[Applications and Applied Mathematics: An International Journal (AAM)](https://digitalcommons.pvamu.edu/aam)

Volume 13 | Issue 2 Article 35

12-2018

Non-existence of Hopf real hypersurfaces in complex quadric with recurrent Ricci tensor

Pooja Bansal
Jamia Millia Islamia

Mohammad H. Shahid
Jamia Millia Islamia

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 1171 – 1183

Non-existence of Hopf real hypersurfaces in complex quadric with recurrent Ricci tensor

[1]Pooja Bansal and [2]Mohammad Hasan Shahid

Department of Mathematics, Faculty of Natural Sciences
Jamia Millia Islamia
New Delhi-110025, India
[1]poojabansal811@gmail.com; [2]hasan_jmi@yahoo.com

Received: February 11, 2018; Accepted: September 27, 2018

Abstract

In this paper, we first introduce the notion of recurrent Ricci tensor which is the generalization of parallel Ricci tensor in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. After then, we investigate real hypersurfaces of the complex quadric Q^m with the condition of recurrent Ricci tensor and give the glimpse of full classification with this condition.

Keywords: Complex quadric; recurrent Ricci tensor; Hopf hypersurfaces

MSC 2010 No: 53C40, 53C55

1. Introduction

The theory of complex hypersurface is an active area of research in the field of differential geometry and was initiated in Smith (1967). After then, Reckziegel gave the concept of complex hypersurface in CP^{m+1} known as complex quadric Q^m (see Reckziegel (1995)).

We say that the Ricci tensor Ric is parallel if it satisfies $\nabla_X Ric = 0$ for $X \in \Gamma(TM)$, (see Suh (2015) and Suh et al. (2014)). The geometrical meaning of this notion is that the eigenspaces of the Ricci tensor Ric are parallel along any curve in M. Here, the eigenspaces of the Ricci tensor are said to be parallel if they are invariant under any parallel displacement along any curves on M in Q^m. On the other hand, it gives that if E is the eigenspace of Ricci tensor Ric then, for any $Y \in E, \nabla_X Y \in E$ along any direction X on M in Q^m.

Many papers have been appeared by considering the real hypersurface of complex quadric Q^m (see Bansal et al. (2017, 2018), Berndt et al. (2013), Klein (2008), Suh (2014, 2015), Suh et al. (2016)). Moreover, Y. J. Suh proved a theorem of non-existence for real hypersurfaces in complex quadric

$\mathcal{Q}^m$ with parallel Ricci tensor (see Suh (2015)).

In our paper, in order to make a generalisation of parallel Ricci tensor, we introduce a notion of *recurrent* Ricci tensor for a real hypersurface M in $\mathcal{Q}^m$ stated by

$$(\nabla_X Ric)Y = \omega(X)Ric(Y),$$

where, ω is 1-form on M and $X, Y \in \Gamma(TM)$.

Consequently, we have the results.

Theorem 1.

There do not exist any Hopf real hypersurfaces M in complex quadric $\mathcal{Q}^m$ with recurrent Ricci tensor and $\mathcal{U}$-principal normal vector field.

Also, motivated by the above given result, we give here one more proposition and theorem as follows:

Proposition 1.

Let M be a Hopf real hypersurfaces in the complex quadric $\mathcal{Q}^m$ with recurrent Ricci tensor and $\mathcal{U}$-isotropic unit normal vector field. Then, M can never be a tube over a totally geodesic $\mathbb{C}P^k$ in $\mathcal{Q}^m(m = 2k)$.

Theorem 2.

Let M be a Hopf real hypersurfaces in the complex quadric $\mathcal{Q}^m(m \geq 4)$, with recurrent Ricci tensor and $\mathcal{U}$-isotropic unit normal vector field. If the shape operator commutes with the structure tensor on the distribution $\mathbb{Q}^\perp$ then, M has 3 distinct constant principal curvatures given by

$$\alpha = 0,\ \gamma = 0,\ \lambda = -\sqrt{3},\ \text{and}\ \mu = \sqrt{3},$$

or

$$\alpha = \sqrt{2(m-3)},\ \gamma = 0,\ \lambda = 0,\ \text{and}\ \mu = -\frac{2}{\sqrt{2(m-3)}}, \tag{1.1}$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi],\ T_\gamma = [A\xi, AN],\ \phi(T_\lambda) = T_\mu,\ \text{with}\ \ dim\, T_\lambda = dim\, T_\mu = m-2.$$

As long as, by virtue of simpleness, all over the paper we denote real hypersurface and vector field by r.h. and v.f. respectively, unless otherwise stated.

2. Preliminaries

For more details of the geometry of the complex quadric we refer to (Berndt et al. (2013), Reckziegel (1995), Suh (2015)). The complex quadric $\mathcal{Q}^m$ is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the equation $z_1^2 + \cdots + z_{m+1}^2 = 0$, where $z_1, \cdots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$ equipped with the induced Riemannian metric g. Then, naturally the canonical Kähler structure (J, g) on $\mathcal{Q}^m$ is induced by Kähler structure on $\mathbb{C}P^{m+1}$ (see Suh (2014)). The 1-dimensional quadric $\mathcal{Q}^1$ is congruent to the round 2-sphere S^2. The 2-dimensional quadric $\mathcal{Q}^2$ is congruent to the Riemannian product $S^2 \times S^2$. For this, we will assume $m \geq 3$ throughout the paper.

Apart from the complex structure J there is one more geometric structure on $\mathcal{Q}^m$, known as complex conjugation A on the tangent spaces of $\mathcal{Q}^m$ which is a parallel rank-two vector bundle $\mathcal{U}$ containing S^1-bundle of real structures. For a point x in $\mathcal{Q}^m$, take a conjugate of x as $\overline{x}$ and let $A_{\overline{x}}$ be the shape operator at $\overline{x}$ of $\mathcal{Q}^m$ in $\mathbb{C}P^{m+1}$. Then, we have $A_{\overline{x}}W = W,\ W \in T_x\mathcal{Q}^m$, that is, A is an involution or $A_{\overline{x}}$ is complex conjugation restricted to $T_x\mathcal{Q}^m$. Now, $T_x\mathcal{Q}^m$ is decomposed as (see Suh (2014)):

$$T_x\mathcal{Q}^m = \mathcal{V}(A_{\overline{x}}) \oplus J\mathcal{V}(A_{\overline{x}}),$$

whereas $\mathcal{V}(A) = \{X \in T_{[x]}Q^{m*} | AX = X\}$ and $J\mathcal{V}(A) = \{X \in T_{[x]}Q^{m*} | AX = -X\}, [x] \in Q^{m*}$ denote the (+1)-eigenspace and (-1)-eigenspace of the involution $A^2 = I$ on $T_{[x]}Q^{m*}, [x] \in Q^{m*}$, respectively.

Now, $W \neq 0 \in T_x\mathcal{Q}^m$ is known as *singular* tangent vector if it is tangent to more than one maximal flat in $\mathcal{Q}^m$. There are two types of singular tangent vectors for the complex quadric Q^m (see Suh (2014)):

(1) If there exists a complex conjugation $A \in \mathcal{U}$ such that $W \in \mathcal{V}(A)$ then, W is singular. Such a singular tangent vector is called *$\mathcal{U}$-principal*.

(2) If there exists a conjugation $A \in \mathcal{U}$ and orthonormal vectors $X, Y \in \mathcal{V}(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$ then, W is singular. Such a singular tangent vector is called *$\mathcal{U}$-isotropic*. Further, there exists $A \in \mathcal{U}$ for $W \in T_x\mathcal{Q}^m$ and orthonormal vectors $X, Y \in \mathcal{V}(A)$ satisfying

$$W = \cos(t)X + \sin(t)JY, \text{ for } t \in [0, \pi/4].$$

Next, we recall some notions for r.h. M of complex quadric $\mathcal{Q}^m$. Let M^n be a r.h. of $\mathcal{Q}^m$ with a connection ∇ induced from the LC connection $\overline{\nabla}$ in $\mathcal{Q}^m$. Then, for a unit normal v.f. N of a r.h. of complex quadric $\mathcal{Q}^m$, a unit v.f. $\xi \in \Gamma(TM)$, known as Hopf v.f., is define by $\xi = -JN$ with dual 1-form $\zeta(X) = g(X, \xi)$. Then, any vector $X \in \Gamma(TM)$ can be written as $JX = \phi X + \zeta(X)N$, where ϕX stands for the tangential element of JX. Here, M associates an induced *almost contact metric structure* (ϕ, ξ, ζ, g) satisfying the following relations (see Blair (1976))

$$\left.\begin{array}{c} \zeta(\xi) = 1, \zeta(X) = g(\xi, X), \\ \phi^2 X + X = \zeta(X)\xi, \\ g(\phi X, \phi Y) + \zeta(X)\zeta(Y) = g(X, Y), g(\phi X, Y) = -g(X, \phi Y). \end{array}\right\} \tag{2.1}$$

Moreover, r.h. M of $\mathcal{Q}^m$ satisfy

$$\nabla_X \xi = \phi \Lambda X,$$

where Λ is the shaper operator of M.

Next, the fundamental Gauss and the Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + \Upsilon(X,Y), \ \overline{\nabla}_X N = -\Lambda X,$$

respectively, for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, whereas $\Gamma(TM)$ denotes set of all v.f. tangent to M and $\Gamma(T^{\perp}M)$ space of normal bundle. Next, Υ is a symmetric bilinear form, known as the second fundamental form. It should be noted that $\Upsilon(X,Y) = g(\Lambda X, Y)N$, we take $\Lambda_N X$ as ΛX for simplicity.

Now, we will take $A \in \mathcal{U}_z$ satisfying $N = \cos(t)Z_1 + \sin(t)JZ_2$, for an orthonormal vectors $Z_1, Z_2 \in \mathcal{V}(A)$ and $0 \leq t \leq \frac{\pi}{4}$ is a function on M (see Proposition 3, Reckziegel (1995)). Since $\xi = -JN$, we have

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \quad AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \quad A\xi = \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

from which it implies $g(\xi, \ AN) = 0$.

Here, we define a maximal $\mathcal{U}$ invariant subspace of $T_zM, z \in M$ as

$$\mathbb{Q}_x = \{X \in T_xM \mid AX \in T_xM \text{ for all } A \in \mathcal{U}\}.$$

Now, we give an important lemma

Lemma 1.

For each $x \in M$, we have

(1) If N_x is $\mathcal{U}$-principal, then $\mathbb{Q}_x = \mathcal{C}_x$.
(2) If N_x is not $\mathcal{U}$-principal, there exist a conjugation $A \in \mathcal{U}$ and orthonormal vectors $X, Y \in \mathcal{V}(A)$ such that $N_x = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathbb{Q}_x = \mathcal{C} \ominus \mathbb{C}(JX + Y)$.

3. Key Lemma

We take M to be a Hopf hypersurface. Then, we have $\Lambda\xi = \alpha\xi$, where α is a smooth function defined by $\alpha = g(\Lambda\xi, \xi)$ on M. Then,

$$\sum_{i=1}^{2m-1} g(Ae_i, e_i) = \text{tr}(A) - g(AN, N) = -g(AN, N),$$

$$\sum_{i=1}^{2m-1} g(AX, e_i)Ae_i = \sum_{i=1}^{2m} g(AX, e_i)Ae_i - g(AX, N)AN = X - g(AX, N)AN,$$

$$\begin{aligned} \sum_{i=1}^{2m-1} g(JAe_i, e_i)JAX &= \sum_{i=1}^{2m} g(JAe_i, e_i)JAX - g(JAN, N)JAX \\ &= -g(JAN, N)JAX, \end{aligned}$$

$$\sum_{i=1}^{2m-1} g(JAX, e_i)JAe_i = \sum_{i=1}^{2m} g(JAX, e_i)JAe_i - g(JAX, N)JAN$$
$$= JAJAX - g(JAX, N)JAN = X - g(JAX, N)JAN.$$

Next, from Suh (2015), the Gauss equation follows

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX \\ &\quad - g(JAX, Z)JAY + g(\Lambda Y, Z)\Lambda X - g(\Lambda X, Z)\Lambda Y, \end{aligned} \tag{3.1}$$

for $X, Y, Z \in \Gamma(TM)$.

Now, on contracting Y, Z in (3.1) and considering the above formualae, the Ricci tensor Ric of Hopf r.h. M is given by

$$\begin{aligned} Ric(X) &= (2m-1)X - X - \phi^2 X - 2\phi^2 X - g(AN, N)AX - X + (\text{tr}\Lambda)\Lambda X \\ &\quad + g(AX, N)AN - g(JAN, N)JAX - X + g(JAX, N)JAN - \Lambda^2 X, \\ &= (2m-1)X - 3\zeta(X)\xi - g(AN, N)AX + g(AX, N)AN \\ &\quad - g(JAN, N)JAX + g(JAX, N)JAN + (\text{tr}\Lambda)\Lambda X - \Lambda^2 X. \end{aligned} \tag{3.2}$$

Now, we can state the consequent lemma.

Lemma 2.

Let M be a r.h. of $\mathcal{Q}^m$ with recurrent Ricci tensor such that all eigenvalues of the Ricci tensor are constant. Then, the Ricci tensor is parallel.

Proof:

Let us assume that all eigenvalues of the Ricci tensor are constant and take Y be a unit eigenvector of the Ricci tensor associating to an eigenvalue λ, that is,

$$RicY = \lambda Y. \tag{3.3}$$

Thus, one can have

$$\begin{aligned} g((\nabla_X Ric)Y, Y) &= g(\nabla_X (RicY) - Ric(\nabla_X Y), Y) \\ &= X\lambda = 0. \end{aligned}$$

Again, by the assumption that M has recurrent Ricci tensor, it yields

$$\begin{aligned} g(\nabla_X RicY, Y) &= \omega(X) g(RicY, Y) \\ &= \omega(X)\lambda. \end{aligned}$$

By combining this relation with the previous relation, we get $\omega(X)\lambda = 0$ for $X \in \Gamma(TM)$, which assert that the Ricci tensor is parallel.

4. Proof of Theorem 1

Here, we suppose that M has $\mathcal{U}$-principal unit normal v.f. N, that is, $AN = N$ or $A\xi = -\xi$ for $A \in \mathcal{U}$ (see Suh et al. (2016)). Thus, we can have

$$g(AN, \xi) = 0 \text{ and } g(AN, N) = 1.$$

We put

$$AY = \mathcal{B}Y + \rho(Y)N,$$

whereas $\mathcal{B}Y$ stands for the tangential element of AY and $\rho(Y) = g(AY, N) = g(Y, N) = 0$. Hence, $AY = \mathcal{B}Y$ for $Y \in \Gamma(TM)$ with $\mathcal{U}$-principal unit normal v.f. N in $\mathcal{Q}^m$. Then, it follows that

$$Ric(X) = (2m-1)X - 2\zeta(X)\xi - AX + h\Lambda X - \Lambda^2 X,$$

whereas $h = \text{tr}(\Lambda)$ denotes the trace of the shape operator Λ of M in $\mathcal{Q}^m$.

Now, assume that Hopf r.h. M of $\mathcal{Q}^m$ has recurrent Ricci tensor

$$(\nabla_X Ric)Y = \omega(X)Ric(Y),$$

which together with $Y = \xi$ and using the fact of Hopf hypersurface follows

$$\begin{aligned}\omega(X)[(2m-2)\xi + \alpha h\xi - \alpha^2\xi] = {}& -2\phi\Lambda X - (\nabla_X A)\xi + \alpha(Xh)\xi \\ & + h(\nabla_X \Lambda)\xi - (\nabla_X \Lambda^2)\xi,\end{aligned}$$

from which we have

$$\begin{aligned}&(-2 + h\alpha - \alpha^2)\phi\Lambda X - [-q(X)N + 2\alpha\zeta(X)N] - h\Lambda\phi\Lambda X + \Lambda^2\phi\Lambda X \\ &= [\omega(X)(2m - 2 + \alpha h - \alpha^2) - \alpha(Xh) - h(X\alpha) + (X\alpha^2)]\xi,\end{aligned}$$

where we have used

$$\begin{aligned}(\nabla_X A)\xi &= \nabla_X A\xi - A\nabla_X \xi \\ &= (\overline{\nabla}_X A)\xi + A(\overline{\nabla}_X)\xi - g(\Lambda X, A\xi)N - A\nabla_X \xi \\ &= q(X)JA\xi + A\sigma(X, \xi) - g(\Lambda X, A\xi)N \\ &= -q(X)N + 2\alpha\zeta(X)N, \\ (\nabla_X \Lambda)\xi &= \nabla_X(\Lambda\xi) - \Lambda\nabla_X\xi = (X\alpha)\xi + \alpha\phi\Lambda X - \Lambda\phi\Lambda X,\end{aligned}$$

and

$$(\nabla_X \Lambda^2)\xi = \nabla_X(\Lambda^2\xi) - \Lambda^2\nabla_X\xi = (X\alpha^2)\xi + \alpha^2\phi\Lambda X - \Lambda^2\phi\Lambda X.$$

On taking scalar product with ξ, we get that the component of Reeb v.f. ξ vanishes. Thus, we are left with

$$(-2 + h\alpha - \alpha^2)\phi\Lambda X - [-q(X) + 2\alpha\zeta(X)]N - h\Lambda\phi\Lambda X + \Lambda^2\phi\Lambda X = 0.$$

By comparing the tangential part, we arrive at

$$(2+\alpha^2-h\alpha)\phi\Lambda X = -h\Lambda\phi\Lambda X + \Lambda^2\phi\Lambda X, \text{ for all } X \in \Gamma(TM). \tag{4.1}$$

Now, on taking $\Lambda X = \lambda X$, (4.1) reduced to

$$(2+\alpha^2-h\alpha)\lambda\phi X = -h\lambda\Lambda\phi X + \lambda\Lambda^2\phi X, \tag{4.2}$$

Also, for a r.h. in $\mathcal{Q}^m, (m \geq 3)$ with $\mathcal{U}$-principal normal v.f. N, we have

$$2\Lambda\phi\Lambda X = \alpha(\phi\Lambda + \Lambda\phi)X + 2\phi X, \tag{4.3}$$

which by the virtue of $\Lambda X = \lambda X$ yields

$$\Lambda\phi X = \Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}\Big)\phi X. \tag{4.4}$$

It follows that

$$\Lambda^2\phi X = \Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}\Big)^2\phi X. \tag{4.5}$$

From this together with (4.3), (4.1) becomes

$$(2+\alpha^2-h\alpha)\lambda\phi X = -h\lambda\Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}\Big)\phi X + \lambda\Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}\Big)^2\phi X.$$

Now, let us consider the cases on λ ($\lambda \neq 0$ and $\lambda = 0$).

case 1: $\lambda \neq 0$. Then, the above relation for $\mu = \Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}\Big)\phi X \neq 0$ implies

$$\mu^2 - h\mu + h\alpha - \alpha^2 - 2 = 0. \tag{4.6}$$

Also, the function λ satisfies

$$\lambda^2 - h\lambda + h\alpha - \alpha^2 - 2 = 0. \tag{4.7}$$

On equating (4.6) and (4.7), we have

$$(\lambda+\mu-h)(\lambda-\mu) = 0.$$

Since, $\lambda \neq \mu$, we get $h = \lambda+\mu$. But, the trace of the shape operator is given by $h = \alpha+(m-1)(\lambda+\mu)$. Thus, we get $h = -\frac{\alpha}{m-2}$, which is a contradiction. Also, (4.3) implies

$$\begin{aligned} 2\Lambda^2\phi\Lambda X &= \alpha(\Lambda\phi\Lambda + \Lambda^2\phi)X + 2\Lambda\phi X \\ &= \alpha\Big[\Big(\frac{\alpha}{2}\Big)(\Lambda\phi+\phi\Lambda)X + \phi X + \Lambda^2\phi X\Big] + 2\Lambda\phi X. \end{aligned} \tag{4.8}$$

Now, on taking account (4.8) into (4.1), we have

$$\begin{aligned} (2+\alpha^2-h\alpha)\phi\Lambda X = &-h\Big[\Big(\frac{\alpha}{2}\Big)(\Lambda\phi+\phi\Lambda)X + \phi X\Big] \\ &+ \Big[\Big(\frac{\alpha^2}{4}\Big)(\Lambda\phi+\phi\Lambda)X + \Big(\frac{\alpha}{2}\Big)\phi X + \Big(\frac{\alpha}{2}\Big)\Lambda^2\phi X + \Lambda\phi X\Big]. \end{aligned} \tag{4.9}$$

After substituting $h = -\frac{\alpha}{m-2}$ in (4.9)

$$\Big[2 + \Big(\frac{m-1}{m-2}\Big)\alpha^2\Big]\phi\Lambda X = \frac{\alpha^2}{2(m-2)}(\Lambda\phi + \phi\Lambda)X + \Big(\frac{\alpha}{m-2}\Big)\phi X + \Big(\frac{\alpha^2}{4}\Big)(\Lambda\phi + \phi\Lambda)X + \Big(\frac{\alpha}{2}\Big)\phi X + \Big(\frac{\alpha}{2}\Big)\Lambda^2\phi X + \Lambda\phi X,$$

which becomes

$$\Big[3 + \Big(\frac{m-1}{m-2}\Big)\alpha^2\Big]g((\phi\Lambda - \Lambda\phi)X, Y) = \Big(\frac{\alpha}{2}\Big)g((\Lambda^2\phi - \phi\Lambda^2)X, Y).$$

Now, for $\Lambda X = \lambda X$ and $\Lambda\phi X = \mu X$ with $\mu \neq \lambda$, this reduces to

$$(\lambda - \mu)\Big[3 + \Big(\frac{m-1}{m-2}\Big)\alpha^2 + \Big(\frac{\alpha}{2}\Big)(\mu + \lambda)\Big]\phi X = 0.$$

Since, λ and μ are distinct, we have $3 + \Big(\frac{2m-3}{2(m-2)}\Big)\alpha^2 = 0$, which again arises a contradiction.

case 2: $\lambda = 0$. Thus, we have $\mu = -\frac{2}{\alpha} \neq 0$. Then,

$$h = \mu = -\frac{2}{\alpha}.$$

Then, (4.9) gives

$$(4 + \alpha^2)\phi\Lambda X = (\Lambda\phi + \phi\Lambda)X + \Big(\frac{2}{\alpha}\Big)\phi X + \Big(\frac{\alpha^2}{4}\Big)(\Lambda\phi + \phi\Lambda)X + \Big(\frac{\alpha}{2}\Big)\phi X + \Big(\frac{\alpha}{2}\Big)\Lambda^2\phi X + \Lambda\phi X,$$

which is equivalent to

$$(\alpha^2 + 5)(\phi\Lambda - \Lambda\phi) = \Big(\frac{\alpha}{2}\Big)(\Lambda^2\phi - \phi\Lambda^2).$$

Again, $\Lambda X = \lambda X = 0$ and $\Lambda\phi X = \mu\phi X$ with $\mu = -\frac{\alpha}{2}$ yields $\alpha^2 + 4 = 0$ which again gives a contradiction. This assert that there do not exist any Hopf r.h. in $\mathcal{Q}^m$ with recurrent Ricci tensor and $\mathcal{U}$-principal normal v.f. This completes the proof of Theorem 1.

5. Proof of Theorem 2

First suppose that M has recurrent Ricci tensor with $\mathcal{U}$-isotropic unit normal v.f. N. Since, unit normal v.f. is $\mathcal{U}$-isotropic, we have the following relations (Suh et al. (2016))

$$g(AN, \xi) = 0,\ g(AN, N) = 0,\ g(A\xi, \xi) = 0,\ g(A\xi, AN) = 0.$$

Thus, we may take

$$AN = \mathcal{B}N + \rho(N)N,$$

whereas $\mathcal{B}N$ stands for the tangential element of AN and $\rho(N) = g(AN, N) = 0$. Hence, $AN = \mathcal{B}N$. From this, it follows that

$$\begin{aligned}\overline{\nabla}_X AN &= (\overline{\nabla}_X A)N + A\overline{\nabla}_X N \\ &= q(X)JAN - A\Lambda X, \qquad (5.1)\\ \nabla_X A\xi &= \overline{\nabla}_X A\xi - \sigma(X, A\xi) \\ &= (\overline{\nabla}_X A)\xi + A\overline{\nabla}_X \xi - g(\Lambda X, A\xi)N \\ &= -q(X)AN + A\phi\Lambda X + \alpha\zeta(X)AN - g(\Lambda X, A\xi)N. \qquad (5.2)\end{aligned}$$

On taking account (5.1) and (5.2) in the hypothesis of our assumption of recurrent Ricci tensor $(\nabla_X Ric)Y = \omega(X)Ric(Y)$, we get

$$\begin{aligned}\omega(X)(RicY) = &-3\zeta(Y)\phi\Lambda X - 3g(\phi\Lambda X, Y)\xi - g(Y, A\Lambda X)AN - g(Y, AN)A\Lambda X \\ &+ [g(Y, A\phi\Lambda X) + \alpha\zeta(X)g(Y, AN)]A\xi + g(AY, \xi)[A\phi\Lambda X \\ &+ \alpha\zeta(X)AN - g(\Lambda X, A\xi)N] + (Xh)\Lambda Y + h(\nabla_X \Lambda)Y - (\nabla_X \Lambda^2)Y.\end{aligned}$$

Moreover, considering $Y = \xi$ with the condition of Hopf hypersurface and using the fact $g(\xi, AN) = 0$ and $g(A\xi, \xi) = 0$, we arrive at

$$\begin{aligned}\omega(X)[(2m-4)\xi + h(\alpha\xi) - \alpha^2\xi] = &-3\phi\Lambda X - g(A\xi, \Lambda X)AN + g(A\xi, \phi\Lambda X)A\xi \\ &+ (Xh)(\alpha\xi) + h(\nabla_X \Lambda)\xi - (\nabla_X \Lambda^2)\xi. \qquad (5.3)\end{aligned}$$

Furthermore, we have the following relations

$$\begin{aligned}(\nabla_X \Lambda)\xi &= \nabla_X(\Lambda\xi) - \Lambda\nabla_X\xi = (X\alpha)\xi + \alpha\phi\Lambda X - \Lambda\phi\Lambda X, \\ (\nabla_X \Lambda^2)\xi &= \nabla_X(\Lambda^2\xi) - \Lambda^2\nabla_X\xi = (X\alpha^2)\xi + \alpha^2\phi\Lambda X - \Lambda^2\phi\Lambda X, \\ g(A\phi\Lambda X, \xi) &= g(\phi\Lambda X, A\xi) = g(J\Lambda X, A\xi) = g(\Lambda X, AN).\end{aligned}$$

From this, it follows that

$$\begin{aligned}-3\phi\Lambda X - g(A\xi, \Lambda X)AN + g(AN, \Lambda X)A\xi + (Xh)(\alpha\xi) + h[(X\alpha)\xi + \alpha\phi\Lambda X - \Lambda\phi\Lambda X] \\ -[(X\alpha^2)\xi + \alpha^2\phi\Lambda X - \Lambda^2\phi\Lambda X] = \omega(X)[(2m-4)\xi + h(\alpha\xi) - \alpha^2\xi].\end{aligned}$$

One can see, on taking scalar product with ξ, the coefficient of Reeb v.f. is zero. Hence,

$$(h\alpha - 3 - \alpha^2)\phi\Lambda X - g(A\xi, \Lambda X)AN + g(AN, \Lambda X)A\xi - h\Lambda\phi\Lambda X + \Lambda^2\phi\Lambda X = 0. \qquad (5.4)$$

Moreover, we have

$$2\Lambda\phi\Lambda X = \alpha(\phi\Lambda + \Lambda\phi)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \qquad (5.5)$$

Remark 1.

It is noted that, (Proposition 6.1, Berndt et al. (2013)) tells if the Λ commutes with ϕ, that is, $(\phi\Lambda = \Lambda\phi)$ then, the normal v.f. of M in $\mathcal{Q}^m$ is $\mathcal{U}$-isotropic.

So, let us assume $\phi\Lambda = \Lambda\phi$. Then, for any $X \in \mathbb{Q}$, $g(X, A\xi) = 0$ and $g(X, AN) = 0$ with $\Lambda X = \lambda X$ in (6.4), we see

$$(3 + \alpha^2 - h\alpha)\lambda + h\lambda^2 + \lambda^3 = 0.$$

case 1: $\lambda \neq 0$, the above equation reduces to

$$\lambda^2 - h\lambda - (3 + \alpha^2 - h\alpha) = 0. \tag{5.6}$$

Also, using $\phi\Lambda = \Lambda\phi$ and $\Lambda X = \lambda X$, (5.5) reduces to

$$\lambda^2 \phi X = \alpha(\lambda\phi X) + \phi X,$$

or equivalently

$$\lambda^2 - \alpha\lambda - 1 = 0. \tag{5.7}$$

From here, we have tr$(\Lambda) = h = \alpha$. On equating (5.5) and (5.6) with $h = \alpha$, we get a contradiction.

Using (Proposition 4.1, Berndt et al. (2013)), all of the above discussion summarize the proof of Proposition 1 which says that the tube over totally geodesic $\mathbb{C}P^k$ in $\mathcal{Q}^m(m = 2k)$ never has recurrent Ricci tensor.

Now, we take the distribution $\mathbb{Q}^\perp$. Then, from the Lemma 1, $\mathbb{Q}^\perp = \mathcal{C} \ominus \mathcal{Q} =$ span $\{AN, A\xi\}$ and assuming $\phi\Lambda = \Lambda\phi$ on the distribution $\mathbb{Q}^\perp$, one can easily get that the orthogonal distribution $\mathbb{Q}^\perp$ is invariant by the shape operator Λ.

Now, we may put $X = AN$ in (5.5). Then,

$$2\Lambda\phi\Lambda AN = \alpha(\phi\Lambda AN + A\phi AN) + 2\phi AN - 2A\xi,$$

which together with the assumption $\phi\Lambda = \Lambda\phi$ on $\mathbb{Q}^\perp$ and considering $\Lambda AN = \lambda AN$ yields

$$\lambda^2 = \alpha\lambda,$$

which arises two cases either $\lambda = 0$ or $\lambda = \alpha$.

Futhermore, since on the distribution $\mathbb{Q}, AN \in \mathbb{Q}$, we have $AX \in T_pM, p \in M$. So, (5.5) with $g(X, A\xi) = 0$ and $g(X, AN) = 0$ for $X \in \mathbb{Q}$ gives

$$2\Lambda\phi\Lambda X = \alpha(\Lambda\phi + \phi\Lambda)X + 2\phi X.$$

Now, we take an orthonormal basis $\{X_i\}_1^{2(m-2)}$ in $\mathbb{Q}$ such that $AX_i = \lambda X_i (i = 1, 2, ...(m-2))$. Then, from above one, we have

$$\Lambda\phi X_i = \Big(\frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\Big)\phi X_i.$$

Since, $\lambda = 0$ or $\lambda = \alpha$, the matrix of the shape operator can be given by block diagonal form

$$\Lambda = BD(\alpha, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3),$$

where, BD denotes for the block diagonal matrix such that $\mathcal{M}_1$ (respectively $\mathcal{M}_2, \mathcal{M}_3$) is the diagonal matrix with the entries $0(\alpha)$ (respectively $\lambda_i,\ \mu_i$) of order 2 (respectively $m-2, m-2$) for $i = 1, 2, ..., m-1$.

Further, for any $X, \phi X \in \mathbb{Q}$, we put $\Lambda X = \lambda X$ and $\Lambda \phi X = \mu \phi X$,

$$\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}. \tag{5.8}$$

Then, (5.4) becomes

$$(h\alpha - 3 - \alpha^2)\lambda\phi X - \lambda g(A\xi, X)AN + \lambda g(AN, X)A\xi - h\lambda\mu\phi X + \lambda\mu^2\phi X = 0,$$

or

$$(3 + \alpha^2 - h\alpha)\lambda + h\lambda\mu - \lambda\mu^2 = 0,$$

where, we have used $g(X, A\xi) = 0$ and $g(X, AN) = 0$. But since $\lambda \neq 0$, we have $(3 + \alpha^2 - h\alpha) + h\mu - \mu^2 = 0$, which is a quadratic equation given as

$$x^2 - hx - (3 + \alpha^2 - \alpha h) = 0, \text{ for } x = \mu. \tag{5.9}$$

Now, equation (5.9) have two distinct principal curvatures denoted by $\lambda,\ \mu$. Thus, tr$(\Lambda) = h = \lambda + \mu = \alpha + (m-2)h$ implies

$$h = \alpha + (m-2)h. \tag{5.10}$$

On the other hand,

$$h\alpha - \alpha^2 = [\alpha + (m-2)(\lambda + \mu)]\alpha - \alpha^2 = (m-2)h\alpha.$$

Furthermore, using $\Lambda A\xi = \alpha A\xi$ and $\Lambda AN = \alpha AN$

$$h = \lambda + \mu = 3\alpha + (m-2)(\lambda + \mu) \tag{5.11}$$

$$= 3\alpha + (m-2)h. \tag{5.12}$$

On combining (5.10) and (5.11), we get $\alpha = 0$ which further gives $h(m-3) = 0$.

Let us take $m \geq 4$ then, the trace of shape operator vanishes, that is, $\lambda = -\mu$. Moreover, from (5.9) we may put $\lambda = -\sqrt{3}\tan r$ and $\mu = \sqrt{3}\cot r$. Further, we know that $\rho = 0$ or $\rho = \alpha$ by the condition of $AN, A\xi$. On summing up all these results, we take $\alpha = 0,\ \rho = 0,\ \lambda = -\sqrt{3}$ and $\mu = \sqrt{3}$ with multiplicities 1, 2, (m - 2) and (m - 2), respectively.

case 2: $\lambda = 0$. Then, (5.8) yields $\mu = -\frac{2}{\alpha}$. Then, the general matrix of Λ becomes

$$\Lambda = BD(\alpha, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3),$$

where, BD denotes for the block diagonal matrix such that $\mathcal{M}_1$ (respectively $\mathcal{M}_2, \mathcal{M}_3$) is the diagonal matrix with the entries $0(\alpha)$ (respectively $0,\ -\frac{2}{\alpha}$) of order 2 (respectively $m-2, m-2$) for $i = 1, 2, ..., m-1$.

Moreover, $h = \lambda + \mu = \alpha + (m-2)\lambda + \mu$ becomes $-\frac{2}{\alpha} = \alpha - \frac{2(m-2)}{\alpha}$ which gives $\alpha = \sqrt{2(m-3)}$. Thus, on distribution $\mathbb{Q}^{\perp}$, we have principal curvatures

$$\begin{cases} \alpha = \sqrt{2(m-3)}, & \text{(with multiplicity 1)}, \\ 0, & \text{(with multiplicity 2)}, \\ 0, & \text{(with multiplicity } (m-2)), \\ -\frac{2}{\alpha} = -\frac{2}{\sqrt{2(m-3)}}, & \text{(with multiplicity } (m-2)). \end{cases}$$

Then, the trace of the shape operator corresponding to an eigenvalue 0 is given by

$$\begin{aligned} h &= \sqrt{2(m-3)} - \frac{2(m-2)}{\sqrt{2(m-3)}} \\ &= -\frac{2}{\sqrt{2(m-3)}}, \end{aligned}$$

and the respective matrix has the block diagonal form $\Lambda = BD(\alpha, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$, where BD denotes for the block diagonal matrix such that $\mathcal{M}_1$ (respectively $\mathcal{M}_2, \mathcal{M}_3$) is the diagonal matrix with the entries 0 (respectively 0, $-\frac{2}{\alpha}$) of order 2 (respectively $m-2, m-2$).

If we have principal curvatures

$$\begin{cases} \alpha = \sqrt{2(m-3)}, & \text{(with multiplicity 1)}, \\ \alpha, & \text{(with multiplicity 2)}, \\ 0, & \text{(with multiplicity } (m-2)), \\ -\frac{2}{\alpha} = -\frac{2}{\sqrt{2(m-3)}}, & \text{(with multiplicity } (m-2)). \end{cases}$$

Then, the trace of the shape operator corresponding to an eigenvalue α is given by

$$\begin{aligned} h &= 3\sqrt{2(m-3)} - \frac{2(m-2)}{\sqrt{2(m-3)}} \\ &= -\frac{2(2m-7)}{\sqrt{2(m-3)}}, \end{aligned}$$

and the respective matrix has the block diagonal form $\Lambda = BD(\alpha, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$, where BD denotes for the block diagonal matrix such that $\mathcal{M}_1$ (respectively $\mathcal{M}_2, \mathcal{M}_3$) is the diagonal matrix with the entries α (respectively 0, $-\frac{2}{\alpha}$) of order 2 (respectively $m-2, m-2$).

So, we write

$$\left. \begin{array}{c} \alpha = \sqrt{2(m-3)} \text{ and } T_\alpha = \{\xi\}, \\ \gamma = 0 \text{ and } T_\gamma = \{A\xi, AN\}, \\ \lambda = 0 \text{ and } \phi T_\lambda = T_\mu, \\ \mu = -\frac{2(m-2)}{\sqrt{2(m-3)}} \text{ and } dimT_\lambda = dimT_\mu = m-2. \end{array} \right\}$$

This completes the proof of Theorem 2.

6. Conclusion

One of the great interest in the submanifold theory is to develop basic fundamental relationships of a submanifold. In this work, using the fundamental equations for r.h, we generalized the result of

Suh (2015). The findings motivate further studies to obtain general geometric fundamental properties for r.h. in the complex quadric Q^m, like (Berndt et al. (2013), Kim et al. (2007), Suh (2015, 2014), Suh et al. (2016)).

Acknowledgement:

The authors wishes to express sincere thanks to the referees for their valuable remarks and suggestions towards the improvement of the manuscript.

REFERENCES

Bansal, P. (2017). Hopf real hypersurfaces in the complex quadric Q^m with recurrent Jacobi operator, Accepted in AISC series, Springer.

Bansal, P. and Shahid, M. H. (2018). Optimization approach for bounds involving generalized normalized δ-Casorati curvatures, AISC series, Springer, Vol. 741, pp. 227-237.

Bansal, P. and Shahid, M. H. (2018). Bounds of generalized normalized δ-Casorati curvatures for real hypersurfaces in the complex quadric, Arab. J. Math., doi.org/10.1007/s40065-018-0223-7.

Berndt, J. and Suh, Y. J. (2013). Real hypersurfaces with isometric Reeb flow in complex quadrics, Internat. J. Math., Vol. 24, 1350050, 18pp.

Blair, D. E. (1976). Contact manifolds in Riemannian Geometry, Lecture Notes in Math, Vol. 509, Springer-Verlag, Berlin.

Klein, S. (2008). Totally geodesic submanifolds in the complex quadric, Differential Geom. Appl., Vol. 26, pp. 79-96.

Kim, U. H., Pérez, J. D., Santos, F. G. and Suh, Y. J. (2007). Real hypersurfaces in complex space forms with ξ-parallel Ricci tensor and structure Jacobi operator, J. of Korean Math. Soc., Vol. 44, pp. 307-326.

Reckziegel, H. (1995). On the geometry of the complex quadric, in: Geometry and Topology of Submanifolds VIII, Brussels/Nordfjordeid, 1995, World Sci. Publ., River Edge, NJ, pp.302-315.

Smith, B. (1967). Differential geometry of complex hypersurfaces, Ann. of Math., Vol. 85, pp. 246-266.

Suh, Y. J. (2014). Real hypersurfaces in the complex quadric with Reeb parallel shape operator, Internat. J. Math., Vol. 25, 1450059, 17pp.

Suh, Y. J. (2015). Real hypersurfaces in the complex quadric with parallel Ricci tensor, Advances in Mathematics, Vol. 281, pp. 886-905.

Suh, Y. J. and Woo, C. (2014). Real hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor, Math Nachr., Vol. 287, pp. 1524-1529.

Suh, Y. J. and Hwang, D. H. (2016). Real hypersurfaces in the complex quadric with commuting Ricci tensor, Sci. China Math., Vol. 59, Issue 11, pp 2185-2198.