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On Indexed Absolute Matrix Summability of an Infinite Series

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Abstract

Some results have been established on absolute index Riesz summability factor of an infinite series. Furthermore, these kind of results can be extended by taking other parameters and an absolute index matrix summability factor of an infinite series or some weaker conditions. In the present paper a new result on generalized absolute index matrix summability factor of an infinite series has been established.

Keywords: Riesz summability; Matrix summability; Index summability method

MSC No.: 40A05, 40D05

1. Introduction

Let $\sum a_n$ be an infinite series and $\{s_n\}$ be its sequence of partial sums. Let $\{p_n\}$ be a sequence of non-negative numbers with $P_n = \sum_{v=0}^{n} p_v \to \infty$, as $n \to \infty$ and $P_{-i} = p_{-i} = 0,\ i \geq 1$. Then, the sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\, s_v \tag{1}$$

defines the $(\bar{N},\ p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.

The series $\sum a_n$ is said to be summable $\left|\bar{N},\ p_n\right|_k,\ k \geq 1,$ if (Bor, 1985)

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty. \tag{2}$$

For a lower triangular matrix $A = (a_{nk})$, we define the matrices $\bar{A} = (\bar{a}_{nk})$ and $\hat{A} = (\hat{a}_{nk})$ as follows:

$$\bar{a}_{nk} = \sum_{v=k}^{n} a_{nv} \tag{3}$$

and $\hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k},\ \hat{a}_{00} = \bar{a}_{00} = a_{00},\ n = 1,2,\ldots$.

Clearly $\bar{A}$ and $\hat{A}$ are lower semi-matrices. For the sequence of partial sums $\{s_n\}$ of the series $\sum a_n$ let

$$A_n(s) = \sum_{k=0}^{n} a_{nk}\ s_k. \tag{4}$$

Then, $A_n(s)$ defines a sequence to sequence transformation $\{s_n\}$ to $\{A_n(s)\}$. From (4) we have

$$A_n(s) = \sum_{k=0}^{n} \bar{a}_{nk}\ a_k\ .$$

Subsequently, we get

$$\Delta A_n(s) = \sum_{n=0}^{n} \hat{a}_{nk}\ a_k,$$

where

$$\Delta A_n(s) = A_n(s) - A_{n-1}(s).$$

Definition 1.1.

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$ if (Sulaiman, 2003)

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{n-1} |\Delta A_n(s)|^k < \infty . \tag{5}$$

Definition 1.2.

If $\{\varphi_n\}$ is a sequence of positive real numbers, then, the series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (Ozarslan and Karakas, 2015)

$$\sum_{n=1}^{\infty} \varphi_n^{n-1} |\Delta A_n(s)|^k < \infty \tag{6}$$

and summable $\varphi - |A, p_n\ ;\delta|_k$, $k \geq 1$, $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\Delta\ A_n(s)|^k < \infty . \tag{7}$$

If we take $\delta = 0$, then $\phi - |A, p_n ;\delta|_k$ summability reduces to $\varphi - |A, p_n|_k$-summability. If we take $\varphi_n = \dfrac{P_n}{p_n}$, then $\phi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$-summability. Also, if we take $a_{nk} = \dfrac{p_k}{P_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $\varphi - |\overline{N}, p_n|_k$ summability. If we take $\varphi_n = \dfrac{P_n}{p_n}$ and $a_{nk} = \dfrac{p_n}{P_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|\overline{N}, p_n|_k$-summability.

Definition 1.3.

A sequence of positive numbers $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (Bari and Steckin, 1956).

2. Known Results

Dealing with summability factors of infinite series, Bor (1996) has proved the following theorem.

Theorem 2.1.

Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = 0(np_n), \text{ as } n \to \infty . \tag{8}$$

Let $\{X_n\}$ be a positive non-decreasing sequence and that there be sequences $\{\beta_n\}$ and $\{\lambda_n\}$ such that

$$|\Delta \lambda_n| \le \beta_n \tag{9}$$

$$\beta_n \to 0, \text{ as } n \to \infty , \tag{10}$$

$$\sum_{n=1}^{\infty} n|\Delta \beta_n| \; X_n < \infty \tag{11}$$

and

$$|\lambda_n| \; X_n = 0(1), \text{ as } n \to \infty . \tag{12}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m), \text{ as } m \to \infty, \tag{13}$$

where $t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v$, then $\sum a_n \lambda_n$ is summable-$\left|\bar{N}, p_n\right|_k, k \ge 1$.

Extending the above theorem to $\left|\bar{N}, p_n, \delta\right|_k, k \ge 1, 0 \le \delta < 1$, and taking a weaker condition almost increasing instead of $\{X_n\}$ to be a positive non-decreasing sequence, Bor (2010) has established the following result:

Theorem 2.2.

Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = 0(np_n), \text{ as } n \to \infty, \tag{14}$$

and

$$\sum_{n=}^{\infty} n^{\delta k} \left| \frac{p_n}{P_n p_{n-1}} \right| = O\left(\frac{v^{\delta k}}{p_v} \right) . \tag{15}$$

If $\{X_n\}$ is an almost increasing sequence satisfying the conditions

$$|\lambda_n| \; X_n = 0(1), \text{ as } n \to \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} n\left|\Delta^2 \lambda_n\right| X_n < \infty, \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} < \infty, \tag{18}$$

$$\sum_{n=1}^{m} \left(\frac{p_n}{P_n}\right)^{\delta k-1} \left|t_n\right|^k = O\left(X_m\right), \text{ as } m \to \infty, \tag{19}$$

where $t_n = \dfrac{1}{n+1}\sum_{v=1}^{n} v a_v$, then $\sum a_n \lambda_n$ is summable-$\left|\bar{N}, p_n, \delta\right|_k, k \geq 1, 0 < \delta \leq 1/k.$

Subsequently, Savas and Rhoades (2007) have extended Theorem-B to $\left|A_k\right|$ summability factor establishing the following theorem.

Theorem 2.3.

Let $A = (a_{nv})$ be a lower triangular matrix with non-negative entries such that

$$\bar{a}_{n0} = 1, \ n = 0,1,2,... \tag{20}$$

$$a_{n-1,v} \geq a_{nv}, \text{ for } n \geq v+1 \tag{21}$$

$$n a_{nn} = O(1) \tag{22}$$

Further, let $\{\lambda_n\}$ be a sequence such that

$$\sum_{n=1}^{m} \left|\Delta\lambda_n\right| = O(1), \tag{23}$$

$$\sum_{n=1}^{m} a_{nn} \left|\lambda_n\right|^k = O(1) \tag{24}$$

and $\{s_n\}$ be a bounded sequence. Then, $\sum a_n \lambda_n$ is summable-$|A|_k, k \geq 1.$

Very recently, Ozarslan and Karakas (communicated) extended Theorem-C, by establishing the following theorem.

Theorem 2.4.

Let $A = (a_{nv})$ be a lower triangular matrix with non-zero diagonal entries such that

$$\bar{a}_{n0} = 1, \ n = 0,1,2,... , \tag{25}$$

$$a_{n-1,v} \geq a_{nv},\ for\ n \geq v+1, \tag{26}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{27}$$

$$\left|\hat{a}_{n,v+1}\right| = O\left(v\left|\Delta_v\left(\hat{a}_{nv}\right)\right|\right). \tag{28}$$

Let $\{X_n\}$ be an almost increasing sequence and sequence $\left\{\frac{\phi_n p_n}{P_n}\right\}$ non-increasing. Let $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences satisfying (8) to (12). If, in addition the condition

$$\sum_{n=1}^{m} \phi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m), \text{ as } m \to \infty \tag{29}$$

is satisfied, where t_n is as defined in Theorem 2.1., then, the series $\sum a_n \lambda_n$ is summable $\phi - |A, p_n|_k$, $k \geq 1$.

3. Main result

In the present paper we generalize the above theorems to $\varphi - |A, p_n; \delta|_k$, $k \geq 1$, $\delta \geq 0$ summability by proving Theorem 3.1.

Theorem 3.1.

Let $A = (a_{nv})$ be a lower triangular matrix with non-zero diagonal entries such that

$$\bar{a}_{n0} = 1,\ n = 0,1,2,\ldots, \tag{30}$$

$$a_{n-1,v} \geq a_{nv}, \text{ for } n \geq v+1, \tag{31}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{32}$$

$$\left|\hat{a}_{n,v+1}\right| = O\left(v\left|\Delta_v\left(\hat{a}_{nv}\right)\right|\right). \tag{33}$$

Let $\{X_n\}$ be an almost increasing sequence and sequences $\{\phi_n^{\delta k+k-1}\}$ and $\left\{\frac{p_n}{P_n}\right\}$ non-increasing. Let, also $\{\lambda_n\}$ and $\{\beta_n\}$ be sequences satisfying (8) to (12). If, in addition, the condition

$$\sum_{n=1}^{m}\phi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k |t_n|^k = O(X_m), \text{ as } m\to\infty \tag{34}$$

is satisfied, the series $\sum a_n\lambda_n$ is summable $\varphi-|A, p_n;\delta|_k$, $k\geq 1, \delta\geq 0$, where t_n is as defined in Theorem 2.1.

For the proof of the theorem, we require the following lemma.

Lemma 3.1. (Mazhar, 1997)

If $\{X_n\}$ is an almost increasing sequence satisfying conditions (10) and (11), then we have

$$n X_n \beta_n = O(1) \tag{35}$$

and

$$\sum_{n=1}^{\infty}\beta_n X_n < \infty.. \tag{36}$$

Proof of the theorem

Let $\{T_n\}$ be the $A=(a_{nv})$ transform of the series $\sum a_n \lambda_n$. Then, using (4), we have

$$\begin{aligned}
\Delta T_n &= \sum_{v=1}^{n}\hat{a}_{nv}\, a_v\, \lambda_v \\
&= \sum_{v=1}^{n}\frac{(\hat{a}_{nv}\lambda_v)}{v}(v a_v) \\
&= \sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v + \frac{a_{nn}\lambda_n}{n}(n+1)t_n,
\end{aligned}$$

(using Abel's partial summation)

$$\begin{aligned}
&= \left(\frac{n+1}{n}\right)a_{nn}t_n + \sum_{v=1}^{n-1}\left(\frac{v+1}{v}\right)\left(\Delta_v\ \hat{a}_{nv}\right)\lambda_v t_v + \sum_{v=1}^{n-1}\left(\frac{v+1}{v}\right)\hat{a}_{n,v+1}(\Delta\lambda_v)t_v \\
&\quad + \sum_{v=1}^{n-1}\frac{1}{v}\hat{a}_{n,v+1}\,\lambda_{v+1}\, t_v \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}
\end{aligned}$$

Using (7) and Minkowski's inequality, in order to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} \phi_n^{\delta k+n-1} \left|T_{n,i}\right|^k < \infty, \text{ for } i = 1,2,3,4.$$

We have

$$\sum_{n=1}^{m} \phi_n^{\delta k+k-1} \left|T_{n,1}\right|^k = O(1) \sum_{n=1}^{m} \phi_n^{\delta k+k-1} a_{nn}^k \ \left|\lambda_n\right|^k \ \left|t_n\right|^k$$

$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \ \left|\lambda_n\right|^{k-1} \left|\lambda_n\right| \left|t_n\right|^k$$

$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \ \left|\lambda_n\right| \left|t_n\right|^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta\left|\lambda_n\right| \sum_{v=1}^{n} \phi_v^{\delta k+k-1} \left(\frac{p_v}{P_v}\right)^k \left|t_v\right|^k + O(1)\left|\lambda_m\right| \sum_{n=1}^{m} \phi_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \left|t_n\right|^k.$$

Using Abel's partial summation

$$= O(1) \sum_{n=1}^{m-1} \left|\Delta\lambda_n\right| \ X_n + O(1) \left|\lambda_m\right| \ X_m, \text{ using (34)}$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \left|\lambda_m\right| X_n, \text{ using (9)}$$

$$= O(1), \text{ as } m \to \infty, \text{ using (36) and (12).}$$

Next,

$$\sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left|T_{n,2}\right|^k = O(1) \sum_{n=2}^{m+1} \phi_n^{\delta n+k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_v\right| \left|t_v\right| \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_v\right|^k \left|t_v\right|^k \right) \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \right)^{k-1}$$

using Holder's inequality with indices k & k' with $\frac{1}{k} + \frac{1}{k'} = 1, k > 1$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \ a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_v\right|^k \left|t_v\right|^k \right)$$

$$= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta n+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v\ (\hat{a}_{nv})\right|\left|\lambda_v\right|^k\left|t_v\right|^k\right), \text{ using (32)}$$

$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|^k\left|t_v\right|^k\sum_{n=v+1}^{m+1}\phi_n^{\delta n+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\left|\Delta_v\ (\hat{a}_{nv})\right|$$

$$= O(1)\sum_{v=1}^{m}\phi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k-1}\left|\lambda_v\right|^k\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left|\Delta_v\left(\hat{a}_{nv}\right)\right|$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k-1}\left|\lambda_v\right|^k\left|t_v\right|^k a_{vv}$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{\delta n+k-1}\left(\frac{p_v}{P_v}\right)^{k}\left|\lambda_v\right|^n\left|t_v\right|^k, \text{ using (32)}$$

$$= O(1), \text{ as above.}$$

Also,

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|T_{n,2}\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta n+k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\beta_v\ \left|t_v\right|\right)^k, \text{ using (9)}$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\beta_v\ \left|t_v\right|^k\right)\times\left(\sum_{v=1}^{k-1}\left|\hat{a}_{n,v+1}\right|\beta_v\right)^{k-1},$$

using Holder's inequality with $\dfrac{1}{k}+\dfrac{1}{k'}=1,\ k>1,$

$$= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}v\left|\Delta_v\left(\hat{a}_{n,v}\right)\right|\beta_v\left|t_v\right|^k\right)\times\left(\sum_{v=1}^{k-1}v\left|\Delta_v\left(\hat{a}_{n,v}\right)\right|\beta_v\right)^{k-1},$$

using (33)

$$= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}v\left|\Delta_v\left(\hat{a}_{nv}\right)\right|\beta_v\left|t_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\phi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}v\left|\Delta_v\left(\hat{a}_{n,v}\right)\right|\beta_v\left|t_v\right|^k$$

$$= O(1) \sum_{v=1}^{m} v\beta_v \left|t_v\right|^k \sum_{n=v+1}^{m+1} \phi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1} \left|\Delta_v\left(\hat{a}_{n,v}\right)\right|$$

$$= O(1) \sum_{v=1}^{m} v\beta_v \left|t_v\right|^k \phi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^{k-1} \sum_{n=v+1}^{m+1} \left|\Delta_\iota\left(\hat{a}_{n,v}\right)\right|$$

$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k-1} v\beta_v \left|t_v\right|^k a_{vv}$$

$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k} v\beta_v \left|t_v\right|^k$$

$$= O(1) \left\{\sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{i=1}^{v} \varphi_i^{\delta k+k-1}\left(\frac{p_i}{P_i}\right)^{k} \left|t_i\right|^k + m\,\beta_m \sum_{i=1}^{m} \varphi_i^{\delta k+k-1}\left(\frac{p_i}{P_i}\right)^{k} \left|t_i\right|^k\right\}$$

$$= O(1) \left\{\sum_{v=1}^{m} \Delta(v\,\beta_v)\, X_v + m\; p_m\; X_m\right\}$$

$$= O(1) \left\{\sum_{v=1}^{m-1} v\left|\Delta\,\beta_v\right| X_v + \sum_{v=1}^{m-1} \beta_v\, X_v + m\,\beta_m\, X_m\right\}$$

$= O(1)$, using (11), (35), and (36).

Finally,

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left|T_{n,4}\right|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\sum_{v}^{n-1} \left|\hat{a}_{n,v+1}\right| \left|\lambda_{v+1}\right| \frac{\left|t_v\right|}{v}\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{n,v}\right)\right| \left|\lambda_{v+1}\right| \left|t_v\right|\right)^k, \text{ using (334)}$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_{v+1}\right|^k \left|t_v\right|^k\right) \times \left(\sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right|\right)^{k-1},$$

using Holder’s inequality

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \underset{a_{nn}}{k-1} \sum_{v=1}^{n-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_{v+1}\right|^{k-1} \left|\lambda_{v+1}\right| \left|t_v\right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \phi_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^{k-1} \sum_{v=1}^{n+1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right| \left|\lambda_{v+1}\right| \left|t_v\right|^k$$

$$= O(1) \sum_{v=1}^{m} \left|\lambda_{v+1}\right| \left|t_v\right|^k \sum_{n=v+1}^{m+1} \phi_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^{k-1} \left|\Delta_v\left(\hat{a}_{nv}\right)\right|$$

$$= O(1)\sum_{v=1}^{m} \phi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k-1} |\lambda_{v+1}|\,|t_v|^k \sum_{n=v+1}^{m+1}\left|\Delta_v\left(\hat{a}_{nv}\right)\right|$$

$$= O(1)\sum_{v=1}^{m} \varphi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k-1} |\lambda_{v+1}|\,|t_v|^k\; a_{nv}$$

$$= O(1)\sum_{v=1}^{m} \varphi_v^{\delta k+k-1}\left(\frac{p_v}{P_v}\right)^{k} |\lambda_{v+1}|\,|t_v|^k$$

$$= O(1), \text{ as above.}$$

This completes the proof of the theorem.

4. Conclusion

If a series is absolute summable, then it is summable by the same summability method. Similarly, if a series is summable by some indexed summability method, then it is absolutely summable by the same method. Different authors established many results on indexed summability of different summability methods of infinite series as well as Fourier series. Some authors established certain results on indexed summability of factored series. The present paper is one of them. We have established a result on generalized matrix indexed summability of a factored series by extending the results of Bor, Savas and Rhoades, as well as Ozarslan and Karakas. For $\delta = 0$, Theorem 2.4. is coming as a particular case of our theorem. By taking $\phi_n = \frac{P_n}{p_n}$ and $\delta = 0$, Theorem 2.3 is coming as a particular case of our theorem. Furthermore, by taking $\phi_n = \frac{P_n}{p_n}$, $a_{nv} = \frac{p_v}{P_n}$, Theorem 2.2 is coming as a particular case of our theorem and taking $\{X_n\}$ as a positive non-decreasing sequence, $\phi_n = \frac{P_n}{p_n}$, $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, Theorem 2.1. follows from our theorem. The theorem can be studied for $\phi - |A, p_n; \delta, \mu|_k$ summability of a factor series. This can be further studied by relaxing the conditions placed on p_n.

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