1.029730, 0)	$(-0.326111, \pm 0.841328)$
0.2	(-1.26074, 0)	(-0.426701, 0)	(1.051120, 0)	$(-0.273333, \pm 0.842931)$
0.25	(-1.25665, 0)	(-0.349058, 0)	(1.07233, 0)	$(-0.221667, \pm 0.843894)$
0.3	(-1.24845, 0)	(-0.274281, 0)	(1.09329, 0)	$(-0.170556, \pm 0.844535)$

Table 2. Locations of the libration points, when $\mu = 0.1$, $k = 0.05$, $l = 0.0001$

α	L_1	L_2	L_3	$L_{4,5}$
0.1	(-1.24581, 0)	(-0.599370, 0)	(1.008220, 0)	$(-0.381667, \pm 0.838120)$
0.2	(-1.23191, 0)	(-0.587848, 0)	(0.972268, 0)	$(-0.363333, \pm 0.810215)$
0.3	(-1.21801, 0)	(-0.573892, 0)	(0.933209, 0)	$(-0.345000, \pm 0.782310)$
0.4	(-1.20412, 0)	(-0.556665, 0)	(0.890260, 0)	$(-0.326667, \pm 0.754404)$
0.5	(-1.19025, 0)	(-0.534906, 0)	(0.842273, 0)	$(-0.308333, \pm 0.726499)$

Table 3. Locations of the libration Points, when $\mu = 0.1$, $\alpha = 0.1$, $l = 0.0001$

k	L_1	L_2	L_3	$L_{4,5}$
0.05	(-1.24581, 0)	(-0.599370, 0)	(1.00822, 0)	$(-0.381667, \pm 0.838120)$
0.10	(-1.23973, 0)	(-0.603344, 0)	(1.00783, 0)	$(-0.396667, \pm 0.829460)$
0.2	(-1.22698, 0)	(-0.6117720, 0)	(1.00705, 0)	$(-0.426667, \pm 0.812139)$
0.4	(-1.19853, 0)	(-0.631032, 0)	(1.0055, 0)	$(-0.486667, \pm 0.777498)$
0.6	(-1.16444, 0)	(-0.654968, 0)	(1.00396, 0)	$(-0.546667, \pm 0.742857)$

Table 4. Locations of the libration points, when $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$

l	L_1	L_2	L_3	$L_{4,5}$
0.0001	(-1.24581, 0)	(-0.59937, 0)	(1.00822, 0)	$(-0.381667, \pm 0.838120)$
0.10	(-1.25451, 0)	(-0.589730, 0)	(1.00510, 0)	$(-0.383102, \pm 0.837933)$
0.2	(-1.28107, 0)	(-0.560767, 0)	(0.995992, 0)	$(-0.3874407, \pm 0.837372)$
0.25	(-1.3012, 0)	(-0.539131, 0)	(0.989391, 0)	$(-0.390637, \pm 0.836951)$
0.3	(-1.32574, 0)	(-0.512951, 0)	(0.981575, 0)	$(-0.394583, \pm 0.836436)$

3.2. Non-collinear Libration Points

The non-collinear libration points are obtained by solving Equations (3) and (4) when $y \neq 0$, that is,

$$n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{2\mu\left\{x-(\mu-1)\right\}(1-\beta)}{r_3 r_4 (r_3+r_4)} = 0, \tag{9}$$

$$n^2 - \frac{(1-\mu)(1-\alpha)}{r_1^3} - \frac{2\mu(1-\beta)(r_3+r_4)}{r_3 r_4 \left\{(r_3+r_4)^2 - 4l^2\right\}} = 0. \tag{10}$$

If we neglect the effect of finite straight segment and Albedo, Equations (9) and (10) agree with classical case of the restricted three-body problem (Szebehely (1967)). These equations are satisfied only when $r_1 = r_2 = 1$. Now, let the solutions of Equations (9) and (10) be

$$\begin{aligned} r_1 &= 1 + \alpha_1, \\ r_2 &= 1 + \alpha_2, \end{aligned} \tag{11}$$

where $\alpha_1, \alpha_2 << 1$.

Using Equation (11) in $r_1^2 = (x-\mu)^2 + y^2$ and $r_2^2 = (x - \mu + 1)^2 + y^2$, we have

$$\begin{aligned} x &= \mu - \tfrac{1}{2} - (\alpha_1 - \alpha_2), \\ y &= \pm\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{\sqrt{3}}(\alpha_1 + \alpha_2)\right). \end{aligned} \tag{12}$$

Using Equations (11) and (12) in Equations (9) and (10), we get

$$\begin{aligned} \alpha_1 &= -\frac{1}{3}\alpha - \frac{(3\mu - 2)}{6(\mu - 1)} l^2, \\ \alpha_2 &= -\frac{1}{3}\beta - \frac{11}{24} l^2. \end{aligned} \tag{13}$$

We have considered only linear terms of α_1 and α_2, and l up to second order. On substituting Equation 13 in Equation 12 and solving them, the coordinates of the non-collinear libration points L_4 and L_5 are obtained as

$$\begin{aligned} x &= \mu - \frac{1}{2} + \frac{1}{3}(\alpha - \beta) + \frac{(\mu+3)}{24(\mu-1)} l^2, \\ y &= \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{1}{3}(\alpha+\beta) + \frac{(23\mu - 19)}{24(\mu-1)} l^2\right\}\right]. \end{aligned}$$

Now by the relation of α and β, we get

$$\begin{aligned} x &= \mu - \frac{1}{2} + \frac{\alpha}{3}\left(1 - \frac{(1-\mu)k}{\mu}\right) + \frac{(\mu+3)}{24(\mu-1)} l^2, \\ y &= \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{\alpha}{3}\left(1 + \frac{(1-\mu)k}{\mu}\right) + \frac{(23\mu - 19)}{24(\mu-1)} l^2\right\}\right]. \end{aligned}$$

It is found that for $y \neq 0$, two libration points exist which are non-collinear in nature. The positions of L_4 and L_5 depend on the length of the finite straight segment as well as Albedo. The infinitesimal mass m_3 makes an equilateral triangle ($r_1 = r_2 = 1$) at L_4 and L_5 with the primaries in classical case of the restricted three-body problem, whereas in this work m_3 makes a scalene triangle ($r_1 \neq r_2$) at L_4 and L_5 with the primaries. The non-collinear libration points L_4 and L_5 are symmetrical with respect to $x-$axis in both the cases.

Thus, we conclude that five libration points exist in the present problem. The numerical interpretation of these libration points are given in Tables 1, 2, 3 and 4.

In Table 1, $\alpha = 0.1$, $k = 0.05$, $l = 0.0001$ and $\mu = 0.1, 0.15, 0.2, 0.25, 0.3$. As μ increases, L_1 moves away from the smaller primary along x-axis, L_2 moves towards the center of mass of the primaries along x-axis, L_3 moves away from the bigger primary along x-axis. The abscissas of L_4 and L_5 increase, the ordinate of L_4 increases while the ordinate of L_5 decreases.

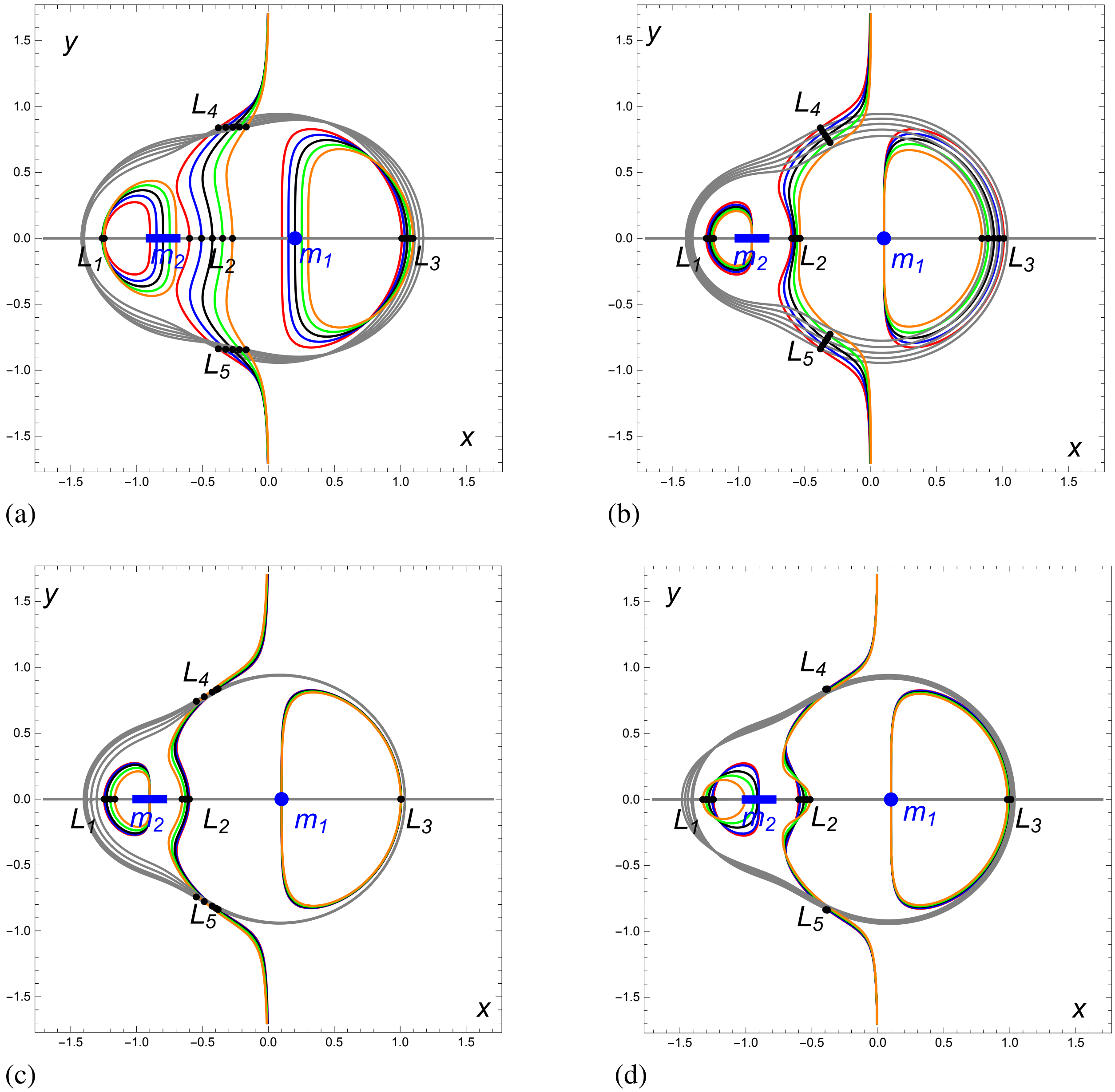

Figure 2. Locations of the libration points (a) $\alpha = 0.1$, $k = 0.05$, $l = 0.0001$ and $\mu = 0.1$ (red, gray), 0.15 (blue, gray), 0.2 (black, gray), 0.25 (green, gray), 0.3 (orange, gray). (b) $\mu = 0.1$, $k = 0.05$, $l = 0.0001$ and $\alpha = 0.1$ (red, gray), 0.2 (blue, gray), 0.3 (black, gray), 0.4 (green, gray), 0.5 (orange, gray). (c) $\mu = 0.1$, $\alpha = 0.1$, $l = 0.0001$ and $k = 0.05$ (red, gray), 0.1 (blue, gray), 0.2 (black, gray), 0.4 (green, gray), 0.6 (orange, gray). (d) $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$ and $l = 0.0001$ (red, gray), 0.1 (blue, gray), 0.2 (black, gray), 0.25 (green, gray), 0.3 (orange, gray).

In Table 2, $\mu = 0.1$, $k = 0.05$, $l = 0.0001$ and $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5$. As α increases, L_1 moves towards the smaller primary along x-axis, L_2 moves towards the center of mass of the primaries along x-axis, L_3 moves towards the bigger primary along x-axis. The abscissas of L_4 and L_5 increase, the ordinate of L_4 decreases while the ordinate of L_5 increases.

In Table 3, the effect of k on the libration points for $\mu = 0.1$, $\alpha = 0.1$, $l = 0.0001$ and $k = 0.05, 0.1, 0.2, 0.4, 0.6$ is given. As k increases, L_1 moves towards the smaller primary along x-axis,

L_2 moves away the center of the mass of the primaries and towards the smaller primary along x-axis, L_3 moves towards the bigger primary along x-axis, the abscissas of L_4 and L_5 decreases, the ordinate of L_4 decreases while the ordinate of L_5 increases.

The effect of l on the libration points is given in Table 4, when $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$ are fixed and $l = 0.0001, 0.1, 0.2, 0.25, 0.3$. As l increases, L_1 moves away from the smaller primary along x-axis, L_2 moves towards the center of the mass of the primaries along x-axis, L_3 moves towards the bigger primary along x-axis, the abscissas of L_4 and L_5 decreases, the ordinate of L_4 decreases while the ordinate of L_5 increases but slowly.

4. Stability of Libration Points

Let (x_0, y_0) be one of the libration point. On putting $x = x_0 + \xi$ and $y = y_0 + \eta$, $\xi, \eta << 1$ in Equation (1), we get

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \xi\Omega^0_{xx} + \eta\Omega^0_{xy}, \\ \ddot{\eta} + 2n\dot{\xi} &= \xi\Omega^0_{yx} + \eta\Omega^0_{yy}, \end{aligned} \tag{14}$$

where “0” in the superscript indicates that derivatives are to be calculated at the libration point (x_0, y_0). The characteristic equation corresponding to Equation (14) is

$$\lambda^4 + \left(4n^2 - \Omega^0_{xx} - \Omega^0_{yy}\right)\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - \left(\Omega^0_{xy}\right)^2 = 0. \tag{15}$$

Table 5. Ω^0_{xx}, and Ω^0_{yy} when $k = 0.05$, $\alpha = 0.1$, $l = 0.0001$

μ	x_1	Ω^0_{xx}	Ω^0_{yy}	$\Omega^0_{xx}\Omega^0_{yy}$
0.01	-1.10428	9.05526	-7.05526	< 0
0.05	-1.2097	7.02238	-5.02238	< 0
0.10	-1.245810	6.28332	-4.28332	< 0
0.15	-1.25859	5.82086	-3.82086	< 0
0.2	-1.26074	5.46992	-3.46992	< 0

Table 6. Ω^0_{xx}, and Ω^0_{yy} when $\mu = 0.1$, $k = 0.05$, $l = 0.0001$

α	x_1	Ω^0_{xx}	Ω^0_{yy}	$\Omega^0_{xx}\Omega^0_{yy}$
0.1	-1.24581	6.28332	-4.28332	< 0
0.2	-1.23191	6.58694	-4.58694	< 0
0.3	-1.21801	6.9296	-4.9296	< 0
0.4	-1.20412	7.31748	-5.31748	< 0
0.5	-1.19025	7.75792	-5.75792	< 0

4.1. Stability of Collinear Libration Points

First, we check the stability of L_1. The partial derivatives at L_1 are given by

$$\begin{aligned}
\Omega^0_{xx} &= n^2 + \frac{2(1-\mu)(1-\alpha)}{(\mu - x_1)^3} + \frac{2\mu\,(\mu-1-x_1)\,(1-\beta)}{\left\{(\mu-1-x_1)^2 - l^2\right\}^2} > 0,\\
\Omega^0_{yy} &= n^2 - \frac{2(1-\mu)(1-\alpha)}{(\mu - x_1)^3} - \frac{\mu\,(\mu-1-x_1)\,(1-\beta)}{\left\{(\mu-1-x_1)^2 - l^2\right\}^2} < 0,\\
\Omega^0_{xy} &= 0.
\end{aligned}$$

The values of Ω^0_{xx} and Ω^0_{yy} for $k = 0.05$, $\alpha = 0.1$, $l = 0.0001$ and $\mu = 0.01, 0.05, 0.1,\ 0.15, 0.2$ are given in Table 5. In Table 6, $\mu = 0.1$, $k = 0.05$, $l = 0.0001$ and $\alpha = 0.1, 0.2, 0.3, 0.4,\ 0.5$ are taken. From Tables 5 and 6, it is clear that $\Omega^0_{xx}\Omega^0_{yy} < 0$.

The four roots of Equation (15) are $\lambda_{1,2} = \pm s$ and $\lambda_{3,4} = \pm it$ $(s, t \in \mathbf{R})$. Thus, the collinear libration point L_1 is unstable for all values of μ. Similarly, we can show that L_2 and L_3 are unstable for all values of μ.

4.2. Stability of Non-collinear Libration Points

In order to check the stability of non-collinear libration points, the second order derivatives at L_4 are

$$\begin{aligned}
\Omega^0_{xx} &= \frac{3}{4} + \left(\frac{3\mu-1}{2}\right)\alpha + \left(\frac{2-3\mu}{2}\right)\beta + \left(\frac{26-49\mu+15\mu^2}{16(1-\mu)}\right)l^2,\\
\Omega^0_{yy} &= \frac{9}{4} + \left(\frac{1-3\mu}{2}\right)\alpha + \left(\frac{3\mu-2}{2}\right)\beta + \left(\frac{22+\mu-15\mu^2}{16(1-\mu)}\right)l^2,\\
\Omega^0_{xy} &= -\frac{3\sqrt{3}}{4}(1-2\mu) + \left(\frac{\mu+1}{2\sqrt{3}}\right)\alpha + \left(\frac{\mu-2}{2\sqrt{3}}\right)\beta + \left(\frac{-50+131\mu-89\mu^2}{16\sqrt{3}(1-\mu)}\right)l^2.
\end{aligned}$$

Substituting these derivatives in Equation (15), we get

$$\lambda^4 + p_1\lambda^2 + p_2 = 0, \tag{16}$$

where

$$\begin{aligned}
p_1 &= (1 + l^2),\\
p_2 &= \frac{27}{4}\mu(1-\mu) + \frac{3}{2}\alpha(1-\mu)\left[k(1-\mu)+\mu\right] + \frac{3}{16}(79\mu - 89\mu^2)l^2.
\end{aligned}$$

In order for the characteristic polynomial (16) to have pure imaginary roots the following conditions have to be satisfied:

(1) $p_1^2 - 4p_2 > 0$
(2) $p_1 > 0$
(3) $p_2 > 0$.

The last two conditions (2) and (3) are always satisfied. Therefore L_4 will be stable, if condition (1) is satisfied.

The discriminant (Δ) of Equation (16) is

$$\begin{aligned}\Delta &= p_1^2 - 4p_2\\ &= 1 - 6k\alpha(-1+\mu)^2 - 3(9+2\alpha)\mu + 3(9+2\alpha)\mu^2 + \frac{1}{4}(8 - 237\mu + 267\mu^2)l^2.\end{aligned}$$

On solving for $\Delta = 0$, we get

$$\mu_{1,2} = \frac{-108 - 237l^2 - 24\alpha + 48k\alpha \pm \sqrt{9(3312 + 14488l^2 + 1600\alpha + 128k\alpha)}}{6(-36 - 89l^2 - 8\alpha + 8k\alpha)}.$$

Since, $0 < \mu < 1/2$, therefore we consider only μ_2. The critical value of μ is $\mu_2 = \mu_c = 0.0385209 - (0.00891745 + 0.22258k)\alpha - 0.0073562l^2$.

Case (i) If $0 < \mu < \mu_c$, $p_1^2 - 4p_2 > 0$, L_4 is stable.

Case (ii) If $\mu = \mu_c$, $p_1^2 - 4p_2 = 0$, L_4 is unstable.

Case (iii) If $\mu_c < \mu < 1/2$, $p_1^2 - 4p_2 < 0$, L_4 is unstable.

Thus, L_4 is stable for $0 < \mu < \mu_c$. Similarly, we can show that L_5 is stable for $0 < \mu < \mu_c$.

5. Zero Velocity Surfaces (ZVSs)

To find the zero velocity surfaces, the Jacobian integral corresponding to Equation (1) is

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C, \tag{17}$$

where C is Jacobian constant. Using Equation (17), C is calculated for fixed values of $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$ and $l = 0.0001$, which is given in Table 7.

Table 7. Jacobian constant when $\mu = 0.1$, $l = 0.0001$, $\alpha = 0.1$ and $k = 0.05$.

Libration points	Coordinates	Jacobian constant
$L_1(x_1, 0)$	(-1.245810, 0)	$C_1 = 3.30811$
$L_2(x_2, 0)$	(-0.59937, 0)	$C_2 = 3.31095$
$L_3(x_3, 0)$	(1.00822, 0)	$C_3 = 2.90031$
$L_4(x_4, y_4)$	(-0.381667, 0.83812)	$C_4 = 2.71779$

In Figure 3, ZVSs are plotted for different values of C and fixed values of μ, α, k and l. In frame (a), ZVSs for $C = 3.31095$ are shown. It is observed that, there exists circular island (in white color) around each of the primaries and the third body can move in this area around the primaries, whereas the circular strip (in yellow color) shows the forbidden region. That is, the third body can move around each of the primaries, but can not move from one primary to other primary. In frame

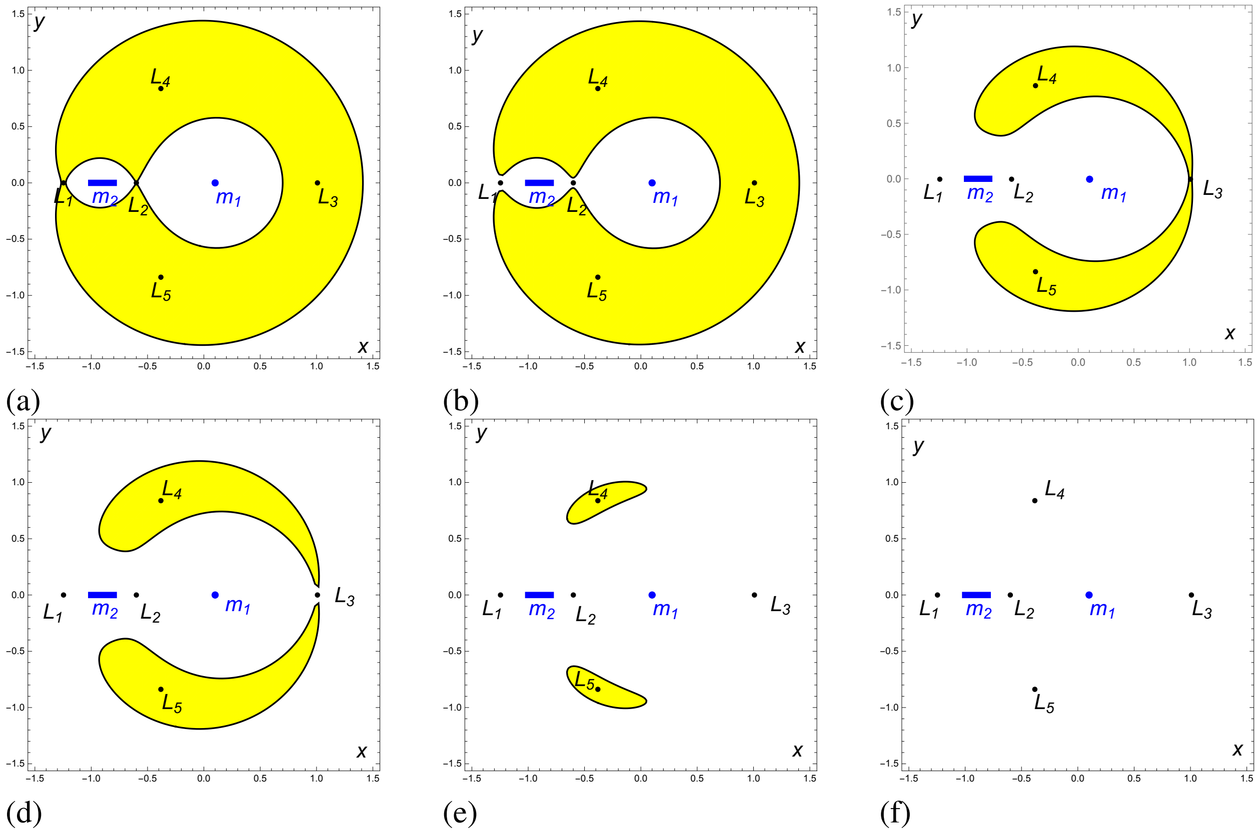

Figure 3. ZVSs of R3BP for $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$ and $l = 0.0001$ (a) $C = 3.31095$ (b) $C = 3.3011$ (c) $C = 2.90094$ (d) $C = 2.90031$ (e) $C = 2.75$ (f) $C = 2.71779$.

(b), $C = 3.3011$ and observed that the path opens at L_1 and L_2. The third body can move from one primary to other primary. In frame (c), $C = 2.90094$ is taken and it is found that the cusp is formed at L_3. The third body can move anywhere in white region.

Further $C = 2.90031$ is taken in frame (d), the path is opened at L_3. The forbidden region splits into two parts, one containing L_4 and other containing L_5. In frame (e), we have taken $C = 2.75$. The forbidden region shrinks to tadpole like shapes. Frame (f) shows ZVSs for $C = 2.71779$ and for this value of C, forbidden region disappears. That is, the third body can move any where in the white region. Thus as we decrease C the possible regions of motion of m_3 increases.

In Figure 4, we show how the regions of possible motion are influenced due to the effect of the parameter α. The regions of motion are plotted for $\mu = 0.1$, $l = 0.0001$, $k = 0.05$, the Jacobian constant $C_{L_4} < C = 2.75 < C_{L_3}$ and for various values of α. Frame (a) shows the ZVSs for $\alpha = 0.1$, the forbidden region (yellow region) occurs around L_4 and L_5 in tadpole shaped region. The third body cannot move in this yellow region while it can move from one primary to another primary. In frame-(b), we have increased the value of $\alpha = 0.15$ and observed that the tadpole shaped region increases. In frame-(c), zero velocity surface is drawn for $\alpha = 0.1735$ and observed that a cusp is formed at L_3. In frame-(d), for $\alpha = 0.2$, the forbidden region increases and contains L_3, L_4 and L_5. In frame-(e), for $\alpha = 0.293$, it is observed that a cusp is formed at L_2. In this case, the third

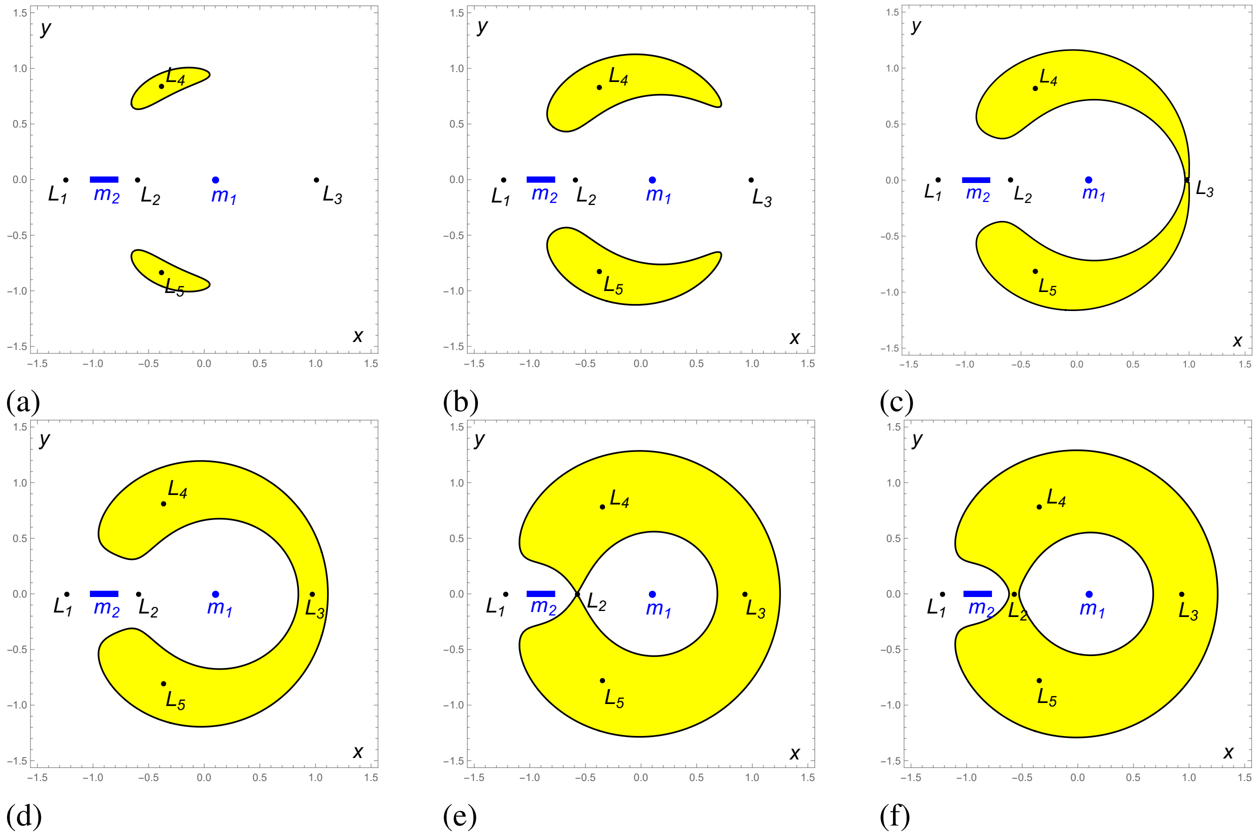

Figure 4. ZVSs of R3BP for $\mu = 0.1$, $k = 0.05$, $l = 0.0001$ and $C = 2.75$ (a) $\alpha = 0.1$ (b) $\alpha = 0.15$ (c) $\alpha = 0.1735$ (d) $\alpha = 0.2$ (e) $\alpha = 0.293$ (f) $\alpha = 0.3$.

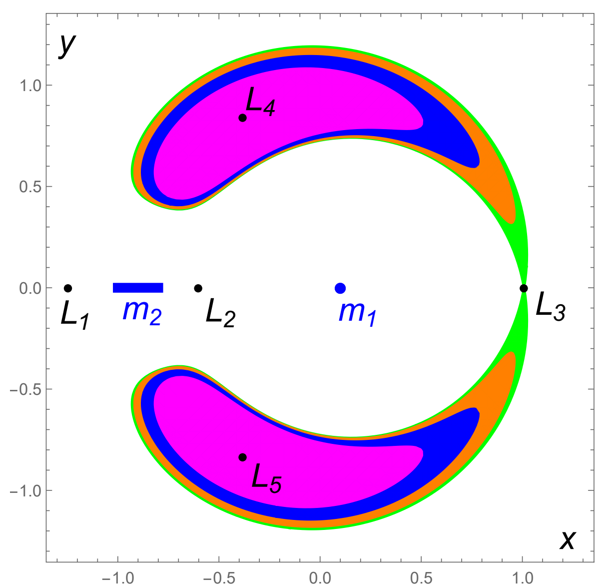

Figure 5. The ZVSs for $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$ and $l = 0.0001$ (green), 0.1 (orange), 0.2 (blue), 0.3 (magenta).

body cannot move from one primary to other primary. The third body can move around m_1 and m_2 in white region. In the last frame-(f), for $\alpha = 0.3$, the forbidden region increases and contains

L_2, L_3, L_4 and L_5. In this case the third body cannot move from one primary to other primary.

In Figure 5, we show how the regions of possible motion are influenced due to the effect of the length of the straight segment. The regions of motion are plotted for $\mu = 0.1$, $\alpha = 0.1$, $k = 0.05$, the Jacobian constant $C = C_{L_3} = 2.90031$ and for various values of length of the straight segment. It is observed that, for $l = 0.0001$ a forbidden region (green region) containing L_4 and L_5 occurs, which forms a cusp at L_3. In this region, the third body cannot move while it can move in white region. Further as we increase length of the straight segment $l = 0.1$ (orange region), $l = 0.2$ (blue region) and $l = 0.3$ (magenta region), the forbidden region decreases and possible region of motion increases.

6. Conclusion and Discussion

The existence and stability of the libration points in the restricted three-body problem under the effect of Albedo when smaller primary is a finite straight segment has been investigated. There exist five libration points, out of which three are collinear and two non-collinear with the primaries. The first collinear libration point L_1 lies at the left of the primary m_2, the second libration point L_2 lies between the primaries and the third libration point L_3 lies at the right of the primary m_1. The non-collinear libration points are symmetric about x-axis and form a scalene triangle with the primaries, while in classical case these points form equilateral triangle with the primaries. The libration points are affected by mass parameter, Albedo parameter and length of the straight segment and this effect displaced the libration points from its actual position as shown in Figure 2. The results of classical problem (Szebehely (1967)) can be obtained from our results by taking $l = \alpha = k = 0$.

Further, we have discussed the stability of the libration points. It is observed that, the collinear libration points are unstable for all values of parameters, whereas the non-collinear libration points are stable for $0 < \mu < \mu_c$, where $\mu_c = 0.0385209 - (0.00891745 + 0.22258k)\alpha - 0.0073562l^2$. Due to the effect of finite straight segment and Albedo, the range of stability decreases in comparison to the classical case of the restricted three-body problem (Szebehely (1967)).

Further, we have discussed the effect of straight segment and Albedo on the possible and forbidden regions of motion of the infinitesimal body. It is observed that when the value of the Jacobian constant C decreases, the region of possible motion increases as shown in Figure 3. Further, we have observed that, as we increase parameter α, the region of possible motion decreases which is shown in Figure 4. Further as we increase the length of the straight segment, the region of possible motion increases as shown in Figure 5. The present study and the corresponding results are applicable in the Sun-Asteroid-Spacecraft system.

REFERENCES

Anselmo, L., Farinella, P., Milani, A. and Nobili, A.M. (1983). Effects of the Earth-reflected Sunlight on the orbit of the LAGEOS Satellite, Astron. Astrophys., Vol. 117, pp. 3 – 8.

Grotte, M.E. and Holzinger, M.J. (2017). Solar sail equilibria with albedo radiation pressure in the circular restricted three-body problem, Advances in Space Research., Vol. 59, No. 4, pp. 1112 – 1127.

Harris, M. and Lyle, R. (1969). Spacecraft radiation torques, NASA Space Vehicle Design Criteria (Guidance and Control) NASA SP-8027.

Idrisi, M.J. (2017). A study of libration points in modified CR3BP under albedo effect when smaller primary is an ellipsoid, J. of Astronaut. Sci., Vol. 64, No. 4, pp. 379 – 398.

Idrisi, M.J. and Ullah, M.S. (2018). Non-collinear libration points in ER3BP with albedo effect and oblateness, J. Astrophys. Astr., Vol. 39, pp. 28.

Jain, R. and Sinha, D. (2014). Stability and regions of motion in the restricted three-body problem when both the primaries are finite straight segments, Astrophys. Space Sci., Vol. 351, pp. 87 – 100.

Mittal, A., Ahmad, I. and Bhatnagar, K.B. (2009). Periodic orbits in the photogravitational restricted problem with the smaller primary an oblate body, Astrophys. Space Sci., Vol. 323, No. 1, pp. 65 – 73.

Pontus, A. (2005). Attitude estimation from magnetometer and Earth-albedo-corrected coarse Sun sensor measurements, J. of Acta Astronaut., Vol. 56, pp. 115 – 126.

Riaguas, A., Elipe, A. and Lopez-Moratalla, T. (2001). Non-linear stability of the equilibria in the gravity field of a finite straight segment, Celes. Mech. Dyn. Astron., Vol. 81, pp. 235 – 248.

Rocco, E.M. (2010). Evaluation of the terrestrial albedo applied to some scientific missions, Space Sci. Rev., Vol. 151, No. 1, pp. 135 – 147.

Sharma, R.K., Taqvi, Z.A. and Bhatnagar, K.B. (2001). Existence and stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and a source of radiation, Indian J. Pure Appl. Math., Vol. 32, No. 2, pp. 255 – 266.

Simmons, J.F.L., McDonald, A.J.C. and Brown, J.C. (1985). The restricted 3-body problem with radiation pressure, Celes. Mech. Dyn. Astron., Vol. 35, No. 2, pp. 145 – 187.

Szebehely, V. (1967). *Theory of orbits, the restricted problem of three bodies*, Academic Press, New York.