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A Method of Deriving Companion Identities Associating *q*-Series

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Abstract

In this paper, we have established two theorems by making use of Euler's *q*-derivative and *q*-shifted operators for a function of one variable and also for function of two variables. We derived several companion identities by applying these theorems on some known *q*-series identities. We deduced several special cases which are also the companion identities in the last section of the paper.

Keywords/Phrases: *q*-series identities; summation and transformation formulae for basic hypergeometric series; *q*-differential operator and Euler's *q*-derivative operator

MSC 2010 No: 33D15, 33D60

1. Introduction

Riese (1997) derived the dual and companion identities in the second chapter of his thesis by making use of the theory of *qWZ*-pairs developed by Wilf and Zeilberger (1990). Riese (1997) used *q*Zeil of the *q*-Zeilberger algorithm (Paule and Riese (1997)) to generate the dual identities and companion identities systematically. Later, Zhang and Yang (2009) verified some of these identities. Somashekara et al. (2011) derived a new summation formula for ${}_2\psi_2$ basic bilateral hypergeometric series by using method of parameter augmentation. Kumar et al. (2012)

established some interesting theorems which verify the special case of companion identity by using parameter augmentation method. In this paper, we have established a method for deriving such identities without use of q-Zeilberger algorithm. For any description about dual identities and companion identities readers may consult Paule and Riese (1997).Finally, we wish to derive results presented in the second chapter of the thesis of Riese (1997) without use of qZeil of the q-Zeilberger algorithm. To achieve the goal we have used the several definitions and known results that are given in the next section.

2. Preliminaries and q-Notations

For any integer n the q- shifted factorial is defined as

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), \quad (1)$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad (2)$$

$$(a;\ q)_\infty = \sum_{n=0}^{\infty} \frac{(-a)^n q^{\binom{n}{2}}}{(q;q)_n}, \quad (3)$$

$$\frac{1}{(a;q)_\infty} = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n}, \quad (4)$$

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n}. \quad (5)$$

Throughout the paper, we have considered $0 < q < 1$ and $-1 < x < 1$.

Definition 2.1. *q*- Derivative

Euler's q- differential operator for a function $F(x)$ is introduced in Rogers (1893, 1894, 1896) as follows

$$D_q\{F(x)\} = \frac{\{F(x)-F(qx)\}}{x}. \quad (6)$$

Definition 2.2. *α, β*-Shift Operators

The q-shifted operator α, β for $F(x)$ is introduced in the Andrews (1971), Roman (1985) defined as

$$\alpha\{F(x)\} = F(qx),\ \beta\{F(x)\} = F(q^{-1}x). \quad (7)$$

Now, we define partial q-derivatives with respect to x and with respect y on the same fashion of Euler's q-derivative for a function $F(x, y)$ as follows

$$F'(x,y) = \frac{F(x,y)-F(qx,y)}{x}. \quad (8)$$

Now, we define partial q-derivatives with respect to x and with respect y on the same fashion of Euler's q-derivative for a function $F(x, y)$ as follows

$$F'(x,y) = \frac{F(x,y)-F(x,q^{-1}y)}{q^{-1}y}. \tag{9}$$

Definition 2.3.

We define a q-derivative formula for a function $F(x, y)$ with respect to x and y

$$D_{xy}F(x,y) = \frac{F(x,yq^{-1})-F(xq,y)}{x-q^{-1}y}. \tag{10}$$

The q-shifted operator 'α' for $F(x, y)$ is defined by

$$\alpha\{F(x,y)\} = F(qx,y). \tag{11}$$

The q-shifted operator 'β' for $F(x, y)$ is defined as under

$$\beta\{F(x,y)\} = F(x,q^{-1}y). \tag{12}$$

The two definitions (11) and (12) coincide to $F(x, y)$ for α^0 and β^0.

3. Main Results

In this section we shall established two theorems and their proof.

Theorem 3.1.

If $D_q F(x) = f(x)$ and $\alpha\{F(x)\}=F(qx)$,

then

$$F(x) - F(xq^m) = x\sum_{n=0}^{m-1} f(xq^n)q^n . \tag{13}$$

***Proof*:**

Taking $D_q F(x)=f(x)$ and replacing $F(qx)$ by $\alpha\{F(x)\}$ in equation (6) and rearranging, we get

$$F(x) = \frac{1}{1-\alpha}\{xf(x)\} \tag{14}$$

Multiplying both sides of (14) by $(1-\alpha^m)$ and then using the equation (7) on the left hand side and sum the right hand side as follows

$$F(x) - F(xq^m) = \sum_{n=0}^{m-1} \alpha^n \{xf(x)\} \tag{15}$$

Now, apply operator defined by equation (7) on equation (15), we get the required result in form of equation (13).

Theorem 3.2.

If $D_{xy}\, F(x, y)=f(x, y)$, $\alpha\{F(x, y)\}=F(qx, y)$ and

d $\beta\{F(x, y)\}=F(x, q^{-1} y)$,

then

$$F(x,y) - F(xq^m, yq^m) = (x-y)\sum_{n=0}^{m-1} f(xq^n, yq^{n+1})q^n. \tag{16}$$

Proof:

Taking $\alpha\{F(x, y)\}=F(qx, y)$ by equation (11) and $\beta\{F(x, y)\}=F(x, q^{-1} y)$ by equation (12) in equation (10) and rearrange as under

$$F(x,y) = \frac{1}{\beta(1-\alpha/\beta)}\left\{(x-q^{-1}y)D_{xy}F(x,y)\right\}. \tag{17}$$

Multiplying both sides of equation (17) by $[1-(\alpha/\beta)^m]$ and then using equation (11) and equation (12), we get

$$\left[1-\left(\frac{\alpha}{\beta}\right)^m\right]F(x,y) = \frac{1}{\beta}\left[\frac{1-\left(\frac{\alpha}{\beta}\right)^m}{1-\alpha/\beta}\right]\left\{(x-q^{-1}y)D_{xy}F(x,y)\right\}. \tag{18}$$

Taking $D_{xy}\, F(x, y)=f(x, y)$ and summing the right hand side of equation (18) by geometric series

$$F(x,y) - F(xq^m, yq^m) = \sum_{n=0}^{m-1} \alpha^n \beta^{-n-1}\{(x-q^{-1}y)f(x,y)\}. \tag{19}$$

Simplifying equation (19) by using equation (11) and equation (12), we get the required equation (16).

4. Applications

In this section, we shall establish some companion identities by making use of theorem 3.1 and theorem 3.2. Identity 4.1 and Identity 4.2 are derived by the application of theorem 3.1 and Identity 4.3 is derived by the application of theorem 3.2.

Identity 4.1.

If

$$F(x) = \frac{1}{(x;q)_\infty} \text{ and}$$

$$F(x) - F(xq^m) = x\sum_{n=0}^{m-1} f(xq^n)q^n,$$

then

$$x\sum_{n=0}^{m-1}(x;\ q)_n\, q^n = 1-(x;\ q)_m. \tag{20}$$

***Proof**:*

We have

$$F(x) = \frac{1}{(x;\,q)_\infty}\ . \tag{21}$$

Using Euler's q-differential operator defined by equation (6) on equation (21), we get

$$D_q F(x) = \frac{1}{(x;\,q)_\infty} = f(x)\ . \tag{22}$$

Setting $x = xq^n$ in equation (22), we get

$$f(xq^n) = \frac{1}{(xq^n;\,q)_\infty} \tag{23}$$

and setting $x = xq^m$ in equation (21), we get

$$F(xq^m) = \frac{1}{(xq^m;\,q)_\infty}\ . \tag{24}$$

Substituting the values of *F(x)*, $F(xq^m)$ and $f(xq^n)$ from equation (21), (24) and (23) respectively in equation(13) and after simplifying, we get the required result (20).

Verification of identity (4.1)

It is verified by analytical method using equation (13). In this method the left hand side of equation (13) becomes dummy and only right hand side is solvable in two different ways. The procedure is as under:

Substituting the value of $f(xq^n)$ from equation (23) in equation (13), we get

$$F(x)-F(xq^m) = x\sum_{n=0}^{m-1}\frac{1}{(xq^n;\,q)_\infty}q^n\ . \tag{25}$$

On simplification, equation (25) can be written as

$$F(x)-F(xq^m) = \frac{x}{(x;\,q)_\infty}\sum_{n=0}^{m-1}(x;\ q)_n q^n\ . \tag{26}$$

Expanding the inner series of equation (25) by using equation (4) as follows

$$F(x)-F(xq^m) = x\sum_{n=0}^{m-1}q^n\sum_{r=0}^{\infty}\frac{(xq^n)^r}{(q;q)_r}\ . \tag{27}$$

Rearranging equation (27) becomes

$$F(x)-F(xq^m)=x\sum_{r=0}^{\infty}\frac{(x)^r}{(q;q)_r}\sum_{n=0}^{m-1}q^{n(r+1)}\ . \tag{28}$$

The inner right series is a geometric series of *'m'* terms with first term 1 and common ratio q^{r+1} after summing this, equation (28) becomes

$$F(x)-F(xq^m)=\sum_{r=0}^{\infty}\frac{x^{r+1}}{(q;q)_{r+1}}\left(1-q^{m(r+1)}\right)\ . \tag{29}$$

Replacing r by $(r\text{-}1)$ in equation (29), we get

$$F(x)-F(xq^m)=\sum_{r=1}^{\infty}\frac{x^r}{(q;q)_r}-\sum_{r=1}^{\infty}\frac{(xq^m)^r}{(q;q)_r}. \tag{30}$$

Taking both series from $r=0$ in the right hand side of equation (30), we get

$$F(x)-F(xq^m)=\sum_{r=0}^{\infty}\frac{x^r}{(q;q)_r}-\sum_{r=0}^{\infty}\frac{(xq^m)^r}{(q;q)_r}. \tag{31}$$

Summing both series of right hand side of equation (31) by equation (4), we get

$$F(x)-F(xq^m)=\frac{1}{(x;q)_\infty}-\frac{1}{(xq^m;q)_\infty}. \tag{32}$$

On combining equation (26) and equation (32), we get the required result (20).

Identity 4.2.

If $F(x)=(x;\ q)_\infty$ and

$$F(x)-F(xq^m)=x\sum_{n=0}^{m-1}f(xq^n)q^n,$$

then

$$x\sum_{n=0}^{m-1}\frac{1}{(x;q)_{n+1}}q^n=\frac{1}{(x;q)_m}-1. \tag{33}$$

***Proof**:*

We have

$$F(x)=(x;\ q)_\infty\ . \tag{34}$$

Using Euler's q-differential operator defined by equation (6) on equation (34), we get

$$D_q F(x) = -(xq;\, q)_\infty = f(x). \tag{35}$$

Taking $x= xq^n$ in equation (35), we get

$$f(xq^n) = -(xq^{n+1};\, q)_\infty\,. \tag{36}$$

Setting $x=xq^m$ in equation (34), we get

$$F(xq^m) = (xq^m;\, q)_\infty\,. \tag{37}$$

Substituting these values of $F(x)$, $F(xq^m)$ and $f(xq^n)$ from equation (34), (36) and (37) respectively in (13), we get the required result (33).

Verification of identity (4.2)

Applying analytical method by using equation (13), the left hand side of equation (13) becomes dummy and only right hand side is solvable in two different ways as discussed in the verification of (4.1).

Identity 4.3.

If

$$F(x,y) = \frac{(yt;q)_\infty}{(xt;q)_\infty} \text{ and}$$

$$F(x,y) - F(xq^m, yq^m) = (x-y)\sum_{n=0}^{m-1} f(xq^n, yq^{n+1})q^n,$$

then

$$t(x-y)\sum_{n=0}^{m-1} \frac{(xt;q)_n}{(yt;q)_{n+1}}\, q^n = 1 - \frac{(xt;q)_m}{(yt;q)_m}. \tag{38}$$

Proof:

We have

$$F(x,y) = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{39}$$

Operating equation (10) on equation (39), we get

$$f(x,y) = t\,\frac{(yt;q)_\infty}{(xt;q)_\infty}\,. \tag{40}$$

Setting $x= xq^n$, $y=yq^{n+1}$ in equation (40), we get

$$f(xq^n, yq^{n+1}) = t\,\frac{(ytq^{n+1};q)_\infty}{(xtq^n;q)_\infty}\,. \tag{41}$$

Setting $x= xq^m$, $y=yq^m$ in (41), we get

$$F(xq^m, yq^m) = \frac{(ytq^m;q)_\infty}{(xtq^m;q)_\infty}. \tag{42}$$

Substituting these values of $F(x, y)$, $F(xq^m, yq^m)$ and $f(xq^n, yq^{n+1})$ in equation (16) , we get the required result (38).

Verification of identity (4.3)

It is verified by analytical method using equation (16). In this method the left hand side becomes dummy and only right hand side is solvable in two different ways. The procedure is illustrated as below:

We have

$$f(xt, yt) = \frac{(yt;q)_\infty}{(xt;q)_\infty}\,. \tag{43}$$

Setting $x= xq^n$, $y=yq^{n+1}$ in equation (43), we get

$$f(xtq^n, ytq^{n+1}) = \frac{(ytq^{n+1};q)_\infty}{(xtq^n;q)_\infty}\,. \tag{44}$$

Substituting the value of $f(xtq^n, ytq^{n+1})$ from equation (44) to equation (16), we get

$$F(xt, yt) - F(xtq^m, ytq^m) = t(x-y)\sum_{n=0}^{m-1}\frac{(ytq^{n+1};q)_\infty}{(xtq^n;q)_\infty}q^n\,. \tag{45}$$

On simplifying equation (45), we get

$$F(xt, yt) - F(xtq^m, ytq^m) = t\,\frac{(yt;q)_\infty}{(xt;q)_\infty}(x-y)\sum_{n=0}^{m-1}\frac{(xt;q)_n}{(yt;q)_{n+1}}q^n\,. \tag{46}$$

Expanding the inner series of equation (45), using equation (5), we have

$$F(xt, yt) - F(xtq^m, ytq^m) = t(x-y)\sum_{n=0}^{m-1}q^n\sum_{r=0}^{\infty}\frac{\left(\frac{yq}{x};q\right)_r}{(q;q)_r}(xtq^n)^r\,. \tag{47}$$

Rearranging the equation (47) as follows

$$F(xt,yt)-F(xtq^m,ytq^m)=t(x-y)\sum_{r=0}^{\infty}\frac{\left(\frac{yq}{x};q\right)_r}{(q;q)_r}(xt)^r\sum_{n=0}^{m-1}q^{n(r+1)}. \quad (48)$$

The inner right series is a geometric series of *'m'* terms with first term 1 and common ratio q^{r+1} after summing and simplifying, the equation (48) becomes

$$F(xt,yt)-F(xtq^m,ytq^m)=\sum_{r=0}^{\infty}\frac{\left(\frac{y}{x};q\right)_{r+1}}{(q;q)_{r+1}}(xt)^{r+1}\left(1-q^{m(r+1)}\right). \quad (49)$$

Replacing *r* by *(r-1)* in equation (49), we get

$$F(xt,yt)-F(xtq^m,ytq^m)=\sum_{r=1}^{\infty}\frac{\left(\frac{y}{x};q\right)_r}{(q;q)_r}x^r-\sum_{r=1}^{\infty}\frac{\left(\frac{y}{x};q\right)_r}{(q;q)_r}(xq^m)^r. \quad (50)$$

Rearranging series from *r=0* in equation (50), we have

$$F(xt,yt)-F(xtq^m,ytq^m)=\sum_{r=0}^{\infty}\frac{\left(\frac{y}{x};q\right)_r}{(q;q)_r}(xt)^r-\sum_{r=0}^{\infty}\frac{\left(\frac{y}{x};q\right)_r}{(q;q)_r}(xtq^m)^r. \quad (51)$$

Summing right hand side of equation (51) by using equation (5), we get

$$F(xt,yt)-F(xtq^m,ytq^m)=\frac{(yt;q)_\infty}{(xt;q)_\infty}-\frac{(ytq^m;q)_\infty}{(xtq^m;q)_\infty}. \quad (52)$$

Simplifying equation (52), we get

$$F(xt,yt)-F(xtq^m,ytq^m)=\frac{(yt;q)_\infty}{(xt;q)_\infty}-\frac{(xt;q)_m}{(yt;q)_m}\frac{(yt;q)_\infty}{(xt;q)_\infty}. \quad (53)$$

On combining equation (46) and equation (53), we get the required result (38).

5. Special Cases

If we set *t*=1 in equation (38)

$$(x-y)\sum_{n=0}^{m-1}\frac{(x;q)_n}{(y;q)_{n+1}}q^n=1-\frac{(x;q)_m}{(y;q)_m}. \quad (54)$$

If we put *y=0* in equation (54), we get the equation (20).

If we put *x=0* in equation (54), we get the equation (33).

The limiting case as $m\to\infty$ the result (38) becomes

$$t(x-y)\sum_{n=0}^{\infty}\frac{(xt;q)_n}{(yt;q)_{n+1}}q^n=1-\frac{(xt;q)_\infty}{(yt;q)_\infty}. \quad (55)$$

If we put t =1 and y=0 in equation (55), we get a result as $\lim m\to\infty$ of equation (20).

If we put t = 1 and x=0 in equation (55), we get a result as $\lim m\to\infty$ of equation (33).

6. Conclusion

In this paper, we have established two main theorems by making use of Euler's q- derivative and q-shifted operators for a function of one variable and two variables. Some companion identities have also been derived by applying these theorems on some known q-series identities. Finally, we deduced some special cases, which also have the companion identities. The main purpose of the paper is to derive results without making use of a qZeil of q-Zeilberger algorithm, which are listed in the second chapter of the thesis of Riese (1997).

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