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Abstract

This paper is concerned with batch arrival queue with an additional second optional service to a batch of customers with dissimilar service rate where the idea of restricted admissibility of arriving batch of customers is also introduced. The server may take two different vacations (i) Emergency vacation-during service the server may go for vacation to an emergency call and after completion of the vacation, the server continues the remaining service to a batch of customers. (ii) Bernoulli vacation-after completion of first essential or second optional service, the server may take a vacation or may remain in the system to serve the next unit, if any. While the server is functioning with first essential or second optional service, it may break off for a short period of time. As a result of breakdown, a batch of customers, either in first essential or second optional service is interrupted. The service channel will be sent to repair process immediately. The repair process presumed to be general distribution. Here, we assumed that the customers just being served before server breakdown wait for the server to complete its remaining service after the completion of the repair process. We derived the queue size distribution at a random epoch and at a departure epoch under the steady state condition. Moreover, various system performance measures, the mean queue size and the average waiting time in the queue have been obtained explicitly. Some particular cases and special cases are determined. A numerical result is also introduced.

Keywords: Bulk service; Second optional service; Emergency vacation; Bernoulli vacation; Breakdown; Repair; Restricted admissibility policy

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1. Introduction

We consider a queueing system in which the service is rendered in bulk, by using the supplementary variable technique. Initially bulk service queues were originated with Bailey (1954). Neuts (1967) studied the "General Bulk Service Rule" in which service starts only when a specified number of customers in the queue is available. Holman et al. (1981) have studied some general bulk service results by using the supplementary variable technique. Briere and Chaudhry (1989) have analyzed single server bulk service queues in computational aspects. Ho woo Lee et al. (1992) have discussed the bulk service queue with single vacation and derived the queue size distribution at a departure epoch. Recently, Jeyakumar and Senthilnathan (2016) have contributed the work on bulk service queue with multiple working vacations. Haghighi and Mishev (2016) discussed the stepwise explicit solution for the joint distribution of queue length of a MAP single-server service queueing system with splitting and varying batch size delayed-feedback.

Queueing system of M^X/G(a,b)/1 type in which the server may provide a second optional service. Such queueing models occur in day-to-day life situations, for example, in a machining process all the arriving customers require the first essential service and only some batch of customers may require the second optional service. Medhi (2002) has considered Poisson arrival queue with a second optional channel. Lotfi and Ke (2008) have focused on bulk quorum queue with a choice of service and optional re-service. Choudhury and Lotfi (2009) investigated an $M/G/1$ queue with an additional second phase of service immediately after the completion of the first essential service and both the service are assumed to be a general distribution. Ayyappan and Shyamala (2013) discussed an M^X/G/1 queue with second optional service, Bernoulli schedule server vacation and random breakdowns. Madan and Choudhury (2004) discussed a Bernoulli vacation schedule under RA-policy. Choudhury and Madan (2007) contributed the work on the Bernoulli vacation queue with a random setup time under the restricted admissibility policy. Madan (2018) discussed the server vacations in a single server queue providing two types of first essential service followed by two types of additional optional service. Dong-Yuh Yang and Yi-Hsuan Chen (2018) have contributed the work on Computation and optimization of a working breakdown queue with the second optional service. They used the matrix-geometric method to compute the stationary probability distribution of the system size and various system performance measures. Pavai Madheswari and Suganthi (2017) examined an M/G/1 retrial queue with second optional service and unreliable server under single exhaustive vacation. Aliakbar Montazer Haghighi and Dimitar Mishev (2013) examined the stochastic three-stage hiring model as a tandem queueing process with bulk arrivals and Erlang Phase-Type Selection $M^{[X]}/M^{(k,K)}/1-M^{[X]}/Er/1-\infty$.

The classical vacation scheme has been investigated by many researchers. The server may take vacation makes the queueing model more natural and flexible in studying real-life situations. Most of the papers the server functioning under any one of the vacation policies: single vacation, multiple vacation, and so on. Ke et al. (2010) discussed some unreliable server queue with different vacation policies. Bagyam and Chandrika (2010) have studied a single service retrial queueing system with emergency vacation. Choudhury and Deka (2012,2015) have studied the concept of two phases of service under Bernoulli vacation. Rajadurai et al. (2016) analyzed single service with working vacations and vacation interruption under Bernoulli schedule.

The service interruptions are happening in many real life situations. In a practical system, we frequently faced the case where the service station may be interrupted before its completion. Fiems, Maetens and Bruneel (2008) have discussed queueing systems with different types of server interruptions. Singh et al. (2016) examined an M^X/G/1 unreliable retrial queue with option of additional service and Bernoulli vacation. Choudhury and Ke (2012) examined an unreliable single service under Bernoulli vacation schedule. Choudhury and Deka (2016) investigated unreliable server queue with two phases of service and Bernoulli vacation under multiple vacation policy. Jiang and Xin (2018) have derived the steady state distribution by matrix-analytic method and spectral expansion method respectively, and also various performance measures and sojourn time distribution of an arbitrary customer.

The rest of the paper is structured as follows. In Section 2, we give the system description of the model. In Section 3 deals with a mathematical description of the queueing model. In Section 4 proposed the definitions, necessary equations and also obtain the transient solution of our model. In Section 5, we finding the Probability Generating Function of the stationary queue length at the random epoch and the corresponding stability condition has been obtained in Section 6. Also, we present the performance measures in the various states of the system, the mean queue size and the average waiting time in the queue are briefly in Section 7. In Section 8, we find the PGF of the stationary queue length at a departure epoch. Some particular cases are given in Section 9. Some special cases are discussed in Section 10. Computational results and graphs are presented in Section 11. At last, the conclusion and further work have been drawn in Section 12.

2. Model Description

In this paper, the authors' best of our knowledge, no works have been found in bulk service queueing systems with service interruptions, second optional service, Bernoulli schedule vacation and emergency vacation, restricted admissibility policy. Hence, to fill up to this gap, the current paper is framed in a very unique procedure in the sense that the concept of bulk service and second optional service is incorporated along with unreliable server, two different types of vacation policies and restricted admissibility policy. The problem is equipped with batch arrival and it is assumed that not all batches are allowed to join the system at all times. In bulk service the server starts service only if a specified minimum say 'a' of customers have accumulated in the queue and he does not take more than 'b' customers for service in one batch. Here, we consider two different vacation mechanisms. After the completion of First Essential Service (FES) or the Second Optional Service (SOS) the server may take a vacation with probability θ or stay in the system with complementary probability $(1-\theta)$ is termed as the Bernoulli vacation. While the server is functioning with the first or second service the server may get an emergency call with service interruption called the emergency vacation. Similarly, when the server is functioning with the first or second service, the service gets interrupted and sent to repair process immediately. After the completion of emergency vacation or repair process the server being served before service interruption waits for the remaining service to complete the service.

3. Mathematical description of the queueing model

To describe the required queueing model, we assume the following.

The arrival process:

Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided bulk service on a first come - first served basis. Let $\lambda c_i dt \quad (i \geq 1)$ be the first order probability that a batch of i customers arrive at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.

The service process:

There is a single server providing service to a batch of customers in First Essential Service (FES). As soon as the FES is completed, then the batch of customers may leave the system with probability $(1-\pi)$ or may get the Second Optional Service (SOS) with probability π $(0 \leq \pi \leq 1)$.

Bernoulli vacation: After attainment of FES or not opted for SOS, the server may take a Bernoulli vacation with probability θ, and with probability $(1-\theta)$ it waits for serving the next batch of customers.

Emergency vacation:

The server may take an emergency vacation when the server is functioning with FES or SOS which is exponentially distributed with rates η_1 for FES and η_2 for SOS.

Breakdown:

While the server is functioning with FES or SOS, it may break down at any time and is assumed to occur according to a Poisson stream with mean breakdown rates α_1 for the FES and α_2 for the SOS.

Repair process:

If the service gets interrupted during FES and SOS the server, enter into the repair process of the respective service.

Restricted admissibility:

We assume that b_1 be the probability that an arriving batch will be allowed to join the system while the server is busy or idle and b_2 be the probability that an arriving batch will be allowed to join the system while the server is on vacation or under repair.

Two types of service time, two different vacation time and repair time follow general distribution. In Table 1, we define some notations used for the Cumulative Distribution Function (CDF), the probability density functions (pdf), the Laplace Stieltjes Transform for two types of service time, two different vacation time, repair time.

Table 1: Some notations for distribution function

Time	CDF	Hazard rate	pdf	LST
First Essential service	$B_1(x)$	$\mu_1(x)$	$b_1(v)=\mu_1(v)e^{-\int_0^v \mu_1(x)dx}$	$b_1^*(s)$
Second Optional service	$B_2(x)$	$\mu_2(x)$	$b_2(v)=\mu_2(v)e^{-\int_0^v \mu_2(x)dx}$	$b_2^*(s)$
Bernoulli vacation	$V(x)$	$\gamma(x)$	$v(t)=\gamma(t)e^{-\int_0^t \gamma(x)dx}$	$v^*(s)$
Emergency vacation on two types of service	$G_i(y)$	$\zeta_i(y)$	$g_i(r)=\zeta_i(r)e^{-\int_0^r \zeta_i(y)dy}$	$g_i^*(s)$
Repair under two types of service	$R_i(y)$	$\beta_i(y)$	$r_i(w)=\beta_i(w)e^{-\int_0^w \beta_i(y)dy}$	$r_i^*(s)$

4. Definitions and Equations Governing the Systems

In this section, we first set up the system state equations for its stationary queue size distribution, by treating elapsed two types of service time, elapsed two different vacation time and the elapsed repair time of the server, for both types of service, as the supplementary variables. Then, we solve the equations and derive the PGFs of the stationary queue size distribution. Let $N(t)$ be the queue size (including one batch of customers being served, if any) at time t, $B_i^0(t)$ be the elapsed service time of the customer for the two types of service at time t, with i = 1, 2 denoting FES and SOS, respectively and $V^0(t)$ be the elapsed vacation time of the server. In addition, let $G_i^0(t)$ be the elapsed emergency vacation time of the server for ith type of service during which emergency call occurs in the system at time t and $R_i^0(t)$ be the elapsed Emergency vacation time and elapsed repair time of the server for ith type of service during which breakdown occurs in the system at time t, where sub-index i = 1 (respectively i = 2) denotes FES (respectively SOS). Further, we introduce the following random variable.

$$Y(t) = \begin{cases} 0, & \text{if the server is idle at time t.} \\ 1, & \text{if the server is busy with FES at time t.} \\ 2, & \text{if the server is busy with SOS at time t.} \\ 3, & \text{if the server is on vacation period at time t.} \\ 4, & \text{if the server is on Emergency vacation during FES at time t.} \\ 5, & \text{if the server is on Emergency vacation during SOS at time t.} \\ 6, & \text{if the server is under repair during FES at time t.} \\ 7, & \text{if the server is under repair during SOS at time t.} \end{cases}$$

Thus the supplementary variable $B_i^0(t)$, $V^0(t)$, $G_i^0(t)$ and $R_i^0(t)$ for i=1, 2 are introduced in order to obtain a bivariate Markov process {N(t), Y(t)} and define the following probabilities as:

$$Q_r(t)dx = P\{N(t) = r, Y(t) = 0\}, \text{for } t \ge 0, \text{and } 0 \le r \le a-1,$$
$$P_{1,n}(x,t)dx = P\{N(t) = n, Y(t) = 1; x \le B_1^0(t) \le x + dx\}, \text{for } t \ge 0, x \ge 0 \text{ and } n \ge 0,$$
$$P_{2,n}(x,t)dx = P\{N(t) = n, Y(t) = 2; x \le B_2^0(t) \le x + dx\}, \text{for } t \ge 0, x \ge 0 \text{ and } n \ge 0,$$
$$V_n(x,t)dx = P\{N(t) = n, Y(t) = 3; x \le V^0(t) \le x + dx\}, \text{for } t \ge 0, x \ge 0 \text{ and } n \ge 0,$$

$$E_{1,n}(x,y,t)dx = P\{N(t) = n, Y(t) = 4; y \le G_1^0(t) \le y + dy / B_1^0(t) = x\}, \text{for } t \ge 0, x, y \ge 0 \text{ and } n \ge 0,$$
$$E_{2,n}(x,y,t)dx = P\{N(t) = n, Y(t) = 5; y \le G_2^0(t) \le y + dy / B_2^0(t) = x\}, \text{for } t \ge 0, x, y \ge 0 \text{ and } n \ge 0,$$
$$R_{1,n}(x,y,t)dx = P\{N(t) = n, Y(t) = 6; y \le R_1^0(t) \le y + dy / B_1^0(t) = x\}, \text{for } t \ge 0, x, y \ge 0 \text{ and } n \ge 0,$$
$$R_{2,n}(x,y,t)dx = P\{N(t) = n, Y(t) = 7; y \le R_2^0(t) \le y + dy / B_2^0(t) = x\}, \text{for } t \ge 0, x, y \ge 0 \text{ and } n \ge 0.$$

The Kolmogorov forward equations to govern the model; where sub index $i = 1,2$ denotes the FES and SOS respectively can be formulated as follows:

$$\frac{\partial}{\partial x} P_{i,0}(x,t) + \frac{\partial}{\partial t} P_{i,0}(x,t) + (\lambda + \mu_i(x) + \alpha_i + \eta_i) P_{i,0}(x,t) = \lambda(1-b_1) P_{i,0}(x,t) + \int_0^\infty R_{i,0}(x,y,t)\beta_i(y)dy + \int_0^\infty E_{i,0}(x,y,t)\zeta_i(y)dy, \tag{1}$$

$$\frac{\partial}{\partial x} P_{i,n}(x,t) + \frac{\partial}{\partial t} P_{i,n}(x,t) + (\lambda + \mu_i(x) + \alpha_i + \eta_i) P_{i,n}(x,t) = \lambda(1-b_1) P_{i,n}(x,t) + \lambda b_1 \sum_{k=1}^{n} c_k P_{i,n-k}(x,t) + \int_0^\infty R_{i,n}(x,y,t)\beta_i(y)dy + \int_0^\infty E_{i,n}(x,y,t)\zeta_i(y)dy, n \ge 1, \tag{2}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+(\lambda+\gamma(x))V_0(x,t)=\lambda(1-b_2)V_0(x,t), \tag{3}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+(\lambda+\gamma(x))V_n(x,t)=\lambda(1-b_2)V_n(x,t)+\lambda b_2\sum_{k=1}^{n}c_kV_{n-k}(x,t),\, n\geq 1, \tag{4}$$

$$\frac{\partial}{\partial y}E_{i,0}(x,y,t)+\frac{\partial}{\partial t}E_{i,0}(x,y,t)+(\lambda+\zeta_i(y))E_{i,0}(x,y,t)=\lambda(1-b_2)E_{i,0}(x,y,t), \tag{5}$$

$$\frac{\partial}{\partial y}E_{i,n}(x,y,t)+\frac{\partial}{\partial t}E_{i,n}(x,y,t)+(\lambda+\zeta_i(y))E_{i,n}(x,y,t)=\lambda(1-b_2)E_{i,n}(x,y,t)$$
$$+\lambda b_2\sum_{k=1}^{n}c_kE_{i,n-k}(x,y,t),\, n\geq 1, \tag{6}$$

$$\frac{\partial}{\partial y}R_{i,0}(x,y,t)+\frac{\partial}{\partial t}R_{i,0}(x,y,t)+(\lambda+\beta_i(y))R_{i,0}(x,y,t)=\lambda(1-b_2)R_{i,0}(x,y,t), \tag{7}$$

$$\frac{\partial}{\partial y}R_{i,n}(x,y,t)+\frac{\partial}{\partial t}R_{i,n}(x,y,t)+(\lambda+\beta_i(y))R_{i,n}(x,y,t)=\lambda(1-b_2)R_{i,n}(x,y,t)$$
$$+\lambda b_2\sum_{k=1}^{n}c_kR_{i,n-k}(x,y,t),\, n\geq 1, \tag{8}$$

$$\frac{d}{dt}Q_0(t)=-\lambda Q_0(t)+\lambda(1-b_1)Q_0(t)+(1-\theta)(1-\pi)\int_0^\infty P_{1,0}(x,t)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty P_{2,0}(x,t)\mu_2(x)dx+\int_0^\infty V_0(x,t)\gamma(x)dx, \tag{9}$$

$$\frac{d}{dt}Q_r(t)=-\lambda Q_r(t)+\lambda(1-b_1)Q_r(t)+\lambda b_1\sum_{k=1}^{r}c_kQ_{r-k}(t)+(1-\theta)(1-\pi)\int_0^\infty P_{1,r}(x,t)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty P_{2,r}(x,t)\mu_2(x)dx+\int_0^\infty V_r(x,t)\gamma(x)dx, 1\leq r\leq a-1. \tag{10}$$

To solve the equations (1) to (10), the following boundary conditions at $x=0$ and $y=0$ are considered,

$$P_{1,0}(0,t)=\lambda b_1\sum_{r=a}^{b}\sum_{k=0}^{a-1}c_{r-k}Q_k(t)+(1-\theta)(1-\pi)\sum_{r=a}^{b}\int_0^\infty P_{1,r}(x,t)\mu_1(x)dx$$
$$+(1-\theta)\sum_{r=a}^{b}\int_0^\infty P_{2,r}(x,t)\mu_2(x)dx+\sum_{r=a}^{b}\int_0^\infty V_r(x,t)\gamma(x)dx, \tag{11}$$

$$P_{1,n}(0,t)=\lambda b_1\sum_{k=0}^{a-1}c_{b+n-k}Q_k(t)+(1-\theta)(1-\pi)\int_0^\infty P_{1,n+b}(x,t)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty P_{2,n+b}(x,t)\mu_2(x)dx+\int_0^\infty V_{n+b}(x,t)\gamma(x)dx,\, n\geq 1, \tag{12}$$

$$P_{2,n}(0,t)=\pi\int_0^\infty P_{1,n}(x,t)\mu_1(x)dx,\, n\geq 0, \tag{13}$$

$$V_n(0,t)=(1-\pi)\theta\int_0^\infty P_{1,n}(x,t)\mu_1(x)dx+\theta\int_0^\infty P_{2,n}(x,t)\mu_2(x)dx,\, n\geq 0, \tag{14}$$

$$E_{i,n}(x,0,t) = \eta_i P_{i,n}(x,t), n \geq 0, i = 1,2, \tag{15}$$

$$R_{i,n}(x,0,t) = \alpha_i P_{i,n}(x,t), n \geq 0, i = 1,2. \tag{16}$$

Further, it is assume that initially there are no adequate number of customers in the system and the server is idle. So the initial conditions are

$$Q_0(0) = 1, Q_r(0) = 0 \text{ for } 1 \leq r \leq a-1,$$

$$P_{i,n}(0) = R_{i,n}(0) = E_{i,n}(0) = V_n(0) = 0 \text{ for } n \geq 0, i = 1,2\,. \tag{17}$$

To solve the above equations, let us introduce the following probability generating functions for $i = 1,2$ and $|z| \leq 1$:

$$P_i(x,z,t) = \sum_{n=0}^{\infty} z^n P_{i,n}(x,t); P_i(z,t) = \sum_{n=0}^{\infty} z^n P_{i,n}(t); C(z) = \sum_{n=1}^{\infty} c_n z^n,$$

$$V(x,z,t) = \sum_{n=0}^{\infty} z^n V(x,t);\ V(z,t) = \sum_{n=0}^{\infty} z^n V(t);\ Q(z) = \sum_{r=0}^{a-1} Q_r z^r,$$

$$E_i(x,y,z,t) = \sum_{n=0}^{\infty} z^n E_{i,n}(x,y,t);\ E_i(x,z,t) = \sum_{n=0}^{\infty} z^n E_{i,n}(x,t),$$

$$R_i(x,y,z,t) = \sum_{n=0}^{\infty} z^n R_{i,n}(x,y,t);\ R_i(x,z,t) = \sum_{n=0}^{\infty} z^n R_{i,n}(x,t). \tag{18}$$

Define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \Re(s) > 0. \tag{19}$$

Taking the Laplace transform of equations (1) to (16) and using (17), we get

$$\frac{\partial}{\partial x}\overline{P}_{i,0}(x,s) + (s + \lambda b_1 + \mu_i(x) + \alpha_i + \eta_i)\overline{P}_{i,0}(x,s) =$$
$$+\int_0^{\infty}\overline{R}_{i,0}(x,y,s)\beta_i(y)dy + \int_0^{\infty}\overline{E}_{i,0}(x,y,s)\zeta_i(y)dy, i = 1,2, \tag{20}$$

$$\frac{\partial}{\partial x}\overline{P}_{i,n}(x,s) + (s + \lambda b_1 + \mu_i(x) + \alpha_i + \eta_i)\overline{P}_{i,n}(x,s) = \lambda b_1 \sum_{k=1}^{n} c_k \overline{P}_{i,n-k}(x,s)$$
$$+\int_0^{\infty}\overline{R}_{i,n}(x,y,s)\beta_i(y)dy + \int_0^{\infty}\overline{E}_{i,n}(x,y,s)\zeta_i(y)dy, n \geq 1, i = 1,2, \tag{21}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s) + (s + \lambda b_2 + \gamma(x))\overline{V}_0(x,s) = 0, \tag{22}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+(s+\lambda b_2+\gamma(x))\overline{V}_n(x,s)=\lambda b_2\sum_{k=1}^{n}c_k\overline{V}_{n-k}(x,s),\,n\geq 1, \tag{23}$$

$$\frac{\partial}{\partial y}\overline{E}_{i,0}(x,y,s)+(s+\lambda b_2+\zeta_i(y))\overline{E}_{i,0}(x,y,s)=0,i=1,2, \tag{24}$$

$$\frac{\partial}{\partial y}\overline{E}_{i,n}(x,y,s)+(s+\lambda b_2+\zeta_i(y))\overline{E}_{i,n}(x,y,s)=\lambda b_2\sum_{k=1}^{n}c_k\overline{E}_{i,n-k}(x,y,s),\,n\geq 1,i=1,2, \tag{25}$$

$$\frac{\partial}{\partial y}\overline{R}_{i,0}(x,y,s)+(s+\lambda b_2+\beta_i(y))\overline{R}_{i,0}(x,y,s)=0,i=1,2, \tag{26}$$

$$\frac{\partial}{\partial y}\overline{R}_{i,n}(x,y,s)+(s+\lambda b_2+\beta_i(y))\overline{R}_{i,n}(x,y,s)=\lambda b_2\sum_{k=1}^{n}c_k\overline{R}_{i,n-k}(x,y,s),\,n\geq 1,i=1,2, \tag{27}$$

$$(s+\lambda b_1)\overline{Q}_0(s)=1+(1-\theta)(1-\pi)\int_0^\infty\overline{P}_{1,0}(x,s)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty\overline{P}_{2,0}(x,s)\mu_2(x)dx+\int_0^\infty\overline{V}_0(x,s)\gamma(x)dx, \tag{28}$$

$$(s+\lambda b_1)\overline{Q}_r(s)=\lambda b_1\sum_{k=1}^{r}c_k\overline{Q}_{r-k}(s)+(1-\theta)(1-\pi)\int_0^\infty\overline{P}_{1,r}(x,s)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty\overline{P}_{2,r}(x,s)\mu_2(x)dx+\int_0^\infty\overline{V}_r(x,s)\gamma(x)dx,1\leq r\leq a-1, \tag{29}$$

$$\overline{P}_{1,0}(0,s)=\lambda b_1\sum_{r=a}^{b}\sum_{k=0}^{a-1}c_{r-k}\overline{Q}_k(s)+(1-\theta)(1-\pi)\sum_{r=a}^{b}\int_0^\infty\overline{P}_{1,r}(x,s)\mu_1(x)dx$$
$$+(1-\theta)\sum_{r=a}^{b}\int_0^\infty\overline{P}_{2,r}(x,s)\mu_2(x)dx+\sum_{r=a}^{b}\int_0^\infty\overline{V}_r(x,s)\gamma(x)dx, \tag{30}$$

$$\overline{P}_{1,n}(0,s)=\lambda b_1\sum_{k=0}^{a-1}c_{b+n-k}\overline{Q}_k(s)+(1-\theta)(1-\pi)\int_0^\infty\overline{P}_{1,n+b}(x,s)\mu_1(x)dx$$
$$+(1-\theta)\int_0^\infty\overline{P}_{2,n+b}(x,s)\mu_2(x)dx+\int_0^\infty\overline{V}_{n+b}(x,s)\gamma(x)dx,\,n\geq 1, \tag{31}$$

$$\overline{P}_{2,n}(0,s)=\pi\int_0^\infty\overline{P}_{1,n}(x,s)\mu_1(x)dx,\,n\geq 0, \tag{32}$$

$$\overline{V}_n(0,s)=(1-\pi)\theta\int_0^\infty\overline{P}_{1,n}(x,s)\mu_1(x)dx+\theta\int_0^\infty\overline{P}_{2,n}(x,s)\mu_2(x)dx,\,n\geq 0, \tag{33}$$

$$\overline{E}_{i,n}(x,0,s)=\eta_i\overline{P}_{i,n}(x,s),\,n\geq 0,i=1,2, \tag{34}$$

$$\overline{R}_{i,n}(x,0,s)=\alpha_i\overline{P}_{i,n}(x,s),\,n\geq 0,i=1,2. \tag{35}$$

By multiplying equations (21), (23), (25) and (27) by z^n and then taking summation over all possible values of n, adding to the equations (20), (22), (24) and (26) respectively, and using the generating functions defined in (18), we get

$$\frac{\partial}{\partial x}\overline{P}_i(x,z,s)+(s+\lambda b_1(1-C(z))+\mu_i(x)+\alpha_i+\eta_i+)\overline{P}_i(x,z,s)=$$
$$\int_0^\infty\overline{R}_i(x,y,z,s)\beta_i(y)dy+\int_0^\infty\overline{E}_i(x,y,z,s)\zeta_i(y)dy,i=1,2, \tag{36}$$

$$\frac{\partial}{\partial x}\overline{V}(x,z,s)+(s+\lambda b_2(1-C(z))+\gamma(x))\overline{V}(x,z,s)=0, \tag{37}$$

$$\frac{\partial}{\partial y}\overline{E}_i(x,y,z,s)+(s+\lambda b_2(1-C(z))+\zeta_i(y))\overline{E}_i(x,y,z,s)=0, i=1,2, \tag{38}$$

$$\frac{\partial}{\partial y}\overline{R}_i(x,y,z,s)+(s+\lambda b_2(1-C(z))+\beta_i(y))\overline{R}_i(x,y,z,s)=0, i=1,2. \tag{39}$$

Multiplying both sides of equation (31) by z^n summing over n from 0 to ∞, and use the equation (30), we get

$$\begin{aligned} z^b\overline{P}_1(0,z,s) &= \lambda b_1\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\overline{Q}_r(s)(z^b-z^{n+r})-z^b\sum_{r=0}^{a-1}(s+\lambda b_1)\overline{Q}_r(s)+z^b \\ &+\lambda b_1\sum_{r=0}^{a-1}C(z)\overline{Q}_r(s)z^r+(1-\theta)(1-\pi)\int_0^\infty\overline{P}_1(x,z,s)\mu_1(x)dx \\ &+(1-\theta)\int_0^\infty\overline{P}_2(x,z,s)\mu_2(x)dx+\int_0^\infty\overline{V}(x,z,s)\gamma(x)dx \\ &+(1-\theta)(1-\pi)\sum_{r=0}^{b-1}(z^b-z^r)\int_0^\infty\overline{P}_{1,r}(x,s)\mu_1(x)dx \\ &+(1-\theta)\sum_{r=0}^{b-1}(z^b-z^r)\int_0^\infty\overline{P}_{2,r}(x,s)\mu_2(x)dx \\ &+\sum_{r=0}^{b-1}(z^b-z^r)\int_0^\infty\overline{V}_r(x,s)\gamma(x)dx. \end{aligned} \tag{40}$$

Similarly from equations (32), (33) (34) and (35), we get

$$\overline{P}_2(0,z,s)=\pi\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s)), \tag{41}$$

$$\overline{V}(0,z,s)=(1-\pi)\theta\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))+\theta\pi\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))\overline{B}_2(\psi_2(z,s)), \tag{42}$$

$$\overline{E}_i(x,0,z,s)=\eta_i\overline{P}_i(x,z,s), i=1,2, \tag{43}$$

$$\overline{R}_i(x,0,z,s)=\alpha_i\overline{P}_i(x,z,s), i=1,2. \tag{44}$$

Solving the partial differential equations (36) to (39), it follows that

$$\overline{P}_i(x,z,s)=\overline{P}_i(0,z,s)e^{-\psi_i(z,s)x-\int_0^x\mu_i(t)dt}, i=1,2 \tag{45}$$

$$\overline{V}(x,z,s)=\overline{V}(0,z,s)e^{-\phi(z,s)x-\int_0^x\gamma(t)dt}, \tag{46}$$

$$\overline{E}_i(x,y,z,s) = \overline{E}_i(x,0,z,s)e^{-\phi(z,s)y-\int_0^y \zeta_i(t)dt}, i=1,2 \tag{47}$$

$$\overline{R}_i(x,y,z,s) = \overline{R}_i(x,0,z,s)e^{-\phi(z,s)y-\int_0^y \beta_i(t)dt}, i=1,2. \tag{48}$$

Integrating equation (47) and (48) from 0 to ∞ with respect to y, we get for i=1,2

$$\overline{E}_i(x,z,s) = \int_0^\infty \overline{E}_i(x,y,z,s)dy = \overline{E}_i(x,0,z,s)\left[\frac{1-\overline{G}_i(\phi(z,s))}{\phi(z,s)}\right], \tag{49}$$

$$\overline{R}_i(x,z,s) = \int_0^\infty \overline{R}_i(x,y,z,s)dy = \overline{R}_i(x,0,z,s)\left[\frac{1-\overline{R}_i(\phi(z,s))}{\phi(z,s)}\right]. \tag{50}$$

Now multiplying both sides of equations (45) to (48) by $\mu_i(x)$, $\gamma(x)$, $\zeta_i(y)$, and $\beta_i(y)$ respectively, and integrating, we obtain

$$\int_0^\infty \overline{P}_i(x,z,s)\mu_i(x)dx = \overline{P}_i(0,z,s)\overline{B}_i(\psi_i(z,s)), \tag{51}$$

$$\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx = \overline{V}(0,z,s)\overline{V}(\phi(z,s)), \tag{52}$$

$$\int_0^\infty \overline{E}_i(x,y,z,s)\zeta_i(y)dy = \overline{E}_i(x,0,z,s)\overline{G}_i(\phi(z,s)), \tag{53}$$

$$\int_0^\infty \overline{R}_i(x,y,z,s)\beta_i(y)dy = \overline{R}_i(x,0,z,s)\overline{R}_i(\phi(z,s)). \tag{54}$$

Again integrating equations (45), (46), (49) and (50) by parts with respect to x and using the equation (41), (42), (43), (44) and (45), we get

$$\overline{P}_1(z,s) = \overline{P}_1(0,z,s)\left[\frac{1-\overline{B}_1(\psi_1(z,s))}{\psi_1(z,s)}\right], \tag{55}$$

$$\overline{P}_2(z,s) = \pi\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))\left[\frac{1-\overline{B}_2(\psi_2(z,s))}{\psi_2(z,s)}\right], \tag{56}$$

$$\overline{V}(z,s) = [(1-\pi)\theta\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s)) + \pi\theta\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))\overline{B}_2(\psi_2(z,s))]$$

$$\left[\frac{1-\overline{V}(\phi(z,s))}{\phi(z,s)}\right], \tag{57}$$

$$\overline{E}_1(z,s)=\eta_1\overline{P}_1(0,z,s)\left[\frac{1-\overline{B}_1(\psi_1(z,s))}{\psi_1(z,s)}\right]\left[\frac{1-\overline{G}_1(\phi(z,s))}{\phi(z,s)}\right], \tag{58}$$

$$\overline{E}_2(z,s)=\eta_2\pi\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))\left[\frac{1-\overline{B}_2(\psi_2(z,s))}{\psi_2(z,s)}\right]\left[\frac{1-\overline{G}_2(\phi(z,s))}{\phi(z,s)}\right], \tag{59}$$

$$\overline{R}_1(z,s)=\alpha_1\overline{P}_1(0,z,s)\left[\frac{1-\overline{B}_1(\psi_1(z,s))}{\psi_1(z,s)}\right]\left[\frac{1-\overline{R}_1(\phi(z,s))}{\phi(z,s)}\right], \tag{60}$$

$$\overline{R}_2(z,s)=\alpha_2\pi\overline{P}_1(0,z,s)\overline{B}_1(\psi_1(z,s))\left[\frac{1-\overline{B}_2(\psi_2(z,s))}{\psi_2(z,s)}\right]\left[\frac{1-\overline{R}_2(\phi(z,s))}{\phi(z,s)}\right]. \tag{61}$$

Inserting the equations (51), (52) into the equation (40), we get

$$\overline{P}_1(0,z,s)=\frac{\left(\begin{array}{l}\lambda b_1\sum_{r=0}^{a-1}C(z)\overline{Q}_r(s)z^r-z^b(s+\lambda b_1)\sum_{r=0}^{a-1}\overline{Q}_r(s)\\ +z^b+\lambda b_1\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_n\overline{Q}_r(s)(z^b-z^{n+r})\\ +\sum_{r=0}^{b-1}(z^b-z^r)((1-\theta)(1-\pi)\int_0^\infty\overline{P}_{1,r}(x,s)\mu_1(x)dx\\ +(1-\theta)\int_0^\infty\overline{P}_{2,r}(x,s)\mu_2(x)dx+\int_0^\infty\overline{V}_r(x,s)\gamma(x)dx)\end{array}\right)}{\left[z^b-\left(K_1(z,s)\overline{B}_1(\psi_1(z,s))+K_2(z,s)\overline{B}_1(\psi_1(z,s))\overline{B}_2(\psi_2(z,s))\right)\right]}, \tag{62}$$

$\phi(z,s)=s+\lambda b_2(1-C(z))$,

$\psi_i(z,s)=s+\lambda b_1(1-C(z))+\alpha_i(1-\overline{R}_i(\phi(z,s)))+\eta_i(1-\overline{E}_i(\phi(z,s))), i=1,2$,
$K_1(z,s)=(1-\theta)(1-\pi)+\theta(1-\pi)\overline{V}(\phi(z,s))$,

$K_2(z,s)=(1-\theta)\pi+\theta\pi\overline{V}(\phi(z,s))$.

Substituting the equation (62) into the equations (55), (56), (57), (58), (59), (60), and (61) and taking the inverse laplace transform of these equations, we get the probability generating fuctions of various states of the system are determined under transient state.

5. The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t). \tag{63}$$

The PGF of the server's state queue size distribution under the steady state conditions are given by

$$P_1(z) = P_1(0,z)\left[\frac{1-\bar{B}_1(\psi_1(z))}{\psi_1(z)}\right], \tag{64}$$

$$P_2(z) = \pi P_1(0,z)\bar{B}_1(\psi_1(z))\left[\frac{1-\bar{B}_2(\psi_2(z))}{\psi_2(z)}\right], \tag{65}$$

$$V(z) = [(1-\pi)\theta P_1(0,z)\bar{B}_1(\psi_1(z)) + \pi\theta P_1(0,z)\bar{B}_1(\psi_1(z))\bar{B}_2(\psi_2(z))]\left[\frac{1-\bar{V}(\phi(z))}{\phi(z)}\right], \tag{66}$$

$$E_1(z) = \eta_1 P_1(0,z)\left[\frac{1-\bar{B}_1(\psi_1(z))}{\psi_1(z)}\right]\left[\frac{1-\bar{G}_1(\phi(z))}{\phi(z)}\right], \tag{67}$$

$$E_2(z) = \eta_2 \pi P_1(0,z)\bar{B}_1(\psi_1(z))\left[\frac{1-\bar{B}_2(\psi_2(z))}{\psi_2(z)}\right]\left[\frac{1-\bar{G}_2(\phi(z))}{\phi(z)}\right], \tag{68}$$

$$R_1(z) = \alpha_1 P_1(0,z)\left[\frac{1-\bar{B}_1(\psi_1(z))}{\psi_1(z)}\right]\left[\frac{1-\bar{R}_1(\phi(z))}{\phi(z)}\right], \tag{69}$$

$$R_2(z) = \alpha_2 \pi P_1(0,z)\bar{B}_1(\psi_1(z))\left[\frac{1-\bar{B}_2(\psi_2(z))}{\psi_2(z)}\right]\left[\frac{1-\bar{R}_2(\phi(z))}{\phi(z)}\right], \tag{70}$$

where

$$P_1(0,z) = \frac{\left(\begin{array}{l}\lambda b_1\sum_{r=0}^{a-1} Q_r(C(z)z^r - z^b) + \lambda b_1 \sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1} c_n Q_r(z^b - z^{n+r}) \\ + \sum_{r=0}^{b-1}(z^b - z^r)((1-\theta)(1-\pi)\int_0^\infty P_{1,r}(x)\mu_1(x)dx \\ + (1-\theta)\int_0^\infty P_{2,r}(x)\mu_2(x)dx + \int_0^\infty V_r(x)\gamma(x)dx)\end{array}\right)}{\left[z^b - \left(K_1(z)\bar{B}_1(\psi_1(z)) + K_2(z)\bar{B}_1(\psi_1(z))\bar{B}_2(\psi_2(z))\right)\right]}, \tag{71}$$

$$\phi(z) = \lambda b_2(1-C(z)),$$

$$\psi_i(z) = \lambda b_1(1-C(z)) + \alpha_i(1-\bar{R}_i(\phi(z))) + \eta_i(1-\bar{E}_i(\phi(z))), i = 1,2$$

$$K_1(z) = (1-\theta)(1-\pi) + \theta(1-\pi)\overline{V}(\phi(z)),$$

$$K_2(z) = (1-\theta)\pi + \theta\pi\overline{V}(\phi(z)).$$

5.1. Queue size distribution at a random epoch

By adding (64), (65), (66), (67), (68), (69) and (70) with idle term, we get the PGF of the queue size distribution at a random epoch.

$$P(z) = P_1(z) + P_2(z) + V(z) + E_1(z) + E_2(z) + R_1(z) + R_2(z) + Q(z)$$

$$P(z) = \frac{\left(\begin{array}{l} [\lambda b_1 \sum_{r=0}^{a-1} Q_r (C(z) z^r - z^b) + \lambda b_1 \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (z^b - z^{n+r}) \\ + \sum_{r=0}^{b-1} (z^b - z^r) W_r] \times [\phi(z)\psi_2(z)(1-\overline{B}_1(\psi_1(z))) \\ + \pi\phi(z)\psi_1(z)\overline{B}_1(\psi_1(z))(1-\overline{B}_2(\psi_2(z))) \\ + (1-\pi)\theta\psi_1(z)\psi_2(z)\overline{B}_1(\psi_1(z))(1-\overline{V}(\phi(z))) \\ + \theta\pi\psi_1(z)\psi_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))(1-\overline{V}(\phi(z))) \\ + \eta_1\psi_2(z)(1-\overline{B}_1(\psi_1(z)))(1-\overline{E}_1(\phi(z))) \\ + \eta_2\pi\psi_1(z)\overline{B}_1(\psi_1(z))(1-\overline{B}_2(\psi_2(z)))(1-\overline{E}_2(\phi(z))) \\ + \alpha_1\psi_2(z)(1-\overline{B}_1(\psi_1(z)))(1-\overline{R}_1(\phi(z))) \\ + \alpha_2\pi\psi_1(z)\overline{B}_1(\psi_1(z))(1-\overline{B}_2(\psi_2(z)))(1-\overline{R}_2(\phi(z)))] \\ + [z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)] \\ \times [\phi(z)\psi_1(z)\psi_2(z)Q(z)] \end{array}\right)}{\begin{array}{l} \left[z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)\right] \\ \times [\phi(z)\psi_1(z)\psi_2(z)] \end{array}}, \tag{72}$$

$$P_r = (1-\theta)(1-\pi)\int_0^\infty P_{1,r}(x)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{2,r}(x)\mu_2(x)dx,$$

$$W_r = \int_0^\infty V_r(x)\gamma(x)dx + P_r.$$

6 . Stability condition

The probability generating function has to satisfy P(1)=1. In order to satisfy this condition, apply L'Hospital's rules and equating the expression to 1,we get

$$X_1 \times [(1+\eta_1 E(G_1)+\alpha_1 E(R_1))E(B_1)+(1+\eta_2 E(G_2)+\alpha_2 E(R_2))\pi E(B_2)+\theta E(V)]$$
$$+[b-\lambda E(I)(\theta b_2 E(V)+T_1 E(B_1)+\pi T_2 E(B_2))]\times \sum_{r=0}^{a-1} Q_r \quad (73)$$
$$=[b-\lambda E(I)(\theta b_2 E(V)+T_1 E(B_1)+\pi T_2 E(B_2))].$$

Next, we calculate the unknown probabilities, W_r, $r=0,1,2,...,b-1$ and then these are related to the idle-server probabilities, Q_r, $r=0,1,2,...,a-1$, then the left hand side of the above expression must be positive. Thus P(1)=1 is satisfied if

$$[z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)] > 0.$$

$$\text{If} \quad \rho = \frac{[\lambda E(I)(\theta b_2 E(V)+T_1 E(B_1)+\pi T_2 E(B_2))]}{b} \quad \text{then} \quad \rho < 1 \quad (74)$$

is the condition to be satisfied for the existence of steady state for the model under consideration. Equation (72) has b+a unknowns. Using the following result, we can express W_r in terms of Q_r in such a way that numerator have only ‘b’ constants. Now equation (72) gives the PGF of the number of customers involving ‘b’ unknowns. By Rouche’s theorem, the expression $[z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)]$ has $b-1$ zeros inside and one on the unit circle $|z|=1$. Since P(z) is analytic within and on the unit circle, the numerator must vanish at these points, which gives ‘b’ equations in ‘b’ unknowns. These equations can be solved by any suitable numerical technique.

6.1. Result: Let W_r can be expressed in terms of Q_r as

$$\sum_{r=0}^{a-1} W_r = \lambda b_1 \sum_{r=0}^{a-1} Q_r - \lambda b_1 \sum_{r=0}^{a-1} Q_r \sum_{k=1}^{a-r-1} c_k,$$

where, W_r is the probabilities of the ‘r’ customers in the queue during idle period and X_1, T_1, T_2, $E(I)$ are given in Section 7.

7. Performance measures

In this section, we derive some system state probabilities, the mean number of customers in the queue (L_q) and the average time a customer spends in the queue (W_q). From (74) we have $\rho<1$, which is stability condition.

7.1. System state probabilities

From equation (64) to (70), by setting $z \to 1$ and applying L’Hospital’s rule whenever necessary, we get the following results

• Let $P_q(1)$ be the steady state probability that the server is busy

$$P_q(1) = P_1(1) + P_2(1) = \frac{\left(X_1 \times \left[E(B_1) + \pi E(B_2)\right]\right)}{[b - \lambda E(I)(\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2))]}.$$

• Let $V_q(1)$ be the steady state probability that the server is on Bernoulli vacation.

$$V_q(1) = V_1(1) + V_2(1) = \frac{\left(X_1 \theta E(v)\right)}{[b - \lambda E(I)(\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2))]}.$$

• Let $E_q(1)$ be the steady state probability that the server is on Emergency vacation.

$$E_q(1) = E_1(1) + E_2(1) = \frac{\left(X_1 \times \left[\eta_1 E(B_1)E(G_1) + \eta_2 \pi E(B_2)E(G_2)\right]\right)}{[b - \lambda E(I)(\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2))]}.$$

• Let $R_q(1)$ be the steady state probability that the server is under repair

$$R_q(1) = R_1(1) + R_2(1) = \frac{\left(X_1 \times \left[\alpha_1 E(B_1)E(R_1) + \alpha_2 \pi E(B_2)E(R_2)\right]\right)}{[b - \lambda E(I)(\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2))]}.$$

7.2. Mean queue size

The mean number of customers in the queue (L_q) under steady state condition is obtained by differentiating (72) with respect to z and evaluating at $z = 1$.

$$L_q = \lim_{z \to 1} \frac{d}{dz} P(z),$$

$$P'(1) = \frac{N^V(1)D^{IV}(1) - D^V(1)N^{IV}(1)}{5[(D^{IV})^2]},$$

where

$$\begin{aligned} N^V = -60(\lambda E(I))^2 [& X_2 \lambda E(I) b_2 T_1 T_2 ((1+\eta_1 E(G_1) + \alpha_1 E(R_1))E(B_1) \\ & + (1+\eta_2 E(G_2) + \alpha_2 E(R_2))\pi E(B_2) + \theta E(V)) \\ & + X_1(\lambda b_2 E(I(I-1)) T_1 T_2 (E(B_1) + \pi E(B_2)) + (1-\pi)\theta T_1 T_2 A_1 \\ & + b_2 (T_1 E(B_1) S_2 + \pi T_2 E(B_2) S_1) + b_2 E(V) \theta (T_2 S_1 + T_1 S_2) \end{aligned}$$

$$+(b_2T_2+b_2T_2\eta_1E(G_1)+b_2T_2\alpha_1E(R_1))(S_1E(B_1)+(\lambda E(I)T_1)^2E(B_1^2))$$
$$+(b_2\pi T_1+b_2T_1\pi\eta_2E(G_2)+b_2T_1\pi\alpha_2E(R_2))(S_2E(B_2)+(\lambda E(I)T_2)^2E(B_2^2))$$
$$+2\pi b_2(\lambda E(I))^2T_1^2T_2E(B_1)E(B_2)+2\theta T_1^2T_2(\lambda E(I))^2b_2E(V)E(B_1)$$
$$+\theta\pi T_1T_2[2(\lambda E(I))^2T_2E(B_2)b_2E(V)+A_1]+b_2\eta_1T_1E(B_1)E(G_1)S_2$$
$$+T_1T_2[\eta_1E(B_1)D_1+\pi\eta_2E(B_2)D_2]+b_2T_2\pi\eta_2E(B_2)E(G_2)S_1$$
$$+2T_1^2b_2T_2E(B_2)\pi\eta_2E(B_1)E(G_2)(\lambda E(I))^2+b_2T_1\alpha_1E(B_1)E(R_1)S_2$$
$$+T_1T_2[\alpha_1E(B_1)C_1+\alpha_2\pi E(B_2)C_2]+b_2T_2\alpha_2\pi E(B_2)E(R_2)S_1$$
$$+2T_1^2\alpha_2\pi E(B_1)(\lambda E(I))^2b_2T_2E(B_2)E(R_2))$$
$$+(b(b-1)-\theta A_1-S_1E(B_1)-\pi S_2E(B_2)$$
$$-2\theta(\lambda E(I))^2b_2E(V)(\pi T_2E(B_2)+T_1E(B_1))$$
$$-(\lambda E(I))^2[T_1^2E(B_1^2)+T_2^2E(B_2^2)]-2\pi(\lambda E(I))^2T_1T_2E(B_1)E(B_2))$$
$$\times(\lambda E(I)b_2T_1T_2)\sum_{r=0}^{a-1}Q_r+(b-\lambda E(I)(\theta b_2E(V)+T_1E(B_1)+\pi T_2E(B_2)))$$
$$\times[\lambda b_2E(I(I-1))T_1T_2\sum_{r=0}^{a-1}Q_r+b_2T_2\sum_{r=0}^{a-1}Q_rS_1$$
$$+2\lambda E(I)T_1T_2b_2\sum_{r=0}^{a-1}rQ_r+S_2b_2T_1\sum_{r=0}^{a-1}Q_r]]$$

$$N^{IV}=-24(\lambda E(I))^3b_2T_1T_2[X_1((1+\eta_1E(G_1)+\alpha_1E(R_1))E(B_1)$$
$$+(1+\eta_2E(G_2)+\alpha_2E(R_2))\pi E(B_2)+\theta E(V))$$
$$+(b-\lambda E(I)(\theta b_2E(V)+T_1E(B_1)+\pi T_2E(B_2)))\sum_{r=0}^{a-1}Q_r]$$

$$D^{IV}=-24(\lambda E(I))^3b_2T_1T_2(b-\lambda E(I)(\theta b_2E(V)+T_1E(B_1)+\pi T_2E(B_2))).$$

$$D^{V}=-60(\lambda E(I))^2[(b(b-1)-\theta A_1-S_1E(B_1)-\pi S_2E(B_2)$$
$$-2\theta(\lambda E(I))^2b_2E(V)(\pi T_2E(B_2)+T_1E(B_1))$$
$$-(\lambda E(I))^2[T_1^2E(B_1^2)+T_2^2E(B_2^2)]-2\pi(\lambda E(I))^2T_1T_2E(B_1)E(B_2))$$
$$\times(\lambda E(I)b_2T_1T_2)+(b-\lambda E(I)(\theta b_2E(V)+T_1E(B_1)+\pi T_2E(B_2)))$$
$$\times[3b_2T_1T_2\lambda E(I(I-1))+b_2(\lambda b_2E(I))^2[T_2\alpha_1E(R_1^2)+$$
$$+T_1\alpha_2E(R_2^2)+T_2\eta_1E(G_1^2)+T_1\eta_2E(G_2^2)]]].$$

$$X_1 = \lambda b_1 \sum_{r=0}^{a-1} Q_r (E(I) + r - b) + \lambda b_1 \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (b - n - r) + \sum_{r=0}^{b-1} (b - r) W_r$$

$$X_2 = \lambda b_1 \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (b(b-1) - (n+r)(n+r-1))$$
$$+ \lambda b_1 \sum_{r=0}^{a-1} Q_r (E(I(I-1)) + 2E(I)r + r(r-1) - b(b-1))$$
$$+ \sum_{r=0}^{b-1} (b(b-1) - r(r-1)) W_r$$

$$A_1 = \lambda b_2 E(I(I-1))E(V) + (\lambda b_2 E(I))^2 E(V^2)$$

$$T_1 = b_1 + b_2[\alpha_1 E(R_1) + \eta_1 E(G_1)]$$

$$T_2 = b_1 + b_2[\alpha_2 E(R_2) + \eta_2 E(G_2)]$$

$$S_1 = \lambda E(I(I-1))T_1 + (\lambda b_2 E(I))^2 [\alpha_1 E(R_1^2) + \eta_1 E(G_1^2)]$$
$$S_2 = \lambda E(I(I-1))T_2 + (\lambda b_2 E(I))^2 [\alpha_2 E(R_2^2) + \eta_2 E(G_2^2)]$$
$$C_1 = \lambda b_2 E(I(I-1))E(R_1) + (\lambda b_2 E(I))^2 E(R_1^2)$$

$$C_2 = \lambda b_2 E(I(I-1))E(R_2) + (\lambda b_2 E(I))^2 E(R_2^2)$$

$$D_1 = \lambda b_2 E(I(I-1))E(G_1) + (\lambda b_2 E(I))^2 E(G_1^2)$$

$$D_2 = \lambda b_2 E(I(I-1))E(G_2) + (\lambda b_2 E(I))^2 E(G_2^2)$$

$$E(I) = C'(1), \quad E(I(I-1)) = C''(1).$$

The average time a customer spends in the queue (W_q) are found by using the Little's formula,

$$W_q = \frac{L_q}{\lambda E(I)}.$$

8 . Queue size distribution at a departure epoch

In this section, we derive the probability generating function of the queue size distribution at a departure epoch of this model is given in the proof of Theorem 1.

Theorem 8.1.

Under the steady-state condition, the PGF of the queue size distribution at a departure epoch of this model is given by

$$P^+(z) = \frac{\left(\begin{array}{l}[\lambda b_1 \sum_{r=0}^{a-1} Q_r (C(z)z^r - z^b) + \sum_{r=0}^{b-1}(z^b - z^r)W_r \\ + \lambda b_1 \sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1} c_n Q_r (z^b - z^{n+r})] \times b(1-\rho) \\ \times \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)\end{array}\right)}{([z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)]X_1)}. \quad (75)$$

Proof:

Following the argument of PASTA (See Wolf (1982)). We state that a departing customer will see 'j' customer in the queue just after a departure if and only if there were 'j' customer in the queue just before the departure.

Let $P^+(z) = \sum_{j=0}^{\infty} p_j^+ z^j$ be the probability that there are 'j' customers in the queue at a departure epoch, we may write

$$p_j^+ = K_0(1-\theta)(1-\pi)\int_0^\infty P_{1,j}(x)\mu_1(x)dx + K_0(1-\theta)\int_0^\infty P_{2,j}(x)\mu_2(x)dx + K_0\int_0^\infty V_j(x)\gamma(x)dx,\ j \geq 0, \quad (76)$$

where K_0 is the normalizing constant.

By multiplying both sides of the equation (76) by z^j summation over j from 0 to ∞, and use the equations (51) and (52) (after applying the Tauberian property), we get on simplification

$$P^+(z) = \frac{K_0\left(\begin{array}{l}[\lambda b_1 \sum_{r=0}^{a-1} Q_r (C(z)z^r - z^b) + \lambda b_1 \sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1} c_n Q_r (z^b - z^{n+r}) \\ + \sum_{r=0}^{b-1}(z^b - z^r)W_r] \\ \times \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)\end{array}\right)}{\left[z^b - \left(K_1(z)\overline{B}_1(\psi_1(z)) + K_2(z)\overline{B}_1(\psi_1(z))\overline{B}_2(\psi_2(z))\right)\right]}. \quad (77)$$

From $P^+(1) = 1$, we get

$$K_0 = \frac{b(1-\rho)}{X_1}, \quad (78)$$

where

$$\rho = \frac{(\lambda E(I)(\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2)))}{b} < 1. \quad (79)$$

Inserting equation (78) into (77) we get the PGF of the queue size distribution at a departure epoch of this model.

Next the mean queue size of this model is given in corollary 1.

Corollary 8.2.

Under the stability conditions, the mean number of customers in the queue at a departure epoch L_d *is given by*

$$L_d = b\rho + \frac{X_2}{2X_1} - \frac{(b-1)}{2(1-\rho)} + \frac{\begin{array}{l}(\lambda E(I))^2[\theta b_2^2 E(V^2) + 2\theta b_2 E(V)(\pi T_2 E(B_2) + T_1 E(B_1)) \\ + b_2^2(\alpha_1 E(R_1^2) + \eta_1 E(G_1^2))E(B_1) \\ + T_1^2 E(B_1^2) + \pi T_2^2 E(B_2^2) + 2\pi T_1 T_2 E(B_1)E(B_2) \\ + b_2^2 \pi(\alpha_2 E(R_2^2) + \eta_2 E(G_2^2))E(B_2)]\end{array}}{2b(1-\rho)} + \frac{(\lambda E(I(I-1)))[\theta b_2 E(V) + T_1 E(B_1) + \pi T_2 E(B_2)]}{2b(1-\rho)}. \tag{80}$$

Proof:

The result follows directly by differentiating (75) with respect to z and then taking limit $z \to 1$ by using the L'Hospital's rule, where X_1, X_2, T_1, T_2, $E(I)$, $E(I(I-1))$ are given in section 7.

9. Particular cases

Case 1:

If single arrival, single service, no second optional service, no emergency vacation, no restricted admissibility is considered, then equation (72) reduces to

$$P(z) = \frac{\begin{array}{l}Q[\phi(z)(1-\bar{B}_1(\psi_1(z))) + \theta\psi_1(z)\bar{B}_1(\psi_1(z))(1-\bar{V}(\phi(z))) \\ + \alpha_1(1-\bar{B}_1(\psi_1(z)))(1-\bar{R}_1(\phi(z)))]\end{array}}{[((1-\theta)+\theta\bar{V}(\phi(z)))\bar{B}_1(\psi_1(z)) - z]\psi_1(z)},$$

where $Q = 1-\rho, \phi(z) = \lambda(1-z)$, $\rho = \lambda((1+\alpha_1)E(B_1) + \theta E(V))$.

These expressions are exactly matched with the results by Choudhury and Deka (2012) by taking single phase of service.

Case 2:

If single arrival, single service, no two types of vacation, no restricted admissibility is considered, then equation (72), (77) and (79) reduces to

$$P(z)=\frac{\begin{gathered}Q[\phi(z)(1-\overline{B}_1(\psi_1(z)))\psi_2(z)\\ +\phi(z)\psi_1(z)\pi\overline{B}_1(\psi_1(z))(1-\overline{B}_2(\psi_2(z)))\\ +\alpha_1\psi_2(z)(1-\overline{B}_1(\psi_1(z)))(1-\overline{R}_1(\phi(z)))\\ +\alpha_2\psi_1(z)\pi\overline{B}_1(\psi_1(z))(1-\overline{B}_2(\psi_2(z)))(1-\overline{R}_2(\phi(z)))]\end{gathered}}{\left([(1-\pi)+\pi\overline{B}_2(\psi_2(z))]\overline{B}_1(\psi_1(z))-z\right)\psi_1(z)\psi_2(z)}$$

and

$$P(z)=\frac{Q(1-z)[(1-\pi)+\pi\overline{B}_2(\psi_2(z))]\overline{B}_1(\psi_1(z))}{[(1-\pi)+\pi\overline{B}_2(\psi_2(z))]\overline{B}_1(\psi_1(z))-z}$$

$$L_d=\rho+\frac{\lambda^2[\alpha_1E(R_1^2)E(B_1)+T_1^2E(B_1^2)+T_2^2\pi E(B_2^2)]}{2(1-\rho)}+\frac{\lambda^2[2\pi T_1T_2E(B_1)E(B_2)+\pi\alpha_2E(R_2^2)E(B_2)]}{2(1-\rho)},$$

where $Q=1-\rho,\quad \rho=\lambda(T_1E(B_1)+\pi T_2E(B_2)),\phi(z)=\lambda(1-z)$
$T_i=1+\alpha_iE(R_i),\psi_i(z)=\phi(z)+\alpha_i(1-\overline{R}_i(\phi(z))),i=1,2$

These expressions agree with the results by Gautam Choudhury and Lotfi Tadj (2009) by without taking delay time to repair.

10. Special cases

Case 1:

Consider that the service time distribution of both services are follows exponential with parameters μ_1, μ_2 then the LST of B_1, B_2 are given by

$$\overline{B}_i(s)=\left(\frac{\mu_i}{\mu_i+s}\right) \text{ and } \quad E(B_i)=\frac{1}{\mu_i}.$$

$$\overline{B}_i(\psi_i(z))=\left(\frac{\mu_i}{\mu_i+\psi_i(z)}\right)\ .,$$

where $\psi_i(z) = \lambda b_1(1-C(z)) + \alpha_i(1-\overline{R}_i(\phi(z))) + \eta_i(1-\overline{E}_i(\phi(z))), i = 1,2$.

$$P(z) = \frac{\left(\begin{aligned} &[\lambda b_1 \sum_{r=0}^{a-1} Q_r (C(z)z^r - z^b) + \lambda b_1 \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (z^b - z^{n+r}) \\ &+ \sum_{r=0}^{b-1} (z^b - z^r) W_r] \times [\phi(z)\psi_2(z)\psi_1(z)(\mu_2 + \psi_2(z)) + \pi\phi(z)\psi_2(z)\psi_1(z)\mu_1 \\ &+ (1-\pi)\theta\psi_1(z)\psi_2(z)\mu_1(\mu_2 + \psi_2(z))(1-\overline{V}(\phi(z))) \\ &+ \theta\pi\psi_1(z)\psi_2(z)\mu_1\mu_2(1-\overline{V}(\phi(z))) + \eta_1\psi_2(z)(\mu_2 + \psi_2(z))\psi_1(z)(1-\overline{E}_1(\phi(z))) \\ &+ \eta_2\pi\psi_1(z)\mu_1\psi_2(z)(1-\overline{E}_2(\phi(z))) + \alpha_1\psi_2(z)(\mu_2 + \psi_2(z))\psi_1(z)(1-\overline{R}_1(\phi(z))) \\ &+ \alpha_2\pi\psi_1(z)\mu_1\psi_2(z)(1-\overline{R}_2(\phi(z)))] + [(\mu_1 + \psi_1(z))(\mu_2 + \psi_2(z))z^b - (K_1(z) \\ &\quad (\mu_2 + \psi_2(z))\mu_1 + K_2(z)\mu_1\mu_2)][\phi(z)\psi_1(z)\psi_2(z)Q(z)] \end{aligned}\right)}{\begin{aligned} &[(\mu_1 + \psi_1(z))(\mu_2 + \psi_2(z))z^b - (K_1(z)\mu_1 \\ &\quad (\mu_2 + \psi_2(z)) + K_2(z)\mu_1\mu_2)] \times [\phi(z)\psi_1(z)\psi_2(z)] \end{aligned}}.$$

where $\rho = \dfrac{[\lambda E(I)(\theta b_2 E(V) + \dfrac{T_1}{\mu_1} + \dfrac{\pi T_2}{\mu_2})]}{b}$.

This is, the PGF of the stationary queue size distribution of $M^{[x]}/M(a,b)/1$ queue with unreliable server, second optional service, two different vacations policy, restricted admissibility policy.

Case 2:

Consider that the service time distribution of both services are follows Erlang-2 with parameter μ_1, μ_2 then the LST of B_1, B_2 are given by

$$\overline{B}_i(s) = \left(\frac{2\mu_i}{2\mu_i + s}\right)^2 \quad \text{and} \quad E(B_i) = \frac{1}{\mu_i}.$$

$$\overline{B}_i(\psi_i(z)) = \left(\frac{\mu_i}{\mu_i + \psi_i(z)}\right) .$$

where $\psi_i(z) = \lambda b_1(1-C(z)) + \alpha_i(1-\overline{R}_i(\phi(z))) + \eta_i(1-\overline{E}_i(\phi(z))), i = 1,2$.

$$P(z)=\frac{\left(\begin{array}{l}[\lambda b_1\sum_{r=0}^{a-1}Q_r(C(z)z^r-z^b)+\lambda b_1\sum_{r=0}^{a-1}\sum_{n=1}^{b-r-1}c_nQ_r(z^b-z^{n+r})\\ +\sum_{r=0}^{b-1}(z^b-z^r)W_r]\times[\phi(z)\psi_2(z)[(2\mu_1+\psi_1(z))^2-(2\mu_1)^2]\\ (2\mu_2+\psi_2(z))^2+\pi\phi(z)\psi_1(z)(2\mu_1)^2[(2\mu_2+\psi_2(z))^2-(2\mu_2)^2]\\ +(1-\pi)\theta\psi_1(z)\psi_2(z)(2\mu_1)^2(2\mu_2+\psi_2(z))^2(1-\overline{V}(\phi(z)))\\ +\theta\pi\psi_1(z)\psi_2(z)(2\mu_1)^2(2\mu_2)^2(1-\overline{V}(\phi(z)))\\ +\eta_1\psi_2(z)(2\mu_2+\psi_2(z))^2[(2\mu_1+\psi_1(z))^2-(2\mu_1)^2](1-\overline{E}_1(\phi(z)))\\ +\eta_2\pi\psi_1(z)(2\mu_1)^2[(2\mu_2+\psi_2(z))^2-(2\mu_2)^2](1-\overline{E}_2(\phi(z)))\\ +\alpha_1\psi_2(z)(2\mu_2+\psi_2(z))^2[(2\mu_1+\psi_1(z))^2-(2\mu_1)^2](1-\overline{R}_1(\phi(z)))\\ +\alpha_2\pi\psi_1(z)(2\mu_1)^2[(2\mu_2+\psi_2(z))^2-(2\mu_2)^2](1-\overline{R}_2(\phi(z)))]\\ +[(2\mu_1+\psi_1(z))^2(2\mu_2+\psi_2(z))^2z^b-(K_1(z)(2\mu_1)^2\\ (2\mu_2+\psi_2(z))^2+K_2(z)(2\mu_1)^2(2\mu_2)^2)]\big[\phi(z)\psi_1(z)\psi_2(z)Q(z)\big]\end{array}\right)}{\begin{array}{l}[(2\mu_1+\psi_1(z))^2(2\mu_2+\psi_2(z))^2z^b-(K_1(z)(2\mu_1)^2\\ (2\mu_2+\psi_2(z))^2+K_2(z)(2\mu_1)^2(2\mu_2)^2)]\times\big[\phi(z)\psi_1(z)\psi_2(z)\big]\end{array}}.$$

where $$\rho=\frac{[\lambda E(I)(\theta b_2E(V)+\frac{T_1}{\mu_1}+\frac{\pi T_2}{\mu_2})]}{b},$$

This is the PGF of the stationary queue size distribution of $M^{[x]}/E_2(a,b)/1$ queue with unreliable server, second optional service, two different vacations policy, restricted admissibility policy.

11. Numerical results

In this section, we present some numerical results using MATLAB in order to illustrate the effect of various parameters in the system performance measures of our system.

1. Batch size distribution of the arrival is geometric with mean 2.
2. Service time of essential and optional service follows Erlang-2 distribution.
3. Bernoulli vacation time, emergency vacation time, both repair time follow exponential distribution.

Let us fix the parameters $a=2$, $b=5$, $\theta=0.4$, $b_1=0.3$, $b_2=0.4$, $\pi=0.3$, $\lambda=1$, $\mu_1=9$, $\mu_2=14$, $\gamma=8$, $\alpha_1=1$, $\alpha_2=1.05$, $\beta_1=1.15$, $\beta_2=1.20$, $\eta_1=1.16$, $\eta_2=1.18$, $\zeta_1=1.20$, $\zeta_2=1.22$, such that the stability condition is satisfied.
Tables 2 to 4 gives computed values of the utilization factor(ρ), the mean queue size(L_q), mean waiting time in the queue(W_q) for our queueing model.

Table 2 clearly shows that the arrival rate (λ) increases, the utilization factor (ρ), the mean queue size (L_q) and the mean waiting time in the queue (W_q) are also increases.

Table 2: The effect of arrival rate (λ) on ρ, L_q, W_q

λ	ρ	L_q	W_q
1.00	0.0629	4.4458	2.2229
1.25	0.0786	5.5684	2.2274
1.50	0.0943	6.8848	2.2949
1.75	0.1100	8.4115	2.4033
2.00	0.1257	10.1666	2.5416
2.25	0.1414	12.1705	2.7045
2.50	0.1572	14.4457	2.8891
2.75	0.1729	17.0175	3.0941
3.00	0.1886	19.9429	3.3238

Table 3 shows that the first essential service rate (μ_1) increases , the utilization factor (ρ), the mean queue size (L_q) and the mean waiting time in the queue (W_q) are decreases.

Table 3: The effect of service rate (μ_1) on ρ, L_q, W_q

μ_1	ρ	L_q	W_q
3	0.1548	12.0056	6.0028
4	0.1203	8.6014	4.3007
5	0.0996	6.9081	3.4541
6	0.0859	5.9068	2.9534
7	0.0760	5.2499	2.6250
8	0.0686	4.7878	2.3939
9	0.0629	4.4458	2.2229
10	0.0583	4.1830	2.0915
11	0.0545	3.9749	1.9875
12	0.0514	3.8063	1.9031

Table 4 shows that the second optional service rate (μ_2) increases, the utilization factor (ρ), the mean queue size (L_q) and the mean waiting time in the queue (W_q) are decreases.

Table 4: The effect of service rate (μ_2) on ρ, L_q, W_q

μ_2	ρ	L_q	W_q
2	0.1047	8.7618	4.3909
3	0.0840	6.3195	3.1598
4	0.0736	5.3537	2.6768
5	0.0674	4.8505	2.4253
6	0.0632	4.5458	2.2729
7	0.0603	4.3429	2.1715
8	0.0580	4.1987	2.0994
9	0.0563	4.0912	2.0456
10	0.0549	4.0081	2.0041
11	0.0538	3.9421	1.9710

In Figure 1 shows that the utilization factor (ρ), the average queue length (L_q) and average waiting time in the queue (W_q) increases for the increasing values of the arrival rate λ.

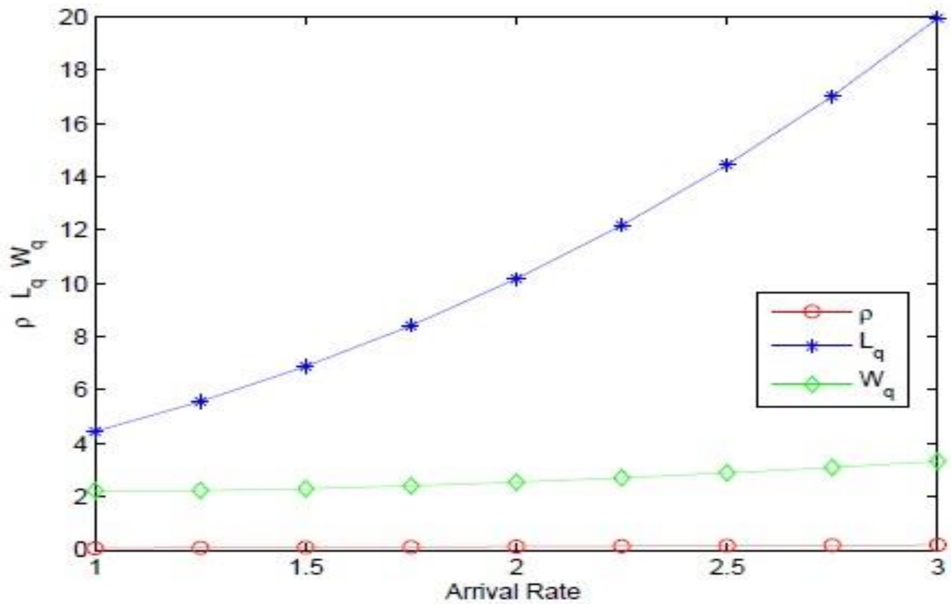

Figure 1: ρ, L_q, W_q versus Arrival rate (λ)

Similarly, In Figure 2 and 3 shows that the utilization factor (ρ), the average queue length (L_q) and average waiting time in the queue(W_q) decreases for the increasing value of service rates μ_1 and μ_2.

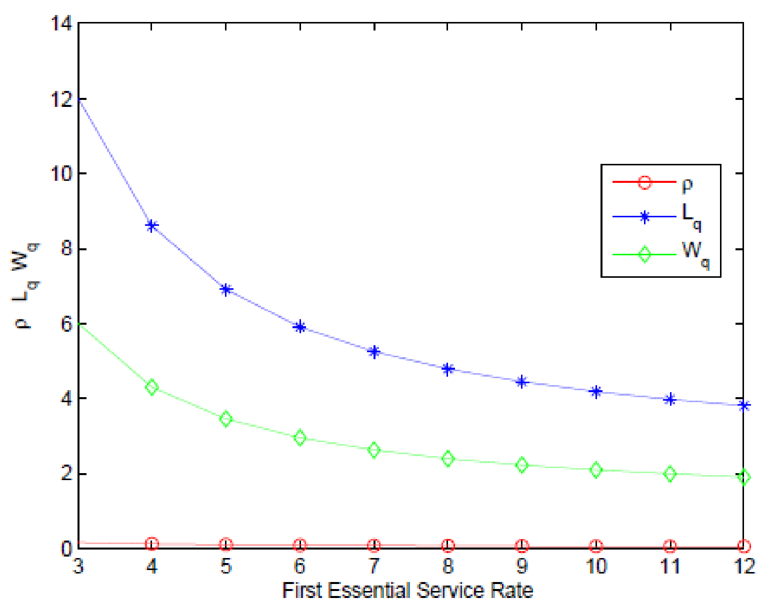

Figure 2: ρ, L_q, W_q versus First essential Service rate (μ_1)

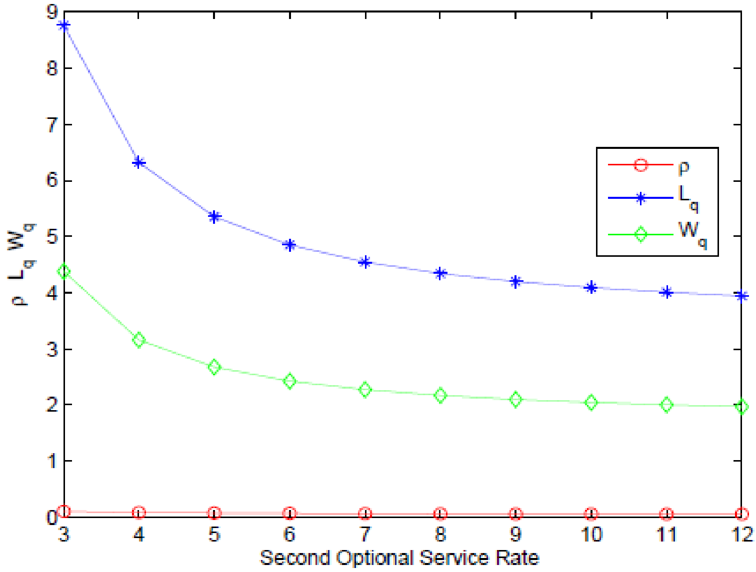

Figure 3: ρ , L_q, W_q versus Second Optional Service rate (μ_2)

12 . Conclusion and further work

In this paper, we have studied an M^X /G(a,b)/1 queueing system with second optional service subject to server breakdown and two different types of vacation under restricted admissibility. We derive the probability generating function of the number of customers in the queue at a random epoch in transient and steady state conditions and also we obtained the queue size distribution at a departure epoch under the steady state conditions. The performance measures of the system state probabilities, the mean queue size and the average waiting time in the queue are determined under steady state condition. Some particular cases are discussed. The results are validated with the help of numerical illustrations. To this end, we can extend this model to J additional options for service under Bernoulli schedule vacaion.

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