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A. Kashuri
Ismail Qemali University

R. Liko
Ismail Qemali University

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Ostrowski type fractional integral operators for generalized $(r; s, m, \varphi)$ −preinvex functions

[1]A. Kashuri and [2]R. Liko

Faculty of Technical Science
Ismail Qemali University
Albania
[1]artionkashuri@gmail.com; [2]rozanaliko86@gmail.com

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Abstract

In the present paper, the notion of generalized $(r; s, m, \varphi)$ −preinvex function is applied to establish some new generalizations of Ostrowski type inequalities via fractional integral operators. These results not only extend the results appeared in the literature but also provide new estimates on these type.

Keywords: Ostrowski type inequality; Hölder's inequality; Minkowski's inequality; power mean inequality; Riemann-Liouville fractional integral; fractional integral operator; s −convex function in the second sense; m −invex

MSC 2010 No.: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n −dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following result is known in the literature as the Ostrowski inequality (Liu et al., 2015), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.

Let $f: I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \forall\, x \in [a, b]. \quad (1)$$

For other recent results concerning Ostrowski type inequalities, see ((Agarval et al., 2016)-(Alomari et al., 2010); (Dragomir et al., 1997)-(Dragomir, 2001); Kashuri et al., 2016; Kashuri et al., 2017; Liu, 2007; Liu, 2009; (Özdemir et al., 2010)-(Pachpatte, 2001); Rafiq et al., 2007; Sarikaya, 2010; Tunç, 2014; Ujević, 2004; Yildiz et al., 2016; Zhongxue, 2008). Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and $n-$times differentiable mappings etc. appeared in a number of papers. In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola, conformable fractional integral operators etc. by many authors see (Abdeljawad, 2015; Katugampola, 2014; Khalil et al., 2014; Purohit et al., 2014; Set et al., 2017). Riemann-Liouville fractional integral operators are the most central between these fractional operators.

Fractional calculus see ((Chu et al., 2017)-(Dahmani et al., 2010); Kashuri et al., 2017; Raina, 2005), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.

Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \;\; x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \;\; b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Ostrowski type inequalities for functions of different classes see (Liu et al., 2016). In see (Raina, 2005), Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\dots}_{\rho,\lambda}(x) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \qquad (\rho, \lambda > 0;\ |x| < \mathbb{R}), \tag{2}$$

where the coefficients $(\sigma(k), k \in \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers. With the help of (2), Raina see (Raina, 2005) and see (Agarwal et al., 2016) defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(x-t)^{\rho}]\varphi(t)dt, \ \ (x > a > 0), \tag{3}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-x)^{\rho}]\varphi(t)dt, \ \ (0 < x < b), \tag{4}$$

where $\lambda, \rho > 0,\ \omega \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. It is easy to verify that $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(x)$ and $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(x)$ are bounded integral operators on $L_1[a,b]$, if

$$\mathfrak{R} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(b-a)^{\rho}] < +\infty.$$

In fact, for $\varphi \in L_1(a,b)$, we have

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)\right\|_1 \le \mathfrak{R}(b-a)^{\lambda}\|\varphi\|_1$$

and

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x)\right\|_1 \le \mathfrak{R}(b-a)^{\lambda}\|\varphi\|_1,$$

where

$$\|\varphi\|_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda = \alpha, \sigma(0) = 1$ and $\omega = 0$ in (3) and (4).

Now, let us evoke some definitions.

Definition 2. (Hudzik et al., 1994)

A function $f: [0, +\infty[\longrightarrow \mathbb{R}$ is said to be $s-$convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y), \tag{5}$$

for all $x, y \geq 0, \lambda \in [0,1]$ and $s \in]0,1]$.

It is clear that a $1-$convex function must be convex on $[0, +\infty[$ as usual. The $s-$convex functions in the second sense have been investigated in see (Hudzik et al., 1994).

Definition 3. (Antczak, 2005)

A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta: K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true see (Antczak, 2005; Yang et al., 2003).

Definition 4. (Pini, 1991)

A function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The aim of this paper is to establish some generalizations of Ostrowski type inequalities using new identity given in Section 2 for generalized $(r; s, m, \varphi)-$preinvex functions via generalized fractional integral operators. In Section 3, some conclusions and future research are given. These results not only extend the results appeared in the literature see (Yildiz et al., 2016) but also provide new estimates on these type.

2. Main Results

Definition 5. (Du et al., 2016)

A set $K \subseteq \mathbb{R}^n$ is said to be $m-$invex with respect to the mapping $\eta: K \times K \times]0,1] \longrightarrow \mathbb{R}^n$, for some fixed $m \in]0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any t $\in [0,1]$.

Remark 1.

In Definition 5, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the $m-$invex set degenerates an invex set on K.

We next recall the definition of generalized $(r; s, m, \varphi)-$preinvex function.

Definition 6. (Kashuri et al., 2017)

Let $K \subseteq \mathbb{R}$ be an open $m-$invex set with respect to the mapping $\eta: K \times K \times]0,1] \longrightarrow \mathbb{R}$ and $\varphi: I \longrightarrow K$ is a continuous function. The function $f: K \longrightarrow (0, +\infty)$ is said to be generalized $(r; s, m, \varphi)-$preinvex with respect to η, if

$$f\big(m\varphi(x)+t\eta(\varphi(y),\varphi(x),m)\big)\le M_r\big(f(\varphi(x)),f(\varphi(y)),m,s;t\big) \tag{6}$$

holds for some fixed $s,m\in\,]0,1]$ and for all $x,y\in I,\ t\in[0,1]$, where

$$M_r\big(f(\varphi(x)),f(\varphi(y)),m,s;t\big)=\begin{cases}\left[m(1-t)^s f^r(\varphi(x))+t^s f^r(\varphi(y))\right]^{\frac{1}{r}}, & r\neq 0;\\ \left[f\,(\varphi(x))\right]^{m(1-t)^s}\left[f\,(\varphi(y))\right]^{t^s}, & r=0,\end{cases}$$

is the weighted power mean of order r for positive numbers $f\,(\varphi(x))$ and $f\,(\varphi(y))$.

Remark 2.

In Definition 6, it is worthwhile to note that the class of generalized $(r;s,m,\varphi)$ −preinvex function is a generalization of the class of s −convex in the second sense function given in Definition 2. For $r=1$, we get the notion of generalized (s,m,φ) −preinvex function see (Kashuri et al., 2016). Also, for $r=1$ and $\varphi(x)=x,\ \forall x\in I$, we get the notion of generalized (s,m) −preinvex function see (Du et al., 2016).

Throughout this paper we denote

$$I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)=\left[\frac{(x-m\varphi(a))^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(x-m\varphi(a))^{\rho}\right]}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}\right]f(x)$$

$$+\left[\frac{(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\rho}]}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}\right]f(x)$$

$$-\frac{\lambda}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}$$

$$\times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;\omega}f\right)(m\varphi(a))+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;\omega}f\right)\big(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)\big)\right].$$

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for generalized $(r;s,m,\varphi)$ −preinvex functions via generalized fractional integral operators, we need the following new interesting lemma:

Lemma 1.

Let $\varphi: I\longrightarrow K$ be a continuous function. Suppose $K\subseteq\mathbb{R}$ be an open m −invex subset with respect to the mapping $\eta: K\times K\times\,]0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in\,]0,1]$ and $\eta\,(\varphi(b),\varphi(a),m)>0$. Assume that $f: K\longrightarrow\mathbb{R}$ is a differentiable function on K°. If $f'\in L_1[m\varphi(a),m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)]$, then we have the following identity involving generalized fractional integral operators:

$$I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)=\int_0^1\theta(t)\,f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)dt, \tag{7}$$

for each $t \in [0,1]$, where $\lambda, \rho > 0,\ \omega \in \mathbb{R}$ and

$$\theta(t) = \begin{cases} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}], & t \in \left[0, \frac{x - m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}\right[; \\ (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}], & t \in \left[\frac{x - m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}, 1\right]. \end{cases}$$

Proof:

Integrating by parts, we get

$$\int_0^1 \theta(t)\, f'\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)dt$$

$$= \int_0^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\, f'\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)dt$$

$$+ \int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\, f'\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)dt$$

$$= t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\frac{f\,\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)}{\eta\,(\varphi(b),\varphi(a),m)}\Bigg|_0^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}$$

$$-\lambda \int_0^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\frac{f\,\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)}{\eta\,(\varphi(b),\varphi(a),m)}dt$$

$$+(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\frac{f\,\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)}{\eta\,(\varphi(b),\varphi(a),m)}\Bigg|_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^1$$

$$-\lambda \int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^1 (1 - t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\frac{f\,\big(m\varphi(a) + t\eta\,(\varphi(b),\varphi(a),m)\big)}{\eta\,(\varphi(b),\varphi(a),m)}dt$$

$$= \left[\frac{\big(x - m\varphi(a)\big)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega\big(x - m\varphi(a)\big)^{\rho}\big]}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}\right] f(x)$$

$$+\left[\frac{(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\rho}]}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}\right]f(x)$$

$$-\frac{\lambda}{\eta^{\lambda+1}(\varphi(b),\varphi(a),m)}$$

$$\times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;\omega}f\right)\left(m\varphi(a)\right)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;\omega}f\right)\left(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)\right)\right].$$

By using Lemma 1, one can extend to the following results.

Theorem 2.

Let $\varphi: I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m −invex subset with respect to the mapping $\eta: A\times A\times\,]0,1] \longrightarrow \mathbb{R}$ for some fixed $s,m \in\,]0,1]$ and $\eta\,(\varphi(b),\varphi(a),m) > 0$. Assume that $f: A \longrightarrow (0,+\infty)$ is a differentiable function on A°. If $0 < r \leq 1$ and f' is generalized $(r;s,m,\varphi)$ −preinvex function on $[m\varphi(a), m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)]$, then the following inequality for generalized fractional integral operators holds:

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right|$$

$$\leq\left\{\begin{array}{c} mf'\left(\varphi(a)\right)^{r}\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\right)^{r} \\ +f'\left(\varphi(b)\right)^{r}\left(\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|\omega|\left(x-m\varphi(a)\right)^{\rho}]\right)^{r}\end{array}\right\}^{\frac{1}{r}}$$

$$+\left\{\begin{array}{c} mf'\left(\varphi(a)\right)^{r}\left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\rho}]\right)^{r} \\ +f'\left(\varphi(b)\right)^{r}\left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\right)^{r}\end{array}\right\}^{\frac{1}{r}}, \quad (8)$$

where $\lambda,\ \rho > 0,\ \omega \in \mathbb{R},\ k = 0,1,2,\ldots,\ \beta(x;a,b)$ is incompleted beta function and

$$\sigma_1(k) = \sigma(k)\beta\left(\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)};\ \lambda+\rho k+1,\frac{s}{r}+1\right);$$

$$\sigma_2(k) = \sigma(k)\left(\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}\right)^{\lambda+\frac{s}{r}+1}\frac{1}{\lambda+\rho k+\frac{s}{r}+1};$$

$$\sigma_3(k) = \sigma(k)\left(\frac{m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x}{\eta\,(\varphi(b),\varphi(a),m)}\right)^{\lambda+\frac{s}{r}+1}\frac{1}{\lambda+\rho k+\frac{s}{r}+1};$$

$$\sigma_4(k) = \sigma(k)\beta\left(\frac{m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x}{\eta\,(\varphi(b),\varphi(a),m)};\ \lambda+\rho k+1,\frac{s}{r}+1\right).$$

Proof:

Let $0 < r \leq 1$. From Lemma 1, generalized $(r; s, m, \varphi)$ −preinvexity of f', Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right|$$

$$\leq \int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} t^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\left|f'\big(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)\big)\right|dt$$

$$+\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^{1} |1-t|^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\left|f'\big(m\varphi(a) + t\eta(\varphi(b),\varphi(a),m)\big)\right|dt$$

$$\leq \int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} t^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{r}+t^{s}f'\big(\varphi(b)\big)^{r}\right]^{\frac{1}{r}}dt$$

$$+\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^{1} (1-t)^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]$$

$$\times\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{r}+t^{s}f'\big(\varphi(b)\big)^{r}\right]^{\frac{1}{r}}dt$$

$$\leq \left\{ mf'\big(\varphi(a)\big)^{r}\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} t^{\lambda}(1-t)^{\frac{s}{r}}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]dt\right)^{r} + f'\big(\varphi(b)\big)^{r}\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} t^{\lambda+\frac{s}{r}}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]dt\right)^{r}\right\}^{\frac{1}{r}}$$

$$+\left\{\begin{aligned}&mf'(\varphi(a))^r\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}(1-t)^{\lambda+\frac{s}{r}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]dt\right)^r\\&+f'(\varphi(b))^r\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}t^{\frac{s}{r}}(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]dt\right)^r\end{aligned}\right\}^{\frac{1}{r}}$$

$$=\left\{\begin{aligned}&mf'(\varphi(a))^r\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\right)^r\\&+f'(\varphi(b))^r\left(\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|\omega|(x-m\varphi(a))^{\rho}]\right)^r\end{aligned}\right\}^{\frac{1}{r}}$$

$$+\left\{\begin{aligned}&mf'(\varphi(a))^r\left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\rho}]\right)^r\\&+f'(\varphi(b))^r\left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\right)^r\end{aligned}\right\}^{\frac{1}{r}}.$$

This completes the proof of the theorem. ■

Corollary 1.

Under the same conditions as in Theorem 2, if we choose $m=s=1$, $\eta\,(\varphi(b),\varphi(a),m)=\varphi(b)-m\varphi(a)$ and $\varphi(x)=x$, we get

$$\left|\left[\frac{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(x-a)^{\rho}]+(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(b-x)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x)-\frac{\lambda}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;\omega}f\right)(a)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;\omega}f\right)(b)\right]\right|$$

$$\le\left\{f'(a)^r\left(\mathcal{F}^{\sigma_1^*}_{\rho,\lambda+1}[|\omega|(b-a)^{\rho}]\right)^r+f'(b)^r\left(\mathcal{F}^{\sigma_2^*}_{\rho,\lambda+1}[|\omega|(x-a)^{\rho}]\right)^r\right\}^{\frac{1}{r}}$$

$$+\left\{f'(a)^r\left(\mathcal{F}^{\sigma_3^*}_{\rho,\lambda+1}[|\omega|(b-x)^{\rho}]\right)^r+f'(b)^r\left(\mathcal{F}^{\sigma_4^*}_{\rho,\lambda+1}[|\omega|(b-a)^{\rho}]\right)^r\right\}^{\frac{1}{r}}, \qquad (9)$$

where

$$\sigma_1^*(k)=\sigma(k)\beta\left(\frac{x-a}{b-a};\ \lambda+\rho k+1,\frac{1}{r}+1\right);$$

$$\sigma_2^*(k)=\sigma(k)\left(\frac{x-a}{b-a}\right)^{\lambda+\frac{1}{r}+1}\frac{1}{\lambda+\rho k+\frac{1}{r}+1};$$

$$\sigma_3^*(k) = \sigma(k)\left(\frac{b-x}{b-a}\right)^{\lambda+\frac{1}{r}+1}\frac{1}{\lambda+\rho k+\frac{1}{r}+1};$$

$$\sigma_4^*(k) = \sigma(k)\beta\left(\frac{b-x}{b-a};\ \lambda+\rho k+1, \frac{1}{r}+1\right).$$

Corollary 2.

If we choose $r = \sigma(0) = 1, \omega = 0$ in Corollary 1, the inequality (9) reduces to inequality (2.1) of see (Yildiz et al., 2016; Theorem 2.1).

Theorem 3.

Let $\varphi: I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m −invex subset with respect to the mapping $\eta: A \times A \times]0,1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in]0,1]$ and $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f: A \longrightarrow (0, +\infty)$ is a differentiable function on $A°$. If $0 < r \leq 1$ and f'^q is generalized $(r; s, m, \varphi)$ −preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality for generalized fractional integral operators holds:

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right| \leq \left(\frac{r}{s+r}\right)^{\frac{1}{q}}\frac{1}{\eta^{\lambda+\frac{s+1}{rq}+\frac{1}{p}}(\varphi(b),\varphi(a),m)}$$

$$\times\left\{\begin{array}{c}\left[\begin{array}{c}m\left[\eta^{\frac{s}{r}+1}(\varphi(b),\varphi(a),m)-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\frac{s}{r}+1}\right]^{r} f'\left(\varphi(a)\right)^{rq}\\ +\left(x-m\varphi(a)\right)^{s+1}f'\left(\varphi(b)\right)^{rq}\end{array}\right]^{\frac{1}{rq}}\\ \times\left(x-m\varphi(a)\right)^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^*}_{\rho,\lambda+1}\left[|\omega|\left(x-m\varphi(a)\right)^{\rho}\right]\\ +\left[\begin{array}{c}m(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{s+1}f'\left(\varphi(a)\right)^{rq}\\ +\left[\eta^{\frac{s}{r}+1}(\varphi(b),\varphi(a),m)-\left(x-m\varphi(a)\right)^{\frac{s}{r}+1}\right]^{r} f'\left(\varphi(b)\right)^{rq}\end{array}\right]^{\frac{1}{rq}}\\ \times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^*}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\rho}]\end{array}\right\}, \quad (10)$$

where $\lambda,\ \rho > 0,\ \omega \in \mathbb{R},\ k = 0,1,2, \ldots$, and

$$\sigma^*(k) = \sigma(k)\left(\frac{1}{p(\lambda+\rho k)+1}\right)^{\frac{1}{p}}.$$

Proof:

Suppose that $q > 1$ and $0 < r \leq 1$. From Lemma 1, generalized $(r; s, m, \varphi)$ −preinvexity of f'^q, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right|$$

$$\leq \int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\left|f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right|dt$$

$$+\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1} |1-t|^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\left|f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right|$$

$$\leq \sum_{k=0}^{+\infty}\frac{\sigma(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)}$$

$$\times\left\{\begin{array}{l}\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}\Big(f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\Big)^{q}dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}(1-t)^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}\Big(f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\Big)^{q}dt\right)^{\frac{1}{q}}\end{array}\right\}$$

$$\leq \sum_{k=0}^{+\infty}\frac{\sigma(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)}$$

$$\times\left\{\begin{array}{l}\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{rq}+t^{s}f'\big(\varphi(b)\big)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}(1-t)^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{rq}+t^{s}f'\big(\varphi(b)\big)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}\end{array}\right\}$$

$$\leq \sum_{k=0}^{+\infty}\frac{\sigma(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)}$$

$$\times\left\{\begin{array}{c}\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}} t^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\\ \times\left[mf'(\varphi(a))^{rq}\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}(1-t)^{\frac{s}{r}}dt\right)^r+f'(\varphi(b))^{rq}\left(\int_0^{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}t^{\frac{s}{r}}dt\right)^r\right]^{\frac{1}{rq}}\\ +\left(\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1(1-t)^{p(\lambda+\rho k)}dt\right)^{\frac{1}{p}}\\ \times\left[mf'(\varphi(a))^{rq}\left(\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1(1-t)^{\frac{s}{r}}dt\right)^r+f'(\varphi(b))^{rq}\left(\int_{\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}}^1t^{\frac{s}{r}}dt\right)^r\right]^{\frac{1}{rq}}\end{array}\right\}$$

$$=\left(\frac{r}{s+r}\right)^{\frac{1}{q}}\frac{1}{\eta^{\lambda+\frac{s+1}{rq}+\frac{1}{p}}(\varphi(b),\varphi(a),m)}$$

$$\times\left\{\begin{array}{c}\left[\begin{array}{c}m\left[\eta^{\frac{s}{r}+1}(\varphi(b),\varphi(a),m)-(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\frac{s}{r}+1}\right]^r f'(\varphi(a))^{rq}\\ +(x-m\varphi(a))^{s+1}f'(\varphi(b))^{rq}\end{array}\right]^{\frac{1}{rq}}\\ \times(x-m\varphi(a))^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^*}_{\rho,\lambda+1}\left[|\omega|(x-m\varphi(a))^{\rho}\right]\\ +\left[\begin{array}{c}m(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{s+1}f'(\varphi(a))^{rq}\\ +\left[\eta^{\frac{s}{r}+1}(\varphi(b),\varphi(a),m)-(x-m\varphi(a))^{\frac{s}{r}+1}\right]^r f'(\varphi(b))^{rq}\end{array}\right]^{\frac{1}{rq}}\\ \times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^*}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\rho}]\end{array}\right\}.$$

This completes the proof of the theorem. ■

Corollary 3.

Under the same conditions as in Theorem 3, if we choose $m=s=1$, $\eta(\varphi(b),\varphi(a),m)=\varphi(b)-m\varphi(a)$ and $\varphi(x)=x$, we get

$$\left|\left[\frac{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(x-a)^{\rho}]+(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(b-x)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x) - \frac{\lambda}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;\omega}f\right)(a)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;\omega}f\right)(b)\right]\right|$$

$$\leq \left(\frac{r}{r+1}\right)^{\frac{1}{q}}\frac{1}{(b-a)^{\lambda+\frac{2}{rq}+\frac{1}{p}}}$$

$$\times\left\{\begin{array}{c}\left[\left[(b-a)^{\frac{1}{r}+1}-(b-x)^{\frac{1}{r}+1}\right]^{r}f'(a)^{rq}++(x-a)^{2}f'(b)^{rq}\right]^{\frac{1}{rq}} \\ \times (x-a)^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^{*}}_{\rho,\lambda+1}[|\omega|(x-a)^{\rho}] \\ +\left[(b-x)^{2}f'(a)^{rq}+\left[(b-a)^{\frac{1}{r}+1}-(x-a)^{\frac{1}{r}+1}\right]^{r}f'(b)^{rq}\right]^{\frac{1}{rq}} \\ \times (b-x)^{\lambda+\frac{1}{p}}\mathcal{F}^{\sigma^{*}}_{\rho,\lambda+1}[|\omega|(b-x)^{\rho}]\end{array}\right\}. \qquad (11)$$

Theorem 4.

Let $\varphi: I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m −invex subset with respect to the mapping $\eta: A \times A \times]0,1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in]0,1]$ and $\eta\,(\varphi(b), \varphi(a), m) > 0$. Assume that $f: A \longrightarrow (0, +\infty)$ is a differentiable function on A°. If $0 < r \leq 1$ and f'^{q} is generalized $(r; s, m, \varphi)$ −preinvex function on $[m\varphi(a), m\varphi(a) + \eta\,(\varphi(b), \varphi(a), m)]$, $q \geq 1$, then the following inequality for generalized fractional integral operators holds:

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right| \leq \left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[|\omega|\left(x-m\varphi(a)\right)^{\rho}\right]\right)^{1-\frac{1}{q}}$$

$$\times\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\left\{\begin{array}{c} mf'\left(\varphi(a)\right)^{rq}\beta^{r}\left(\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)};\,\lambda q+\rho k+1,\frac{s}{r}+1\right) \\ +f'\left(\varphi(b)\right)^{rq}\left[\left(\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}\right)^{\lambda q+\rho k+\frac{s}{r}+1}\frac{1}{\lambda q+\rho k+\frac{s}{r}+1}\right]^{r}\end{array}\right\}^{\frac{1}{r}}\right]^{\frac{1}{q}}$$

$$+\left(\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left[\begin{array}{c}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)]\\ \left\{\begin{array}{c} mf'(\varphi(a))^{rq}\left[\left(\frac{m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x}{\eta\,(\varphi(b),\varphi(a),m)}\right)^{\lambda q+\rho k+\frac{s}{r}+1}\frac{1}{\lambda q+\rho k+\frac{s}{r}+1}\right]^{r} \\ +f'(\varphi(b))^{rq}\beta^{r}\left(\frac{m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x}{\eta\,(\varphi(b),\varphi(a),m)};\ \lambda q+\rho k+1,\frac{s}{r}+1\right)\end{array}\right\}^{\frac{1}{r}}\end{array}\right]^{\frac{1}{q}}, \quad (12)$$

where $\lambda,\rho>0,\ \omega\in\mathbb{R},\ k=0,1,2,\ldots,$ and

$$\sigma_1(k)=\sigma(k)\left(\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}\right)\frac{1}{\rho k+1};$$

$$\sigma_2(k)=\sigma(k)\left(\frac{m\varphi(a)+\eta\,(\varphi(b),\varphi(a),m)-x}{\eta\,(\varphi(b),\varphi(a),m)}\right)\frac{1}{\rho k+1}.$$

Proof:

Suppose that $q\geq 1$ and $0<r\leq 1$. From Lemma 1, generalized $(r;s,m,\varphi)$ −preinvexity of f'^{q}, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\lambda,\rho,\omega,m,a,b)\right|$$

$$\leq\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\left|f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right|dt$$

$$+\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}|1-t|^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\left|f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right|$$

$$\leq\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda q}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)t^{\rho}]\left(f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right)^{q}dt\right)^{\frac{1}{q}}$$

$$\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}]\,dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1} (1-t)^{\lambda q}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)(1-t)^{\rho}] \times\left(f'\big(m\varphi(a)+t\eta\,(\varphi(b),\varphi(a),m)\big)\right)^{q} dt\right)^{\frac{1}{q}}$$

$$\leq\left(\sum_{k=0}^{+\infty}\frac{\sigma\,(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)}\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\rho k}dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\sum_{k=0}^{+\infty}\frac{\sigma\,(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)} \times\int_{0}^{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}} t^{\lambda q+\rho k}\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{rq}+t^{s}f'\big(\varphi(b)\big)^{rq}\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}$$

$$+\left(\sum_{k=0}^{+\infty}\frac{\sigma\,(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)}\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}(1-t)^{\rho k}dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\sum_{k=0}^{+\infty}\frac{\sigma\,(k)|\omega|^{k}\eta^{\rho k}(\varphi(b),\varphi(a),m)}{\Gamma(\lambda+\rho k+1)} \times\int_{\frac{x-m\varphi(a)}{\eta\,(\varphi(b),\varphi(a),m)}}^{1}(1-t)^{\lambda q+\rho k}\left[m(1-t)^{s}f'\big(\varphi(a)\big)^{rq}+t^{s}f'\big(\varphi(b)\big)^{rq}\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}$$

$$\leq\left(\mathcal{F}_{\rho,\lambda+1}^{\sigma_1}\left[|\omega|\big(x-m\varphi(a)\big)^{\rho}\right]\right)^{1-\frac{1}{q}}$$

$$\times\left[\begin{array}{c}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)] \\ \left\{\begin{array}{c} mf'(\varphi(a))^{rq}\beta^{r}\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)};\ \lambda q+\rho k+1,\frac{s}{r}+1\right) \\ +f'(\varphi(b))^{rq}\left[\left(\frac{x-m\varphi(a)}{\eta(\varphi(b),\varphi(a),m)}\right)^{\lambda q+\rho k+\frac{s}{r}+1}\frac{1}{\lambda q+\rho k+\frac{s}{r}+1}\right]^{r}\end{array}\right\}^{\frac{1}{r}}\end{array}\right]^{\frac{1}{q}}$$

$$+\left(\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|\omega|(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left[\begin{array}{c}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[|\omega|\eta^{\rho}(\varphi(b),\varphi(a),m)] \\ \left\{\begin{array}{c} mf'(\varphi(a))^{rq}\left[\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x}{\eta(\varphi(b),\varphi(a),m)}\right)^{\lambda q+\rho k+\frac{s}{r}+1}\frac{1}{\lambda q+\rho k+\frac{s}{r}+1}\right]^{r} \\ +f'(\varphi(b))^{rq}\beta^{r}\left(\frac{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x}{\eta(\varphi(b),\varphi(a),m)};\ \lambda q+\rho k+1,\frac{s}{r}+1\right)\end{array}\right\}^{\frac{1}{r}}\end{array}\right]^{\frac{1}{q}}.$$

This completes the proof of the theorem. ■

Corollary 4.

Under the same conditions as in Theorem 4, if we choose $r=m=s=1$, $\eta(\varphi(b),\varphi(a),m)=\varphi(b)-m\varphi(a)$ and $\varphi(x)=x$, we get

$$\left|\begin{array}{c}\left[\frac{(x-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(x-a)^{\rho}]+(b-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(b-x)^{\rho}]}{(b-a)^{\lambda+1}}\right]f(x) \\ -\frac{\lambda}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x-;\omega}f\right)(a)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,x+;\omega}f\right)(b)\right]\end{array}\right|$$

$$\leq\left(\mathcal{F}^{\sigma_1^*}_{\rho,\lambda+1}[|\omega|(x-a)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left[f'(a)^{q}\mathcal{F}^{\sigma_2^*}_{\rho,\lambda+1}[|\omega|(b-a)^{\rho}]+f'(b)^{q}\mathcal{F}^{\sigma_3^*}_{\rho,\lambda+1}[|\omega|(x-a)^{\rho}]\right]^{\frac{1}{q}}$$

$$+\left(\mathcal{F}^{\sigma_4^*}_{\rho,\lambda+1}[|\omega|(b-x)^{\rho}]\right)^{1-\frac{1}{q}}$$

$$\times\left[f'(a)^{q}\mathcal{F}^{\sigma_5^*}_{\rho,\lambda+1}[|\omega|(b-x)^{\rho}]+f'(b)^{q}\mathcal{F}^{\sigma_6^*}_{\rho,\lambda+1}[|\omega|(b-a)^{\rho}]\right]^{\frac{1}{q}}, \quad (13)$$

where

$$\sigma_1^*(k) = \sigma(k)\left(\frac{x-a}{b-a}\right)\frac{1}{\rho k+1};$$

$$\sigma_2^*(k) = \sigma(k)\beta\left(\frac{x-a}{b-a};\ \lambda q+\rho k+1, 2\right);$$

$$\sigma_3^*(k) = \sigma(k)\left(\frac{x-a}{b-a}\right)^{\lambda q+2}\frac{1}{\lambda q+\rho k+2};$$

$$\sigma_4^*(k) = \sigma(k)\left(\frac{b-x}{b-a}\right)\frac{1}{\rho k+1};$$

$$\sigma_5^*(k) = \sigma(k)\left(\frac{b-x}{b-a}\right)^{\lambda q+2}\frac{1}{\lambda q+\rho k+2};$$

$$\sigma_6^*(k) = \sigma(k)\beta\left(\frac{b-x}{b-a};\ \lambda q+\rho k+1, 2\right).$$

Corollary 5.

If we choose $\sigma(0) = 1, \omega = 0$ in Corollary 4, the inequality (13) reduces to inequality (2.4) of see (Yildiz et al., 2016; Theorem 2.3).

3. Conclusion

In the present paper, the notion of generalized $(r; s, m, \varphi)$ −preinvex function was applied to establish some new generalizations of Ostrowski type inequalities via fractional integral operators. These results not only extended the results appeared in the literature but also provided new estimates on these type. Motivated by this new interesting class of generalized $(r; s, m, \varphi)$ −preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski, Hermite−Hadamard and Simpson type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k −fractional integrals, local fractional integrals, fractional integral operators, q −calculus, (p, q) −calculus, time scale calculus and conformable fractional integrals.

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Biographical Note of Authors:

The authors are Assistant Professors of Department of Mathematics, Faculty of Technical Science, University “Ismail Qemali”, Vlora, Albania. Their research interests include Mathematical Inequalities, Fractional Calculus, Approximation Theory and Applied Mathematics. They have published more than 15 research articles in reputed international journals.