Conformable Derivative Operator in Modelling Neuronal Dynamics

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Abstract

This study presents two new numerical techniques for solving time-fractional one-dimensional cable differential equation (FCE) modeling neuronal dynamics. We have introduced new formulations for the approximate-analytical solution of the FCE by using modified homotopy perturbation method defined with conformable operator (MHPMC) and reduced differential transform method defined with conformable operator (RDTMC), which are derived the solutions for linear-nonlinear fractional PDEs. In order to show the efficiencies of these methods, we have compared the numerical and exact solutions of fractional neuronal dynamics problem. Moreover, we have declared that the proposed models are very accurate and illustrative techniques in determining to approximate-analytical solutions for the PDEs of fractional order in conformable sense.

1. Introduction

Many scientist pay attention to fractional ordinary/partial differential equations on a day-to-day basis. During the last few decades, they have especially used the FDEs in modelling and describing certain problems such as diffusion processes, biology, chemistry, engineering, economic, material sciences and other areas of application. In recent years, some special

analytical-approximate solution methods such as Adomian decomposition Evirgen et al. (2011), Ilie et al. (2018), Yavuz et al. (2018c), Yavuz et al. (2019), homotopy decomposition Atangana et al. (2013), homotopy perturbation transform Singh et al. (2015), local fractional Laplace variational iteration Jafari et al. (2016), separation of variables Bishehniasar et al. (2017), fractional homotopy analysis transform Kumar et al. (2014), Laplace perturbation method Yavuz et al. (2018), optimal homotopy asymptotic method, Ilie et al. (2019), rational approximation Özdemir et al. (2017), Turut et al. (2016), Turut et al. (2013), inverse Laplace homotopy Yavuz et al. (2018b), improved G'/G-expansion Biazar et al. (2011), mesh-free radial basis function interpolation Usta (2017) and other methods Ali et al. (2016), Baskonus et al. (2015), Bildik et al. (2006), Çenesiz et al. (2017), Evirgen et al. (2012), Hristov (2016), Koca et al. (2016), Kurulay et al. (2013), Morales-Delgado et al. (2016), Özdemir et al. (2009), Yavuz et al. (2018a), Yavuz et al. (2016), Yokus et al. (2018) have been used in order to obtain solutions of fractional partial differential equations (FPDEs). Khalil et al. (2014) defined the conformable derivative operator in 2014 and this operator has been applied to many fractional PDEs in different fields. In addition, many authors have been applied this derivative operator to their studies such as Abdeljawad et al. (2016), Al-Salti et al. (2017), Atangana et al. (2016), Atangana et al. (2016), Avcı et al. (2017), Avci et al. (2017), Caputo et al. (2015), Eroğlu et al. (2017), Evirgen (2017), Gómez-Aguilar et al. (2016), Hristov (2017), Koca et al. (2017), Koca et al. (2016), Scherer et al. (2011), Yavuz (2018). After that, new improvements and applications of the conformable operator have been developed by Abdeljawad, (2015), Anderson et al. (2015), Atangana et al. (2015), Avcı et al. (2017), Avci et al. (2017), Batarfi et al. (2015), Eroğlu et al. (2017), Yavuz (2018).

The fractional cable equation (FCE) can be given in its general form as Liu et al. (2009):

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{I}_{*t}^{1-\gamma_1}\left(K \frac{\partial^2 u(x,t)}{\partial x^2} \right) + \mu_0^2 \, {}_0\mathfrak{I}_{*t}^{1-\gamma_2} u(x,t) + f(x,t), \tag{1}$$

with the initial condition

$$u(x,0) = g(x), \quad 0 \le x \le L \tag{2}$$

and the boundary conditions

$$u(0,t) = \varphi(t), \quad u(L,t) = \psi(t), \quad 0 \le t \le T, \tag{3}$$

where $0 < \gamma_1, \gamma_2 < 1$, $K > 0$ and μ_0^2 are constants, and ${}_0\mathfrak{I}_{*t}^{1-\gamma_1} u(x,t)$ is the conformable derivative operator of order $1-\gamma_1$. In the literature, there are some processes of approximate solutions of the FCE. Conformable ADM and conformable VIM Yavuz et al. (2017), implicit numerical methods (INM) Liu et al. (2009), the implicit compact difference scheme (ICDS) Hu et al. (2012), and explicit numerical methods (ENM) Quintana-Murillo et al. (2011) have been applied to the FCE.

In this study, we consider the following non-homogeneous fractional cable equation for the special case:

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$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{I}_{*t}^{1-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2} - {}_0\mathfrak{I}_{*t}^{1-\alpha}u(x,t) + f(x,t), \quad 0 < \alpha \leq 1, \tag{4}$$

with the special initial condition

$$u(x,0) = 0, \quad 0 \leq x \leq 1 \tag{5}$$

and the special boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leq t \leq T, \tag{6}$$

where

$$f(x,t) = 2\sin\pi x\left(t + (\pi^2 + 1)\frac{t^{1+\alpha}}{\Gamma(\alpha + 2)}\right).$$

The exact solution of equations (1) - (3) is given by $u(x,t) = t^2\sin\pi x$ Liu et al. (2009).

The main purpose of this study is to redefine MHPM and RDTM for the solution of the FCE by using the conformable derivative. We have solved FCE of fractional order by using the recommended methods and we have compared the numerical and approximate-analytical solutions in terms of figures and tables. Therefore, we have fulfilled the purpose. When looking at the results, it is obvious that these methods are very effective and accurate for solving fractional cable differential equation (FCDE).

2. Conformable Derivative Operator

Definition 2.1.

Given a function $f : [0,\infty) \to R$, then, the conformable derivative of f order $\alpha \in (0,1]$ is defined by

$$\mathfrak{I}_{*t}^{\alpha}(f)(t) = \lim_{\varepsilon \to 0}\frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$ Khalil et al. (2014).

Theorem 2.2.

Let $\alpha \in (0,1]$ and f, g be α − differentiable at a point $t > 0$. Then, Khalil et al. (2014);

     i. $\mathfrak{I}_{*t}^{\alpha}(af + bg) = a\mathfrak{I}_{*t}^{\alpha}(f) + b\mathfrak{I}_{*t}^{\alpha}(g)$ for all $a,b \in R,$

     ii. $\mathfrak{I}_{*t}^{\alpha}(t^k) = kt^{k-\alpha}$, for all $k \in R,$

iii. $\mathfrak{I}_{*t}^{\alpha}\left(f\left(t\right)\right)=0$ for all constant functions $f\left(t\right)=k$,

iv. $\mathfrak{I}_{*t}^{\alpha}\left(fg\right)=f\,\mathfrak{I}_{*t}^{\alpha}\left(g\right)+g\,\mathfrak{I}_{*t}^{\alpha}\left(f\right)$,

v. $\mathfrak{I}_{*t}^{\alpha}\left(f\,/\,g\right)=\dfrac{g\,\mathfrak{I}_{*t}^{\alpha}\left(f\right)-f\,\mathfrak{I}_{*t}^{\alpha}\left(g\right)}{g^{2}}$, and

vi.   If $f\left(t\right)$ is differentiable, then $\mathfrak{I}_{*t}^{\alpha}\left(f\left(t\right)\right)=t^{1-\alpha}\dfrac{d}{dt}f\left(t\right)$.

Definition 2.3.

Let f be an $n-$ times differentiable at t. Then, the conformable derivative of f order α is defined as Anderson et al. (2015), Khalil et al. (2014):

$$\mathfrak{I}_{*t}^{\alpha}\left(f\left(t\right)\right)=\lim_{\varepsilon\to 0}\frac{f^{\left(\lceil\alpha\rceil-1\right)}\left(t+\varepsilon t^{\left(\lceil\alpha\rceil-\alpha\right)}\right)-f^{\left(\lceil\alpha\rceil-1\right)}\left(t\right)}{\varepsilon},$$

for all $t>0$, $\alpha\in\left(n,n+1\right]$. Here, $\lceil\alpha\rceil$ is the smallest integer that is greater than or equal to α.

Lemma 2.4.

Let f be an $n-$ times differentiable at t. Then,

$$\mathfrak{I}_{*t}^{\alpha}\left(f\left(t\right)\right)=t^{\lceil\alpha\rceil-\alpha}f^{\lceil\alpha\rceil}\left(t\right),$$

for all $t>0$, $\alpha\in\left(n,n+1\right]$ Khalil et al. (2014).

3. Modified homotopy perturbation method in conformable sense

In this section, we illustrate the solution strategies that are generated by modified homotopy perturbation method in conformable-type derivative (CMHPM). Now we introduce a solution algorithm in an effective way for the general linear FPDEs. In this regard, firstly, we consider the following linear fractional equation:

$$\mathfrak{I}_{*t}^{\alpha}u\left(x,t\right)=L\left(u,u_{x},u_{xx}\right)+v\left(x,t\right),\quad t>0,\tag{7}$$

where L is a linear operator, v is a known analytical function and $\mathfrak{I}_{*t}^{\alpha}$, $m-1<\alpha\leq m$, shows the conformable derivative of order α. We also have the following initial condition

$$u^{(k)}\left(x,0\right)=f_{k}\left(x\right),\quad k=0,1,\ldots,m-1.$$

Considering the mentioned technique above, the following homotopy can be derived as:

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$$\frac{\partial^m u}{\partial t^m} - v_1(x,t) = p\left(\frac{\partial^m u}{\partial t^m} + L(u, u_x, u_{xx}) - \mathfrak{I}^\alpha_{*t} u(x,t) + v_2(x,t)\right), \ p \in [0,1], \qquad (8)$$

where $v(x,t) = v_1(x,t) + v_2(x,t)$.

Here, the function $v(x,t)$ is divided into two parts, namely $v_1(x,t)$ and $v_2(x,t)$. The suggestion is that only the part $v_1(x,t)$ is assigned to the zero*th* component u_0, whereas the remaining part $v_2(x,t)$ is combined with u_1.

If we take the homotopy parameter $p = 0$, then equation (8) expresses the following linear equations,

$$\frac{\partial^m u}{\partial t^m} = v_1(x,t).$$

In case of $p = 1$, equation (8) represents the main original differential equation of fractional order in equation (7). Therefore, we get the solution of equation (8) by using a power series of p:

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots. \qquad (9)$$

Substituting (9) into (8) and equating the terms with identical powers of p, we can obtain a series of linear equations of the form

$$p^0 : \frac{\partial^m u_0}{\partial t^m} = v_1(x,t), \ u_0^{(k)}(x,0) = f_k(x),$$

$$p^1 : \frac{\partial^m u_1}{\partial t^m} = \frac{\partial^m u_0}{\partial t^m} + L(u_0) - \mathfrak{I}^\alpha_{*t} u_0 + v_2(x,t), \ u_1^{(k)}(x,0) = 0,$$

$$p^2 : \frac{\partial^m u_2}{\partial t^m} = \frac{\partial^m u_1}{\partial t^m} + L(u_1) - \mathfrak{I}^\alpha_{*t} u_1, \ u_2^{(k)}(x,0) = 0,$$

$$p^3 : \frac{\partial^m u_3}{\partial t^m} = \frac{\partial^m u_2}{\partial t^m} + L(u_2) - \mathfrak{I}^\alpha_{*t} u_2, \ u_3^{(k)}(x,0) = 0,$$

$$\vdots$$

At the end of the solution steps, we approximate the solution as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

4. Reduced differential transform method in conformable sense

Now we need some basic definitions and properties of RDTM with conformable-type derivation. Throughout the study, we represent the original function with the lowercase $u(x,t)$ and the fractional reduced differential transformed function with the uppercase $U_h^\alpha(x)$ in conformable sense.

Definition 4.1.

We consider the analytic and differentiated continuously function $u(x,t)$ with respect to time t and space variable x. Then, the fractional reduced differential transformed function of $u(x,t)$ is defined as Acan et al. (2017)

$$U_h^\alpha(x) = \frac{1}{\alpha^h h!}\left[\left({}_\alpha\mathfrak{J}_{*t}^{(h)}u\right)\right]_{t=t_0},$$

where $\alpha, (0 < \alpha \le 1)$ is the fractional parameter of the conformable-type operator, and the $t-$dimensional spectrum function $U_h^\alpha(x)$ shows the CFRD transformed function.

Definition 4.2.

Let $U_h^\alpha(x)$ be the transformed function of $u(x,t)$. Then, the inverse transformed function of $U_h^\alpha(x)$ is defined as

$$u(x,t) = \sum_{h=0}^{\infty} U_h^\alpha(x)(t-t_0)^{\alpha h} = \sum_{h=0}^{\infty}\frac{1}{\alpha^h h!}\left[{}_\alpha\mathfrak{J}_{*t}^{(h)}u\right]_{t=t_0}(t-t_0)^{\alpha h}.$$

In addition, transformed functions of the initial conditions are defined as

$$U_h^\alpha(x) = \begin{cases} \dfrac{1}{(\alpha h)!}\left[\dfrac{\partial^{\alpha h}}{\partial t^{\alpha h}}u(x,t)\right]_{t=t_0}, & \text{if } \alpha h \in \mathbb{Z}^+, \\ 0, & \text{if } \alpha h \notin \mathbb{Z}^+, \end{cases} \quad \text{for } h = 0,1,2,\ldots,\left(\frac{n}{\alpha}-1\right),$$

where n is the order of conformable PDE.

Now we consider the following general linear fractional differential equation:

$$\mathfrak{J}_{*t}^\alpha u(x,t) = Lu(x,t) + v(x,t), \tag{10}$$

with the initial condition

$$u(x,0) = f(x). \tag{11}$$

According to the CRDTM, we can construct the following result:

$$\alpha(h+1)U_{h+1}^\alpha(x) = LU_h^\alpha(x) + V_h^\alpha(x). \tag{12}$$

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By using the initial condition (11), we get

$$U_0^\alpha(x) = f(x). \tag{13}$$

Substituting (13) into (12) and by straightforward iterative calculations, we have the following $U_h^\alpha(x)$ functions for values $h = 0, 1, 2, 3, \ldots, n.$ Then, the inverse transformed function of the $\left\{U_h^\alpha(x)\right\}_{h=0}^n$ gives the approximate solution as:

$$\tilde{u}_n(x,t) = \sum_{h=0}^{\infty} U_h^\alpha(x) t^{h\alpha},$$

where n shows the order of approximate solution. Moreover, the exact solution of equation (10) is given by:

$$u(x,t) = \lim_{n \to \infty} \tilde{u}_n(x,t).$$

The main transformations of CFRDT that are used extensively and that can be derived from Definition 4.1 and Definition 4.2 are listed in Table 1.

Table 1. Transformations of some original functions.

Original Function	Transformed Function
$u(x,t)$	$U_h^\alpha(x) = \dfrac{1}{\alpha^h h!}\left[\left({}_\alpha\mathfrak{I}_{*t}^{(h)} u\right)\right]_{t=t_0}$
$u(x,t) = av(x,t) \pm bw(x,t)$	$U_h^\alpha(x) = aV_h^\alpha(x) \pm bW_h^\alpha(x)$
$u(x,t) = v(x,t)w(x,t)$	$U_h^\alpha(x) = \displaystyle\sum_{r=0}^{h} V_r^\alpha(x)W_{h-r}^\alpha(x)$
$u(x,t) = \mathfrak{I}_{*t}^\alpha v(x,t)$	$U_h^\alpha(x) = \alpha(h+1)V_{h+1}^\alpha(x)$
$u(x,t) = x^m(t-t_0)^n$	$U_h^\alpha(x) = x^m\delta\left(h - \dfrac{n}{\alpha}\right), \quad \delta\left(h - \dfrac{n}{\alpha}\right) = \begin{cases} 1, & if\ h = \dfrac{n}{\alpha}, \\ 0, & if\ h \neq \dfrac{n}{\alpha}. \end{cases}$

5. Solution of the fractional cable equation

In this section of the study, we apply the suggested methods in Section 3 and Section 4 to the fractional cable equation (4) with its initial condition (5) and its boundary conditions (6), which is one of the most important equations in the biology literature in modeling of neuronal dynamics.

5.1. Solution by MHPM defined with the conformable-type derivation

Firstly, we solve the fractional cable equation by using CMHPM.

Let $L_\alpha = \mathfrak{S}_{*t}^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha} = t^{1-\alpha}\dfrac{\partial}{\partial t}$ be a linear operator and $L_\alpha^{-1} = \displaystyle\int_0^t \dfrac{1}{\zeta^{1-\alpha}}(.)\,d\zeta$ be inverse of the

linear operator. Then if we apply the operator $\mathfrak{S}_{*t}^{-1}$ to both sides of equation (4), we get

$$u(x,t) = \mathfrak{S}_{*t}^{-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2} - \mathfrak{S}_{*t}^{-\alpha}u(x,t) + \mathfrak{S}_{*t}^{-1}f(x,t). \tag{14}$$

Now, applying the operator $\mathfrak{S}_{*t}^\alpha$ to both sides of equation (14), we obtain

$$\mathfrak{S}_{*t}^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + \mathfrak{S}_{*t}^\alpha \mathfrak{S}^{-1}f(x,t). \tag{15}$$

Considering the initial condition (5) and according to the homotopy (8) and where $v_1(x,t) = 0$, $v_2(x,t) = v(x,t)$ are taken, we can write the iterations of the perturbation series as:

$$
\begin{aligned}
&\frac{\partial u_0}{\partial t} = 0,\ u_0(x,0) = 0,\\
&\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{\partial^2 u_0}{\partial x^2} - u_0 - \mathfrak{S}_{*t}^\alpha u_0 + \mathfrak{S}_{*t}^\alpha \mathfrak{S}^{-1}f(x,t),\ u_1(x,0) = 0,\\
&\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial^2 u_1}{\partial x^2} - u_1 - \mathfrak{S}_{*t}^\alpha u_1,\ u_2(x,0) = 0,\\
&\ \vdots
\end{aligned}
\tag{16}
$$

By solving the equations in (16) according to $u_0,\ u_1,\ u_2$ and u_3, the first several components of the CMHPM solution for equation (4), are given by:

$$u_0(x,t) = 0,$$

$$u_1(x,t) = 2\sin\pi x\left(\frac{t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{t^3}{3\Gamma(2+\alpha)}\right),$$

$$
\begin{aligned}
u_2(x,t) = {}& 2\sin\pi x\left(\frac{t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{t^3}{3\Gamma(2+\alpha)}\right)\\
& -2(\pi^2+1)\sin\pi x\left(\frac{t^{4-\alpha}}{(4-\alpha)(3-\alpha)} + (\pi^2+1)\frac{t^4}{4\cdot3\cdot\Gamma(2+\alpha)}\right)\\
& -2\sin\pi x\left(\frac{t^{4-2\alpha}}{4-2\alpha} + (\pi^2+1)\frac{t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)}\right),
\end{aligned}
$$

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$$u_3(x,t) = 2\sin \pi x \left(\frac{t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{t^3}{3\Gamma(2+\alpha)} \right)$$

$$-4(\pi^2+1)\sin \pi x \left(\frac{t^{4-\alpha}}{(4-\alpha)(3-\alpha)} + (\pi^2+1)\frac{t^4}{4\cdot 3\cdot\Gamma(2+\alpha)} \right)$$

$$-4\sin \pi x \left(\frac{t^{4-2\alpha}}{4-2\alpha} + (\pi^2+1)\frac{t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)} \right)$$

$$+2(\pi^2+1)^2\sin \pi x \left(\frac{t^{5-\alpha}}{(5-\alpha)(4-\alpha)(3-\alpha)} + (\pi^2+1)\frac{t^5}{5\cdot 4\cdot 3\cdot\Gamma(2+\alpha)} \right)$$

$$+2(\pi^2+1)\sin \pi x \left(\frac{t^{5-2\alpha}}{(5-2\alpha)(4-2\alpha)} + (\pi^2+1)\frac{t^{5-\alpha}}{(5-\alpha)(4-\alpha)\Gamma(2+\alpha)} \right)$$

$$+2(\pi^2+1)\sin \pi x \left(\frac{t^{5-2\alpha}}{(5-2\alpha)(3-\alpha)} + (\pi^2+1)\frac{t^{5-\alpha}}{3(5-\alpha)\Gamma(2+\alpha)} \right)$$

$$+2\sin \pi x \left(\frac{t^{5-3\alpha}}{5-3\alpha} + (\pi^2+1)\frac{t^{5-2\alpha}}{(5-2\alpha)\Gamma(2+\alpha)} \right),$$

$$\vdots$$

continuing in this way, the remaining steps of the homotopy can be obtained. Then the numerical solution of equation (4) is presented by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$= 2\sin \pi x \left(\frac{3t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{3t^3}{3\Gamma(2+\alpha)} - \frac{3(\pi^2+1)t^{4-\alpha}}{(4-\alpha)(3-\alpha)} - \frac{3(\pi^2+1)^2 t^4}{4\cdot 3\cdot\Gamma(2+\alpha)} \right.$$

$$-\frac{3t^{4-2\alpha}}{4-2\alpha} - \frac{3(\pi^2+1)t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)} + \frac{(\pi^2+1)^2 t^{5-\alpha}}{(5-\alpha)(4-\alpha)(3-\alpha)} + \frac{(\pi^2+1)^3 t^5}{5\cdot 4\cdot 3\cdot\Gamma(2+\alpha)}$$

$$+\frac{(\pi^2+1)t^{5-2\alpha}}{(5-2\alpha)(4-2\alpha)} + \frac{(\pi^2+1)^2 t^{5-\alpha}}{(5-\alpha)(4-\alpha)\Gamma(2+\alpha)} + \frac{(\pi^2+1)t^{5-2\alpha}}{(5-2\alpha)(3-\alpha)}$$

$$\left. +\frac{(\pi^2+1)^2 t^{5-\alpha}}{3(5-\alpha)\Gamma(2+\alpha)} + \frac{t^{5-3\alpha}}{5-3\alpha} + \frac{(\pi^2+1)t^{5-2\alpha}}{(5-2\alpha)\Gamma(2+\alpha)} + \cdots \right).$$

Then, the exact solution of the equation (4) with its initial condition (5) and its boundary conditions (6) for special case of $\alpha = 1$, is obtained with CMHPM as $u(x,t) \cong t^2 \sin \pi x$.

5.2. Solution by RDTM defined with the conformable-type derivation

Secondly, we apply the proposed method to the fractional cable equation. Considering the equation with the conformable operator, we get

$$\mathfrak{I}_{*t}^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + t^{1-\alpha} f(x,t).$$

By taking the transformed function in Definition 4.1, it can be obtained that

$$\alpha(h+1)U_{h+1}^{\alpha}(x) = \frac{\partial^2 U_h^{\alpha}(x)}{\partial x^2} - U_h^{\alpha}(x) + t^{1-\alpha} f(x,t),$$

$$\alpha(h+1)U_{h+1}^{\alpha}(x) = \frac{\partial^2 U_h^{\alpha}(x)}{\partial x^2} - U_h^{\alpha}(x) + 2\sin\pi x\left(\delta\left(h - \frac{2-\alpha}{\alpha}\right) + (\pi^2+1)\frac{\delta\left(h - \frac{2}{\alpha}\right)}{\Gamma(2+\alpha)}\right),$$

where the $t-$dimensional spectrum function $U_h^{\alpha}(x)$ is the conformable reduced differential transform function. From the initial condition (5) we have $U_0^{\alpha}(x) = 0$. Moreover, we obtain the following $U_h^{\alpha}(x)$ functions as follows:

$$U_1^{\alpha}(x) = 0,$$
$$U_2^{\alpha}(x) = \sin\pi x,$$
$$U_3^{\alpha}(x) = -\frac{(\pi^2+1)}{2+\alpha}\sin\pi x + \frac{2(\pi^2+1)}{\Gamma(3+\alpha)}\sin\pi x,$$
$$U_4^{\alpha}(x) = \frac{(\pi^2+1)^2}{(2+2\alpha)(2+\alpha)}\sin\pi x - \frac{2(\pi^2+1)^2}{(2+2\alpha)\Gamma(3+\alpha)}\sin\pi x,$$
$$\vdots$$

Then, the inverse transformation of the set of values $\left\{U_h^{\alpha}(x)\right\}_{h=0}^{n}$ allows the following approximate solution

$$\tilde{u}_n(x,t) = \sum_{h=0}^{\infty} U_h^{\alpha}(x) t^{h\alpha}$$

$$= t^2\sin\pi x - \frac{(\pi^2+1)}{2+\alpha}t^{2+\alpha}\sin\pi x + \frac{2(\pi^2+1)}{\Gamma(3+\alpha)}t^{2+\alpha}\sin\pi x + \cdots.$$

Finally, for $\alpha = 1$, the exact solution is given by $u(x,t) = t^2\sin\pi x$.

In Figure 1, we demonstrate the solution functions of the fractional cable equation according to the mentioned methods and the comparison with the exact solution. In Figure 2, we represent the comparison of the solutions obtained with conformable reduced differential transform method and conformable modified homotopy perturbation method. In Table 1, we show the $u(x,t)$ solutions for various values of α and x. Figure 1, Figure 2 and Table 1 say that the CRDTM gives better results than the CMHPM in the solution of the fractional cable equation.

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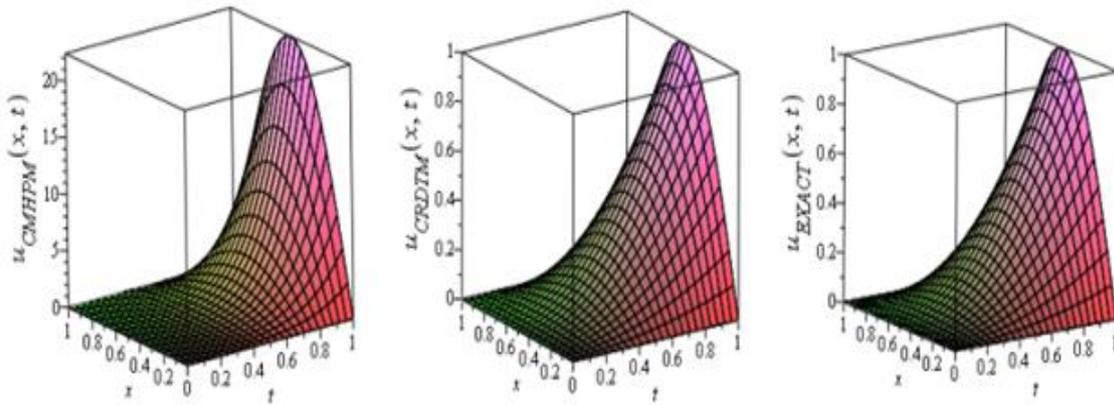
Figure 1. CMHPM, CRDTM and exact solutions for values $(x,t) = [0,1] \times [0,1]$
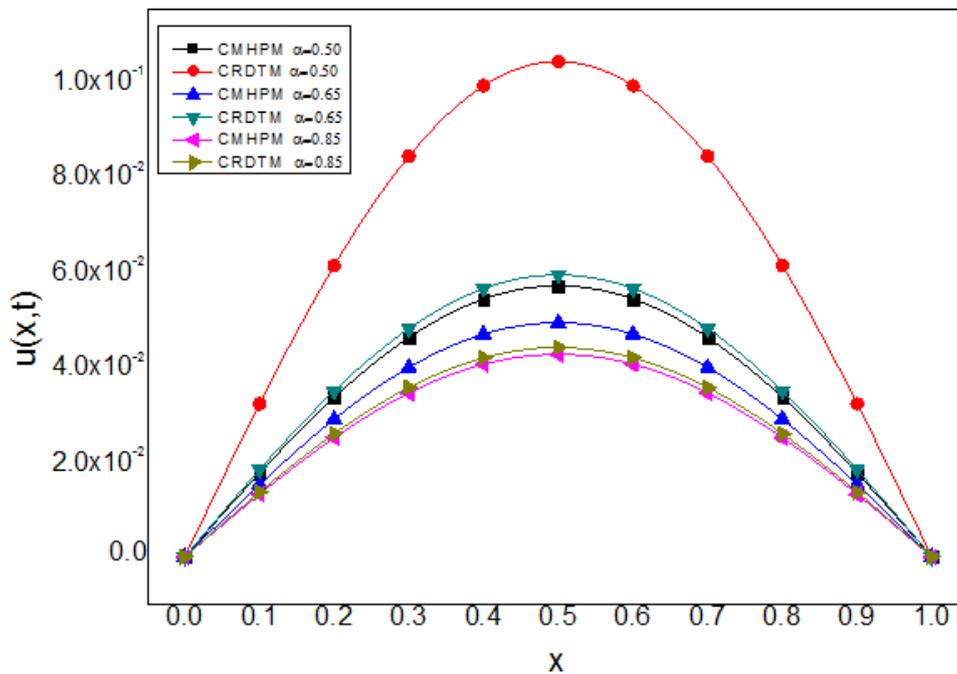
Figure 2. Comparison of the solutions obtained with CMHPM and CRDTM

Table 2. $u(x,t)$ solutions for various values of α and x

x	$\alpha = 0.30$		$\alpha = 0.70$		$\alpha = 0.95$		$\alpha = 1$		
	CMHPM	CRDTM	CMHPM	CRDTM	CMHPM	CRDTM	CMHPM	CRDTM	Exact
0.1	0.021064	0.125279	0.014491	0.016421	0.013502	0.012678	0.014477	0.012360	0.012360
0.2	0.040066	0.238295	0.027564	0.031236	0.025684	0.024115	0.027537	0.023511	0.023511
0.3	0.055147	0.327986	0.037938	0.042992	0.035351	0.033192	0.037901	0.032360	0.032360
0.4	0.064829	0.385570	0.044599	0.050541	0.041557	0.039020	0.044556	0.038042	0.038042
0.5	0.068165	0.405413	0.046895	0.053142	0.043696	0.041028	0.046849	0.040000	0.040000
0.6	0.064829	0.385570	0.044599	0.050541	0.041557	0.039020	0.044556	0.038042	0.038042
0.7	0.055147	0.327986	0.037938	0.042992	0.035351	0.033192	0.037901	0.032360	0.032360
0.8	0.040066	0.238295	0.027564	0.031236	0.025684	0.024115	0.027537	0.023511	0.023511
0.9	0.021064	0.125279	0.014491	0.016421	0.013502	0.012678	0.014477	0.012360	0.012360
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

6. Conclusion

This study deals with the solutions of the time-fractional cable equation by using two approximate-analytical solution methods based on the conformable-type derivative operator. In the present work, firstly, we have redefined MHPM and RDTM by using conformable derivative operator. This derivative definition is a convenient definition in the exact solution procedure of fractional differential equations. Conformable derivatives are easier to apply to fractional differential equations, as its derivative definition does not include any integral terms. Then we have demonstrated the efficiencies and accuracies of the recommended methods by applying them to the fractional cable equation which is a special equation models the neuronal dynamics. The successful applications of the suggested methods prove that these solution methods are in complete settlement with the corresponding exact solutions. In conclusion, a table and some figures which compare the numerical and analytical solutions are provided to show that the CRDTM and CMHPM are the powerful and efficient techniques in finding the numerical solution of the conformable time fractional cable equation. Especially, it is clear that the CRDTM gives better results than the CMHPM in the solution of the fractional cable equation.

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