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Abstract

In this article, a generalization of the new Weibull-Pareto (NWP) distribution is derived. The quadratic equation (QRTM rank transmutation map) studied by Shaw and Buckley (2007), has been used to develop the generalization. The proposed distribution includes as special cases with the new Weibull-Pareto distribution (NWP), transmuted Weibull distribution (TW), transmuted Rayleigh (TR) distribution and transmuted exponential (TE) distribution. Various structural properties of the new distribution, including of moments, quantiles, moment generating function, mean deviations, reliability analysis, order statistics and Renyi entropy are derived. The maximum likelihood estimation method has been proposed for the estimation of the parameters of the TNWP distribution. The usefulness of the derived model is illustrated using two data sets and it is proved that TNWP distribution is a better distribution than other distributions based on some goodness of fit measures. Therefore, we conclude that the new model attracts applications in several areas such as engineering, survival data, economics and others.

Keywords: Weibull-Pareto distribution; Reliability function; Entropy; Moment generating function; Maximum likelihood estimation

MSC 2010 No.: 62E15, 62E99

1. Introduction

Several continuous probability distributions have been expansively utilized in literature for modeling of real data sets in numerous areas, such as engineering, economics, biomedical studies and environmental sciences. However, generalized forms of these probability distributions are obviously needed for applied areas. Therefore, in the last few years, several generalized distributions have been proposed based on different modification methods. These modification methods require the addition of one or more parameters to base model which could provide better adaptability in the modeling of real-life data. Numerous generalized families are also proposed by authors for the derivation of new generalized distribution. For example, McDonald-G by McDonald (1984), Marshall–Olkin generated by Marshall and Olkin (1997), beta-G by Eugene et al. (2002), transmuted-G by Shaw and Buckley (2007), gamma-G (type I) by Ristić and Balakrishnan (2012), Weibull-G by Bourguignon et al. (2014), Logistic-X by Tahir et al. (2016), Odd Log-logistic by Cordeiro et al. (2017), Odd Fréchet-G by Haq and Elgarhy (2018) and many more (see Tahir and Cordeiro 2016) for the derivation of new models.

Pareto distribution is power law distribution that originally developed by (Vilfredo Pareto). This distribution is used in the description of social, scientific, geophysical and actuarial and many other types of observable phenomena. Recently some extensions of Pareto distribution are considered. For example, Akinsete et al. (2008) introduced beta-Pareto distribution with various properties. Shawky and Abu-Zinadah (2009) developed exponentiated Pareto distribution, Exponential Pareto Distribution (Al-Kadim and Boshi 2013), Kumaraswamy Pareto by Bourguignon et al. (2013), Transmuted Pareto distribution by (Merovci and Puka 2014) and Weibull-Pareto by (Tahir et al. 2016). Further, Nasiru and Luguterah (2015) introduced the new Weibull-Pareto (NWP) distribution.

The cumulative distribution function (cdf) of NWP distribution is defined as,

$$G(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}; \qquad \theta, \beta, \delta > 0. \tag{1}$$

and its probability density function (pdf) is

$$g(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, \quad \theta, \beta, \delta > 0. \tag{2}$$

The shape parameter is β, whereas θ and δ are scale parameters.

In this study, we derive subject distribution by utilizing the quadratic rank transmutation map (QRTM) considered by Shaw and Buckley (2007) and we drive several mathematical properties of a new generalized distribution. We refer this new distribution as a transmuted new Weibull-Pareto distribution (TNWP) distribution. Using this approach various generalized distributions were generated. For example, Khan and King (2013) presented transmuted modified Weibull distribution. Afify et al. (2014) studied transmuted complementary Weibull geometric distribution. Merovci (2013) studied transmuted Rayleigh distribution. Haq (2016) derived transmuted exponentiated inverse Rayleigh distribution, Granzotto et al. ((2014transmuted Log-Logistic distribution while Tiana et al. (2014) developed the transmuted linear Exponential distribution. Haq et al. (2016) derived and studied transmuted Power function distribution. Aryal and Tsokos (2011) introduced another generalization of the Weibull distribution called transmuted Weibull distribution and Haq et al. (2017) derived transmuted Weibull Fréchet distribution.

For an arbitrary baseline cdf $G(x)$, Shaw and Buckley (2007) defined the TG family with cdf and pdf given by

$$F(x) = (1+\lambda)\,G(x) - G^2(x),\ |\lambda| \le 1, \tag{3}$$

and

$$f(x) = g(x)[1+\lambda-2\lambda G(x)],\ |\lambda| \le 1, \tag{4}$$

respectively, where $|\lambda| \le 1$, $f(x)$ and $g(x)$ are the corresponding *pdf's*. If we put $\lambda = 0$ in (3) and (4) it reduces to the parent model.

Rest of the article is prearranged as follows. Section 2 shows the demonstration of the transmuted new Weibull-Pareto (TNWP) distribution and discusses its limiting behavior. In Section 3, reliability behavior of the new model is presented. We derive the statistical properties in Section 4. Expressions of order statistics are derived in Section 5. In Section 6, maximum likelihood method is presented for parameter estimation. To access accuracy of derived distribution, real data applications are given in Section 5 and finally, the concluding remarks are given in the last section.

2. Transmuted New Weibull-Pareto Distribution

The TNWP distribution derived from incorporating equations (1) & (2) into equations (3) and (4). The cdf and pdf of the TNWP are:

$$F_{\mathrm{TNWP}}(x) = \left(1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left(1 + \lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right). \tag{5}$$

$$f_{\mathrm{TNWP}}(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right);\ \ \theta,\beta,\delta > 0, |\lambda| \le 1. \tag{6}$$

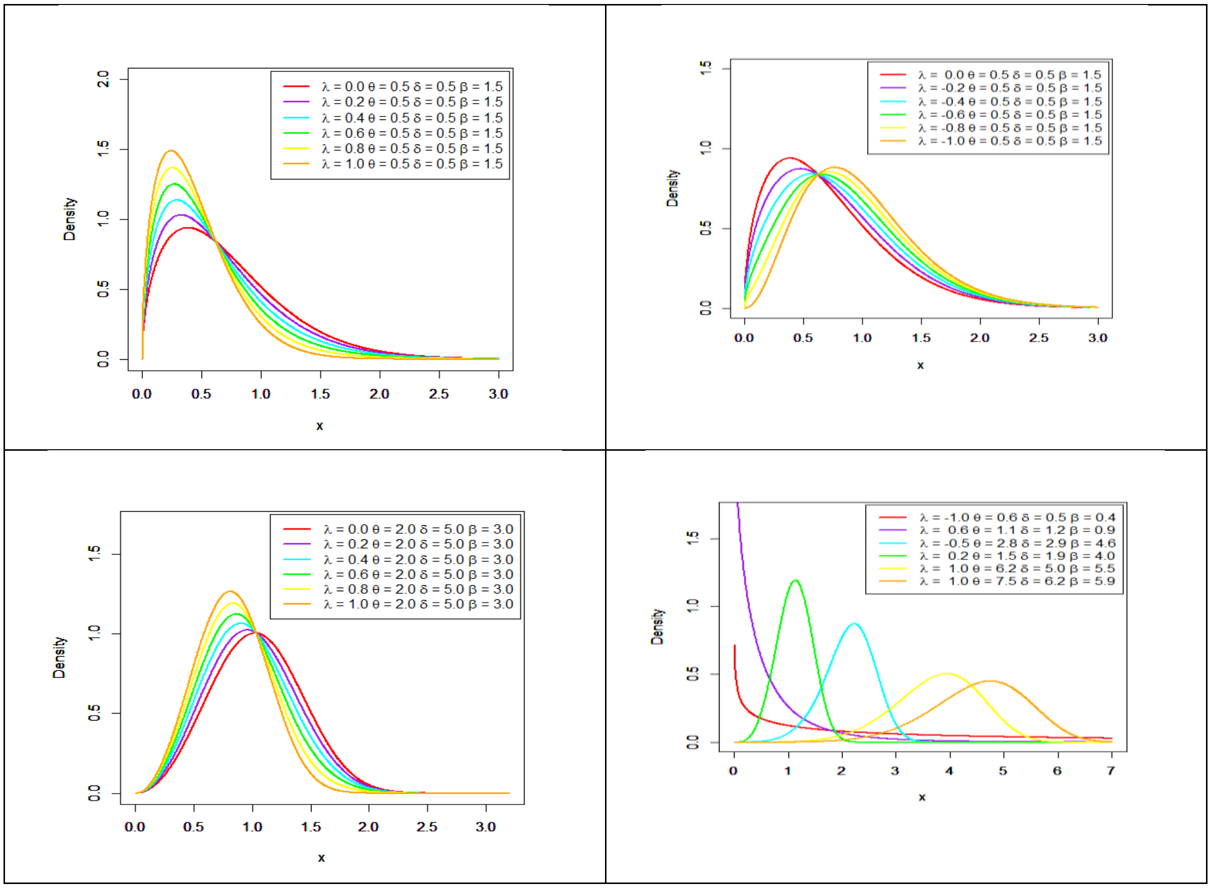

Figure 1: pdf curves for selected values of parameters

The proposed distribution is an extremely adaptable model that approaches to several distinct distributions with the different combination of parameters. The new model has 11 submodels special cases of widely known and unknown probability models.

Corollary 1.

We get the following special cases if X is a random variable specifying with pdf of TNWP

No.	**Reduced Model**	**Parameters**				**Author**
		θ	**β**	**δ**	**λ**	
1	NWP	θ	β	δ	0	Nasiru and Luguterah (2015)
2	TME	θ	1	δ	λ	New
3	TMR	θ	2	δ	λ	New
4	ME	θ	1	δ	0	New
5	MR	θ	2	δ	0	New
6	TW	θ	β	1	λ	Aryal and Tsokos (2011)
7	W	θ	β	1	0	Waloddi Weibull (1951)
8	TE	θ	1	1	λ	Shaw et al.(2007)
9	TR	θ	2	1	λ	Merovci, F. (2013)
10	R	θ	2	1	0	Lord Rayleigh (1880)

Lemma 2.1.

Limiting behavior of pdf of TNWP distribution for $x \to 0$,

$$\lim_{x\to 0} f_{\mathrm{TNWP}}(x) = \begin{cases} \infty, & \beta < 1, \\ \dfrac{\beta\delta}{\theta}(1+\lambda), & \beta = 1\,, \\ 0\ , & \beta < 1, \end{cases}$$

and limiting behavior of pdf of TNWP distribution for $x \to \infty$,

$$\lim_{x\to\infty} f_{\mathrm{TNWP}}(x) = 0,\ \ for\ all\ \beta.$$

3. Reliability Analysis

In this section, reliability analysis such as survival function (*sf*) and hazard function (*hf*) of TNWP are studied.

3.1. Survival Function

Survival function is the likelihood that a system will survive beyond a specified time. Reliability function of the TNWP denoted by $S_{\mathrm{TNWP}}(x)$, can be derived as $S_{\mathrm{TNWP}}(x) = 1 - F_{\mathrm{TNWP}}(x)$,

$$S(x;\ \theta,\beta,\delta,\lambda) = e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right). \tag{7}$$

3.2. Hazard Rate Function

Hazard function can be defined as a conditional density, given that the event has not yet occurred prior to time t. Hazard function of the TNWP is derived by $h_{\mathrm{TNWP}}(x) = f_{\mathrm{TNWP}}(x)/ R_{\mathrm{TNWP}}(x)$,

$$h_{\mathrm{TNWP}}(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}\frac{\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}{\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)}. \tag{8}$$

The pattern of hazard function with different values of parameters θ, β, δ and λ is illustrated in Figure (2)

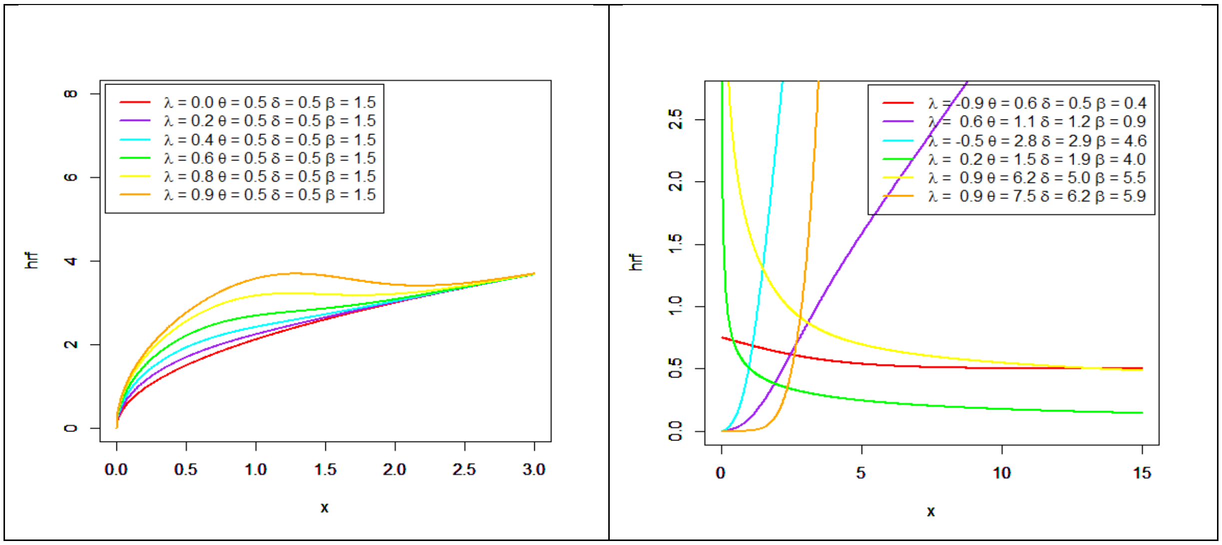

Figure 2: Hazard curves for selected parameter values

From the above figure of hazard function, the following can be perceived:

1) When $\beta > 1$, hazard function is an increasing function of x. So, in this case, TNWP is right for modeling the components wears speedier per time t.

2) When $\beta < 1$, hazard function is a decreasing function of x, means TNWP is right for modeling the components wears slowly per time *t*.

3) When $\beta = 1$ and $\lambda = 0$, hazard rate function is constant. So, TNWP is also be used for modeling the components with constant hazard rate.

4. Some Structural Properties

4.1. Moments

The rth non-central moments of TNWP is defined by:

$$E(X^r) = \theta^r \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-r}{\beta}}\right). \tag{9}$$

Proof:

$$E(X^r) = \int_0^\infty x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right) dx.$$

$$= (1-\lambda)\int_0^\infty x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx + \lambda\int_0^\infty 2x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^\beta} dx. \tag{10}$$

Consider first integral of (10)

$$I_1 = \int_0^\infty x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} dx.$$

Making substitution

$$t = \delta\left(\frac{x}{\theta}\right)^\beta, \qquad dt = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} dx.$$

We have

$$I_1 = \int_0^\infty \left(\theta\left(\frac{t}{\delta}\right)^{\frac{1}{\beta}}\right)^r e^{-t} dt.$$

$$= \theta^r \delta^{\frac{-r}{\beta}} \int_0^\infty (t)^{\frac{r}{\beta}} e^{-t} dt.$$

$$= \theta^r\, \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right). \tag{11}$$

Similarly, for second integral part of (10)

$$I_2 = \int_0^\infty 2x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^\beta} dx,$$

making substitution

$$t = 2\delta\left(\frac{x}{\theta}\right)^\beta, dt = 2\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} dx.$$

We have

$$I_2 = \int_0^\infty \left(\theta\left(\frac{t}{2\delta}\right)^{\frac{1}{\beta}}\right)^r e^{-t} dt.$$

$$= \theta^r (2\delta)^{\frac{-r}{\beta}} \int_0^\infty (t)^{\frac{r}{\beta}} e^{-t} dt.$$

$$= \theta^r (2\delta)^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right). \tag{12}$$

Substituting (11) and (12) in (10), we get

$$E(X^r) = \theta^r \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-r}{\beta}}\right).$$

When $r = 1$,

$$\mathrm{E(X)} = \theta\delta^{\frac{-1}{\beta}}\Gamma\left(\frac{\beta+1}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right).$$

If $r = 2$,

$$\mathrm{E(X^2)} = \theta^2\delta^{\frac{-2}{\beta}}\Gamma\left(\frac{\beta+2}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right).$$

So, the variance is can be presented as

$$\mathrm{Var(X)} = \theta^2\delta^{\frac{-2}{\beta}}\left[\Gamma\left(\frac{\beta+2}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-2}{\beta}}\right) - \left\{\Gamma\left(\frac{\beta+1}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right)\right\}^2\right].$$

4.2. Incomplete Moments

For TNWP, rth incomplete moment is defined by

$$M_r(z) = \theta^r\delta^{\frac{-r}{\beta}}\left[(1-\lambda)\gamma\left(\frac{\beta+r}{\beta},\delta\left(\frac{z}{\theta}\right)^\beta\right)+\lambda 2^{\frac{-r}{\beta}}\left(\frac{\beta+r}{\beta},2\,\delta\left(\frac{z}{\theta}\right)^\beta\right)\right]. \tag{13}$$

Proof:

$$M_r(z) = \int_0^z x^r\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)dx\,.$$
$$= (1-\lambda)\int_0^z x^r\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^\beta}dx + \lambda\int_0^z 2x^r\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}dx.$$

Making substitution

$t = \delta\left(\frac{x}{\theta}\right)^\beta, dt = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}dx$ and $x = \theta\left(\frac{t}{\delta}\right)^{\frac{1}{\beta}}$. If $x = 0$, t = 0 and if $x = z$, t= $\delta\left(\frac{z}{\theta}\right)^\beta$,

and further simplifying, we have

$$M_r(z) = \theta^r\delta^{\frac{-r}{\beta}}\left[(1-\lambda)\gamma\left(\frac{\beta+r}{\beta},\delta\left(\frac{z}{\theta}\right)^\beta\right)+\lambda 2^{\frac{-r}{\beta}}\gamma\left(\frac{\beta+r}{\beta},2\,\delta\left(\frac{z}{\theta}\right)^\beta\right)\right].$$

4.3 Moment Generating Function

For TNWP, moment generating function (*mgf*) is defined by

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \theta^i \delta^{\frac{-i}{\beta}} \Gamma\left(\frac{\beta+i}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-i}{\beta}}\right). \quad (14)$$

Proof:

By definition

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)dx.$$

Using Taylor series

$$M_x(t) = \int_0^{\infty}\left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\cdots+\frac{t^nx^n}{n!}+\cdots\right)f(x)dx = \sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i).$$

$$M_x(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \theta^i \delta^{\frac{-i}{\beta}} \Gamma\left(\frac{\beta+i}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-i}{\beta}}\right).$$

4.4. Quantile Function and Random Number Generation

By inverting the cumulative distribution function, the quantile function for TNWP is derived as

$$Q(\varsigma) = \theta\left\{\frac{1}{\delta}\ln\left(\frac{2\lambda}{-(1-\lambda)+\sqrt{(1-\lambda)^2-4\lambda(q-1)}}\right)\right\}^{\frac{1}{\beta}}. \quad (15)$$

We will obtain the distribution median by setting $\varsigma = 0.5$ in (15)

$$Q(0.5) = \theta\left\{\frac{1}{\delta}ln\left(\frac{2\lambda}{-(1-\lambda)+\sqrt{1+\lambda^2}}\right)\right\}^{\frac{1}{\beta}}.$$

The random number x of the TNWP$(x,\theta,\beta,\delta,\lambda)$ is defined by the following relation

$F_{TNWP}(x) = \zeta$, where $\zeta \sim (0, 1)$, then

$$\left(1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left(1+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right) = \zeta.$$

Solving for x, we have

$$X = \theta\left\{\frac{1}{\delta}\ln\left(\frac{2\lambda}{-(1-\lambda)+\sqrt{(1-\lambda)^2-4\lambda(q-1)}}\right)\right\}^{\frac{1}{\beta}}. \quad (16)$$

4.5. Mean deviation

For the TNWP, the mean deviation about mean can be defind as

$$E(|x-\mu|)= \int_0^\infty |x-\mu| f(x)\ dx = 2\mu F(\mu) - 2\int_0^\mu x f(x)\ dx.$$

So, we have

$$E(|x-\mu|) = 2\theta\delta^{\frac{-1}{\beta}}\left[\Gamma\left(\frac{\beta+1}{\beta}\right)\left(1-e^{-\delta\left(\frac{\mu}{\theta}\right)^\beta}\right)\left(1+\lambda e^{-\delta\left(\frac{\mu}{\theta}\right)^\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) - \left\{(1-\lambda)\gamma\left(\frac{\beta+1}{\beta},\delta\left(\frac{\mu}{\theta}\right)^\beta\right) + \lambda(2)^{\frac{-1}{\beta}}\gamma\left(\frac{\beta+1}{\beta},2\,\delta\left(\frac{\mu}{\theta}\right)^\beta\right)\right\}\right]. \quad (17)$$

The mean deviation about median of the distribution is given by

$$E(|x-m|) = \int_0^\infty |x-m| f(x)\ dx = \mu - 2\int_0^m x f(x)\ dx.$$

$$= \theta\delta^{\frac{-1}{\beta}}\left[\Gamma\left(\frac{\beta+1}{\beta}\right)\left(1-\lambda+\lambda 2^{\frac{-1}{\beta}}\right) - 2\left\{(1-\lambda)\gamma\left(\frac{\beta+1}{\beta},\delta\left(\frac{m}{\theta}\right)^\beta\right) + \lambda(2)^{\frac{-1}{\beta}}\gamma\left(\frac{\beta+1}{\beta},2\,\delta\left(\frac{m}{\theta}\right)^\beta\right)\right\}\right]. \quad (18)$$

4.6. Rényi entropy

For TNWP, the Rényi entropy can be obtained using the expression.

$$I(\alpha) = \frac{1}{1-\alpha}\log\left[\int_0^\infty f^\alpha(x)\,dx\right].$$

where α is the order of the entropy measure

$$f^\alpha(x) = \left[\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^\beta}\right)\right]^\alpha.$$
$$= \left[(1-\lambda)\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^\beta} + 2\lambda\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-2\delta\left(\frac{x}{\theta}\right)^\beta}\right]^\alpha.$$

$$f^\alpha(x) = (1-\lambda)^\alpha\left(\frac{\beta\delta}{\theta}\right)^\alpha\left(\frac{x}{\theta}\right)^{\alpha(\beta-1)} e^{-\alpha\delta\left(\frac{x}{\theta}\right)^\beta} + (2\lambda)^\alpha\left(\frac{\beta\delta}{\theta}\right)^\alpha\left(\frac{x}{\theta}\right)^{\alpha(\beta-1)} e^{-2\alpha\delta\left(\frac{x}{\theta}\right)^\beta}.$$

Now,

$$\int_0^\infty f^\alpha(x) = \left[(1-\lambda)^\alpha\int_0^\infty\left(\frac{x}{\theta}\right)^{\alpha(\beta-1)} e^{-\alpha\delta\left(\frac{x}{\theta}\right)^\beta}dx + (2\lambda)^\alpha\int_0^\infty\left(\frac{x}{\theta}\right)^{\alpha(\beta-1)} e^{-2\alpha\delta\left(\frac{x}{\theta}\right)^\beta}dx\right].$$

Making substitution

$$t=\delta\left(\frac{x}{\theta}\right)^{\beta},\ x=\theta\left(\frac{t}{\alpha\delta}\right)^{\frac{1}{\beta}} \text{ and } dx=\frac{\theta}{\alpha\delta\beta}\left(\frac{t}{\alpha\delta}\right)^{\frac{1}{\beta}-1} \text{dt.}$$

and get the expression

$$=\left(\frac{\theta}{\beta\delta^{\frac{1}{\beta}}}\right)^{(1-\alpha)}\left(\frac{1}{\alpha}\right)^{\alpha\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}}\Gamma\left(\alpha-\frac{\alpha}{\beta}+\frac{1}{\beta}\right)\ \left\{(1-\lambda)^{\alpha}+(\lambda)^{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\beta}(1-\alpha)}\right\}.$$

So the final expression of renyi entropy is,

$$I(\alpha)=\frac{1}{1-\alpha}\log\left[\left(\frac{\theta}{\beta\delta^{\frac{1}{\beta}}}\right)^{(1-\alpha)}\left(\frac{1}{\alpha}\right)^{\alpha\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}}\Gamma\left(\alpha-\frac{\alpha}{\beta}+\frac{1}{\beta}\right)\left\{(1-\lambda)^{\alpha}+(\lambda)^{\alpha}\left(\frac{1}{2}\right)^{\frac{1}{\beta}(1-\alpha)}\right\}\right]. \tag{19}$$

5. Order Statistics

Let a random sample of size n, and $X_1, X_2, \ldots, X_n$ from TNWP$(x,\theta,\beta,\delta,\lambda)$ having cumulative distribution function and the corresponding probability density function, as in (5) and (6), correspondingly. Then random variables $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are refereed the order statistics has probability density function of the rth order statistic, $X_{(r)}$, as

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!\,(n-r)!}f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r},$$

for r=1,2,3,…,n.

rth order statistic pdf for TNWP is

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!(n-r)!}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left[\left(1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left(1+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]^{r-1}\left[(1-\lambda)e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}+\lambda e^{-2\delta\left(\frac{x}{\theta}\right)^{\beta}}\right]^{n-r}. \tag{20}$$

Largest order statistic pdf for TNWP is

$$f_{X_{(n)}}(x)=n\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left[\left(1-e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left(1+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]^{n-1}, \tag{21}$$

and smallest order statistic pdf for TNWP is

$$f_{X_{(1)}}(x) = n\frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\left[\left(1-\lambda+\lambda e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}\right)\right]^{n-1}. \quad (22)$$

6. Maximum Likelihood Estimation

Consider the random sample x_1, x_2,.., x_n of size n from TNWP(x, θ, β, δ, λ) with probability density function in (6). Let X~TNWP(x, θ, β, δ, λ) and let $(\theta, \beta, \delta, \lambda)^T$ is the vector of the distribution parameters. The log-likelihood function of TNWP for $(\theta, \beta, \delta, \lambda)^T$ can be defined as

$$L(x_1, x_2, \dots, x_n, \theta, \beta, \delta, \lambda) = \prod_{i=1}^{n} f_{\text{TNWP}}(x, \theta, \beta, \delta, \lambda).$$

$$L = \frac{\beta^n\delta^n}{\theta^n}\prod_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta-1} e^{-\delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}}\prod_{i=1}^{n}\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right). \quad (23)$$

So, we have log-likelihood function Ł = ln$L$

$$Ł = n(\ln\beta + \ln\delta - \ln\theta) + (\beta-1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right) - \delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + \sum_{i=1}^{n}\ln\left(1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right). \quad (24)$$

Differentiating equation (21) on θ, β, δ & λ then equating to zero, MLEs of θ, β, δ and λ are attained

$$\frac{\partial Ł}{\partial\theta} = \frac{-n\beta}{\theta} + \frac{\beta\delta}{\theta}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + \frac{2\lambda\beta\delta}{\theta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\theta}\right)^{\beta} e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} = 0. \quad (25)$$

$$\frac{\partial Ł}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left[1-\delta\left(\frac{x_i}{\theta}\right)^{\beta}\right]\ln\left(\frac{x_i}{\theta}\right) - 2\lambda\delta\sum_{i=1}^{n}\frac{e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} = 0. \quad (26)$$

$$\frac{\partial Ł}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} - 2\lambda\sum_{i=1}^{n}\frac{e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} = 0. \quad (27)$$

$$\frac{\partial Ł}{\partial\lambda} = \sum_{i=1}^{n}\frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}-1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}} = 0. \quad (28)$$

The maximum likelihood estimators $(\hat{\theta},\hat{\beta},\hat{\delta},\hat{\lambda})^T$ of TNWP can be found by equating (22) - (25) to zero and these equations are not expressed in close form. So, iterative methods must be employed to find the value of parameters that "solves" this equation (22) - (25). The solution of these equation will yield the maximum likelihood estimators $\widehat{\theta}$, $\widehat{\beta}$, $\hat{\delta}$ and $\hat{\lambda}$ of TNWP. All derivatives of second order exist for four parameters of new transmuted Weibull Pareto distribution TNWP(x, θ, β, δ, λ) pdf.

7. Application

We compare our proposed distribution with the new Weibull-Pareto distribution (NWP), the Kumaraswamy Pareto distribution (Kw-P), the transmuted Pareto distribution (TP) and the Pareto Distribution (P). By utilizing the maximum likelihood method, we estimate distribution parameters. The goodness of fit measures including the minus log-likelihood function (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC) are calculated to make a comparison among the fitted distributions. Generally, the lesser values of the above-mentioned measures, the better and the most appropriate fit to the information data set.

Data Set 1:

This data set is based on "exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada". For years 1958–1984. This information set contains 72 exceedances. Pereira et al. (2012) and Merovcia et al. (2014) analyzed this data.

Data Set 2:

The Floyd River located in James USA is used as a third data set. The Floyd River flood rates for years 1935–1973". The descriptive statistics of both data sets is given in Table 1.

Table 1: Descriptive statistics

	Min.	**Q1**	**Median**	**Mean**	**Q3**	**Max.**
Data-1	0.100	2.125	9.500	12.200	20.12	64.00
Data-2	318	1590	3570	6771	6725	71500

The maximum likelihood estimates and the goodness of fit measures are computed by R software which is presented in the table below.

Table 2: Comparison of maximum likelihood estimates for Data-1

Model	ML Estimates	SE	-2LL	AIC	CAIC	BIC
TNWP	β = 0.895296 δ = 0.029834 θ =0.225537 λ =-0.032583	0.110855 0.072399 0.629521 0.383984	502.991	510.991	511.588	520.097
NWP	β = 1.060742 δ = 0.006776 θ =0.112524	0.095007 0.003732 0.075207	506.173	512.173	512.526	519.003
TP	a=0.349941 λ=-0.952412 x_0=min(x)=0.1	0.031058 0.047363 -	572.402	576.402	576.576	580.955
Kw-p	a=2.85531 b=85.84682 k=0.05284 x_0=0.1	0.33710 60.4210 0.01850 -	542.4	548.4	549.0	555.3
P	a=0.243863 x_0=0.1	0.028739	606.128	608.128	608.185	610.405

Table 3: Comparison of maximum likelihood estimates for the Data-2

Model	ML Estimates	SE	-2LL	AIC	CAIC	BIC
TNWP	β = 0.895296 δ = 0.029834 θ =0.225537 λ =-0.032583	0.110855 0.072399 0.629521 0.383984	761.194	768.194	768.527	767.521
NWP	β = 0.802545 δ = 0.003573 θ =5.155509	0.081194 0.001284 4.287028	764.807	770.807	771.141	772.134
Kw-p	a=0.5850 λ=−0.910 x_0=min(x)=318	0.072 0.089	770.698	776.698	777.014	778.025
EP	a=1.504007 b=66.33311 k=0.024819 x_0=318	0.130404 0.004786 0.070744 -	778.577	784.577	785.262	789.567
P	a=0.412 x_0=318	0.066	785.619	789.619	789.953	792.947

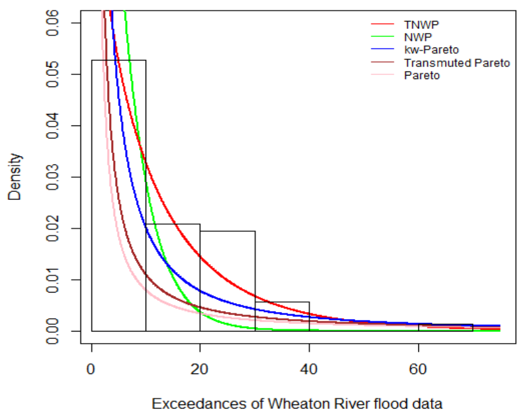

Figure 3: Estimated pdfs (data-1)

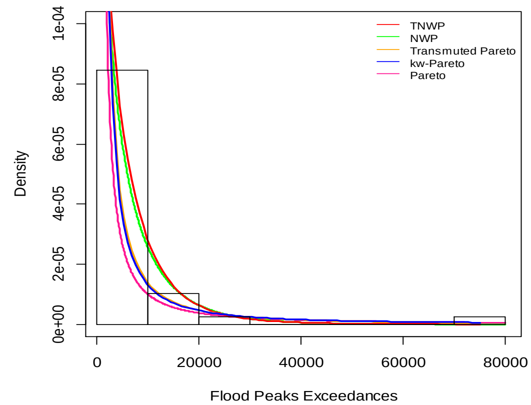

Figure 4: Estimated pdfs (data-2)

The comparison has been made for TNWP model with NWP, Kw-P, TP and Pareto models in Tables (3) and (4). These tables show that our new TNWP having least -2LL, AIC, CAIC and BIC values as compared to all other known distributions. So, we can say that TNWP can be selected as top distribution for given data sets. Moreover, estimated pdf plots of all fitted distributions also show the similar results. Finally, it can be concluded that our new model "transmuted new Weibull-Pareto distribution" gives a better fit and hence can be picked as the best distribution model for all two sets of data.

8. Conclusion

We proposed and studied a new lifetime distribution. The derived model is flexible with variable hazard rate. The Shapes of hazard rate function indicates that the new developed model is a competitive model for other probability models having constant, increasing and decreasing failure rate. Moreover, TNWP has special cases, which style it of divergent statistical significance from other models. We used two real data sets for application purpose. The derived model is more flexible and give better fit than well-known existing models. So, we can conclude that our new distribution can be used an alternative models for moddling in various areas. For example, engineering, failure analysis, survival analysis, weather forecasting, extreme value theory, economics (income inequality) and others.

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