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Analytical solutions for the Black-Scholes equation

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Abstract

In this paper, the Black-Sholes equation (BS) has been applied successfully with the Cauchy-Euler method and the method of separation of variables and new analytical solutions have been found. The linear partial differential equation (PDE) transformed to linear ordinary differential equation (ODE) as well. We acquired three types of solutions including hyperbolic, trigonometric and rational solutions. Descriptions of these methods are given and the obtained results reveal that three methods are tools for exploring partial differential models.

Keywords: Black-Sholes equation; Partial differential equation; Ordinary differential equation; Cauchy-Euler method; Separation of variables

MSC 2010 No.: 65D19, 65H10, 35A20, 35A24, 35C08, 35G50

1. Introduction

In this paper, an application of the proposed method to the Black-Scholes partial differential equation is illustrated. Recently, the Black-Scholes equation considered by Bohner and Zheng (2009):

$$u_t + ax^2u_{xx} + bxu_x - ru = 0, \quad x > 0, \quad t \in [0, T], \tag{1}$$

where $a = \sigma^2/2$ and $b = r - \delta$, r is the risk-free rate, σ is the volatility, δ is the dividend yield, and $u(x,t)$ is the value of the option for a market price x at time t before the expiry time T, together with the terminal condition,

$$u(x,t) = g(x), \tag{2}$$

where $g(x)$ is assumed to have derivatives of all orders. It is clear that solutions to the nonlinear partial differential equation (1) would be of great interest to the financial world. Equation (1) may satisfy also other kinds of options, like the barrier option. A barrier option can be considered an exotic option and as such has features that makes it more complex than the ”vanilla” option, O'Hara (2011), Kwok (2008) and O'Hara et al. (2013). The complete group classification of a generalization of the Black-Scholes-Merton model is carried out by making use of the underlying equivalence and additional equivalence transformations by Bozhkov and Dimas (2014). A theoretical analysis for the Black-Scholes equation together applying the decomposition method has been presented by Lesnic (2006). Moreover, Chanane (2011) obtained the solutions of a class of partial differential equations and its application to the Black-Scholes equation. Edelstein and Govinder (2009) focused on classical point symmetries and also potential symmetries and obtained new abundant exact solutions to the Black-Scholes equation. Company et al. used the numerical solution of Black-Scholes option pricing partial differential equations by means of semi-discretization technique Company et al. (2008). Likewise, in Bohner and Zheng (2009) a theoretical analysis for the Black-Scholes equation has been presented and the analytical solution of the Black-Scholes equation is obtained by using the Adomian approximate decomposition technique.

Partial differential equations (PDEs) find special applicability within many scientific and mathematical disciplines. These play an important role in the fields of applied mathematics and engineering such as mechanics, physics, chemistry, potential theory, dynamics, ecology etc. So instead of using current models of partial differential equations, we can transform PDEs to ordinary differential equations. Hence there occurs a need to use solitary wave variable that would appropriately transform PDEs to ODEs and solve them. Sometimes, when these equations are generally difficult to solve analytically; thereby, a numerical method is needed. However, several analytical methods exists for finding exact solutions of PDEs. Many research papers dealing with analytical methods exist in open literature and some of them are reviewed and cited here for better understanding of the physical problems. The research of traveling wave solutions of some nonlinear evolution equations derived from such fields played an important role in the analysis of some phenomena, such as the homotopy perturbation method Dehghan and Manafian (2009), the variational iteration method Dehghan et al. (2010a), the homotopy analysis method Dehghan et al. (2010b), the Adomian decomposition method Luo (2006), the tanh-coth method Manafian and Lakestani (2016a), the Exp-function method (Dehghan et al. (2011); Manafian and Lakestani (2015a); Manafian (2015)), the G'/G-expansion method (Manafian and Lakestani (2015b); Manafian et al. (2016a)), the homogeneous balance method Zhao (2006), the formal linearization method Mirzazadeh and Eslami (2015), the improved $\tan(\phi(\xi))$-expansion method (Manafian and Lakestani (2016a); Manafian and Lakestani (2016b); Manafian 2016; Manafian et al. (2016c); Manafian and Lakestani (2015c); Manafian et al. (2016b); Aghdaei and Manafian(2016); Manafian and Lakestani (2016c); Manafian and Lakestani (2016d); Manafian and Lakestani (2016e)) and so on.

In this paper, we have two goals. First, we introduce Cauchy-Euler method for solving Black-Sholes equation, which is an analytical method. Next, we obtain the exact solutions of the BS equation with the method of separation of variables.

The outline of this paper is organized as follows: In Section 2, we investigate applications of the BS equation with the Cauchy-Euler method and the method of separation of variables. Also, conclusion is given in Section 3.

2. Applications of the BS Equation

In this section, we apply the Cauchy-Euler method and the method of separation of variables for searching exact solution of the Black-Sholes equation.

2.1. Solving the BS Equation by Cauchy-Euler Method

Consider the Black-Scholes equation (1) together with the terminal condition (2). Let $y = \ln(x)$ or $x = e^y$ and $D = \frac{d}{dy}$ and by denoting $v(y,t) = u(x,t)$ we get to,

$$x\frac{\partial u}{\partial x} = Dv, \quad x^2\frac{\partial^2 u}{\partial x^2} = D(D-1)v, \quad x^k\frac{\partial^k u}{\partial x^k} = D(D-1)\ldots(d-k+1)v. \tag{3}$$

Then, equation (1) becomes

$$v_t + av_{yy} + (b-a)v_y - rv = 0, \ \ y > 0, \quad t \in [0,T], \tag{4}$$

while the final condition becomes

$$v(y,T) = z(y), \tag{5}$$

where $z(y) = g(e^y)$. By considering wave variable $\xi = kx + ct$, the equation (4) gets transformed to the following ordinary differential equation,

$$cv' + ak^2v'' + (b-a)kv' - rv = 0, \tag{6}$$

or

$$ak^2v'' + (c + k(b-a))v' - rv = 0. \tag{7}$$

From above equation, we would expect this general solution to be of the form

$$v(\xi) = c_1v_1(\xi) + c_2v_2(\xi) + v_p(\xi), \tag{8}$$

where c_1, c_2, v_1, v_2 are constant coefficients and common solutions, respectively, to ODE and v_p is personal solution.

Remark 1:

The functions $v_1(\xi)$ and $v_2(\xi)$ are linearly independent on an interval I, if the only solution of

$c_1 v_1(\xi) + c_2 v_2(\xi) = 0$, for all $\xi \in I$, is $c_1 = c_2 = 0$.

By considering $e^{m\xi}$ for homogenous case to equation (7), we obtain the quadratic formula in the following form;

$$ak^2m^2 + (c + k(b-a))m - r = 0. \tag{9}$$

Using the quadratic formula, the solutions of (9) are given by

$$m = \frac{-B+\sqrt{\Delta}}{2A}, \quad \Delta = B^2 - 4AC = (c + k(b-a))^2 + 4ark^2, \tag{10}$$

where $A = ak^2$, $B = c + k(b-a)$ and $C = -r$. We have three types of exact solutions of (7) as follows:

Case I:

When $\Delta > 0$, we obtain the hyperbolic function solution

$$v(y,T) = c_1 e^{\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(ky+ct)} + c_2 e^{\frac{-c-k(b-a)-\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(ky+ct)}, \tag{11}$$

and by using the final condition (5), then solution of equation (1) becomes

$$\begin{aligned} u(x,t) = \quad & g(x) e^{\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(t-T)} \\ & + c_2 \left[e^{\frac{-c-k(b-a)-\sqrt{(c+k(\ b-a))^2+4ark^2}}{2ak^2}(k\ln(x)+ct)} - e^{\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(-k\ln(x)+c(t-2T))} \right]. \end{aligned} \tag{12}$$

Case II:

When $\Delta < 0$, we have the trigonometric function solution

$$\begin{aligned} v(y,T) = {} & c_1 \cos\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(ky+ct) \right) \\ & + c_2 \sin\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(ky+ct) \right), \end{aligned} \tag{13}$$

and by using the final condition (5), then solution of the (1) becomes

$$u(x,t) = g(x)\frac{\cos\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(k\ln(x)+ct)\right)}{\cos\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(k\ln(x)+cT)\right)}$$

$$+c_2\sin\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(\ln(x)+ct)\right)$$

$$-c_2\tan\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(k\ln(x)+cT)\right)$$

$$\times\cos\left(\frac{-c-k(b-a)+\sqrt{(c+k(b-a))^2+4ark^2}}{2ak^2}(k\ln(x)+ct)\right). \tag{14}$$

Case III: When $\Delta = 0$, we have the rational function solution

$$v(y,T) = e^{\frac{-c-k(b-a)}{2ak^2}(ky+ct)}[c_1 + c_1(ky+ct)], \tag{15}$$

and by using the final condition (5), then solution of the (1) becomes

$$u(x,t) = e^{\frac{-c-k(b-a)}{2ak^2}(k\ln(x)+ct)}\left\{g(x)e^{\frac{-c+k(b-a)}{2ak^2}(k\ln(x)+cT)} + c_2c(t-T)\right\}. \tag{16}$$

2.2. The Method of Separation of Variables for BS Equation

In this section, by using the separation of variables, the following function

$$u(x,t) = \emptyset(x)\psi(t), \tag{17}$$

will be a solution to a linear homogeneous partial differential equation in x and t. By substituting (17) in equation (1) we get the form

$$\frac{\psi\prime(t)}{\psi(t)} + r = ax^2\frac{\emptyset\prime\prime(x)}{\emptyset(x)} + bx\frac{\emptyset\prime(x)}{\emptyset(x)}. \tag{18}$$

It follows that there exists a constant λ such that

$$\frac{\psi\prime(t)}{\psi(t)} + r = ax^2\frac{\emptyset\prime\prime(x)}{\emptyset(x)} + bx\frac{\emptyset\prime(x)}{\emptyset(x)} = \lambda, \tag{19}$$

where the λ is called the separation constant and is arbitrary. Then, we have

$$ax^2\emptyset''(x) + bx\emptyset'(x) - \lambda\emptyset(x) = 0, \tag{20}$$

$$\psi'(t) = (r-\lambda)\psi(t). \tag{21}$$

We have already written the general solutions of the ODE (20):

I. If $\Delta = (b-a)^2 + 4a\lambda < 0$, then $\emptyset(x) = x^{\frac{a-b}{2a}}\left[\alpha\cos\left(\frac{\sqrt{\Delta}}{2a}\ln(x)\right) + \beta\sin\left(\frac{\sqrt{\Delta}}{2a}\ln(x)\right)\right]$.

II. If $\Delta = 0$, then $\emptyset(x) = x^{\frac{a-b}{2a}}[\alpha + \beta\ln(x)]$.

III. If $\Delta => 0$, then $\emptyset(x) = x^{\frac{a-b}{2a}}\left[\alpha x^{\frac{\sqrt{\Delta}}{2a}} + \beta x^{-\frac{\sqrt{\Delta}}{2a}}\right]$.

where α and β are arbitrary real numbers. Also, we have the following solution of the ODE (21);

$\psi(t) = ce^{(r-\lambda)t}$, where c is an arbitrary real number. By applying above conditions, we get to exact solution $u(x,t)$ of equation (1) as:

Case I:

$$u(x,t) = ce^{(r-\lambda)t}x^{\frac{a-b}{2a}}\left[\alpha\cos\left(\frac{\sqrt{\Delta}}{2a}\ln(x)\right) + \beta\sin\left(\frac{\sqrt{\Delta}}{2a}\ln(x)\right)\right]. \tag{22}$$

Case II:

$$u(x,t) = ce^{(r-\lambda)t}x^{\frac{a-b}{2a}}[\alpha + \beta\ln(x)]. \tag{23}$$

Case III:

$$u(x,t) = ce^{(r-\lambda)t}x^{\frac{a-b}{2a}}\left[\alpha x^{\frac{\sqrt{\Delta}}{2a}} + \beta x^{-\frac{\sqrt{\Delta}}{2a}}\right], \tag{24}$$

where α, β and c are arbitrary real numbers.

Remark 2:

In general, consider the semi-linear partial differential equation as

$$u_t(x,t) + F(u(x,t)) = 0, \tag{25}$$

where

$$F(u(x,t)) = \sum_{k=0}^{m} a_k x^k \frac{\partial^k u(x,t)}{\partial x^k}, \tag{26}$$

$y = \ln(x)$ or $x = e^y$ and $D = \frac{d}{dy}$ and by denoting $v(y,t) = u(x,t)$ we get to

$$F(v(y,t)) = \sum_{k=0}^{m} a_k D(D-1)\ldots(d-k+1)v(y,t). \tag{27}$$

Now, the equation (25) will be as

$$v_t(y,t) + \sum_{k=0}^{m} a_k D(D-1)\dots(d-k+1)v(y,t) = 0, \tag{28}$$

or

$$\begin{aligned}&v_t(y,t) + a_0 v(y,t) + v_y(y,t)(a_1 - a_2 + 2a_3 - 6a_4 + 24a_5 - 120a_6 + \cdots)\\&+v_{yy}(y,t)(a_2 - 3a_3 + 11a_4 - 50a_5 + 274a_6 - 1764a_7 + \cdots)\\&+ v_{yyy}(y,t)(a_3 - 6a_4 + 35a_5 - 225a_6 + 1624a_7 - 13132a_8 + \cdots)\\&+ \cdots + v^{(m-1)}(y,t)\left(a_{m-1} - \frac{m(m-1)}{2}a_m\right) + v^{(m)}(y,t)a_m = 0.\end{aligned} \tag{29}$$

By considering wave variable $= kx + ct$, the equation (29) transformed to the following m^{th} order ordinary differential equation,

$$cv' + a_0 v + A_1 v' + A_2 v'' + \cdots + A_{m-1}v^{(m-1)} + A_m v^{(m)} = 0, \tag{30}$$

or

$$a_0 v + (c + A_1)v' + A_2 v'' + \cdots + A_{m-1}v^{(m-1)} + A_m v^{(m)} = 0, \tag{31}$$

where equation (31) is an m^{th} order linear ordinary differential equation. Likewise, the coefficients of equation (31) are as

$$\begin{aligned}A_1 &= \sum_{k=0}^{m}(-1)^k (k-1)!\, a_k,\\A_2 &= a_2 - 3a_3 + 11a_4 - 50a_5 + 274a_6 - 1764a_7 + \cdots,\\A_3 &= a_3 - 6a_4 + 35a_5 - 225a_6 + 1624a_7 - 13132a_8 + \cdots,\\&\cdots\\&\cdots\\A_{m-1} &= a_{m-1} - \frac{m(m-1)}{2}a_m,\\A_m &= a_m.\end{aligned} \tag{32}$$

By considering $e^{\lambda\xi}$ for homogenous case to equation (7), we obtain the mth order formula in the following form

$$A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_2\lambda^2 + (c + A_1)\lambda + a_0 = 0. \tag{33}$$

By solving the (33), based on numerical or analytical methods, we can obtain solutions for the equation (31). For example, consider $m = 3$ in (33), therefore

$$A_3\lambda^3 + A_2\lambda^2 + (c + A_1)\lambda + a_0 = 0, \tag{34}$$

include solutions as

$$\lambda_1 = \frac{\frac{1}{6A_3}\sqrt[3]{36cA_2A_3 + 36A_1A_2A_3 - 108a_0{A_3}^2 - 8{A_2}^3 + 12A_3\sqrt{3\Sigma}} - \frac{2}{3}\left(3cA_3 + 3A_1A_3 - {A_2}^2\right)}{A_3\sqrt[3]{36cA_2A_3 + 36A_1A_2A_3 - 108a_0{A_3}^2 - 8{A_2}^3 + 12A_3\sqrt{3\Sigma}}} - \frac{A_2}{3A_3}, \tag{35}$$

$$\lambda_2 = \frac{\frac{-1}{12A_3}\sqrt[3]{36cA_2A_3+36A_1A_2A_3-108a_0{A_3}^2-8{A_2}^3+12A_3\sqrt{3\Sigma}}+\frac{1}{3}\left(3cA_3+3A_1A_3-{A_2}^2\right)}{A_3\sqrt[3]{36cA_2A_3+36A_1A_2A_3-108a_0{A_3}^2-8{A_2}^3+12A_3\sqrt{3\Sigma}}} - \frac{A_2}{3A_3}$$

$$\pm\frac{\sqrt{-3}}{2}\frac{\frac{1}{6A_3}\sqrt[3]{36cA_2A_3+36A_1A_2A_3-108a_0{A_3}^2-8{A_2}^3+12A_3\sqrt{3\Sigma}}+\frac{2}{3}\left(3cA_3+3A_1A_3-{A_2}^2\right)}{A_3\sqrt[3]{36cA_2A_3+36A_1A_2A_3-108a_0{A_3}^2-8{A_2}^3+12A_3\sqrt{3\Sigma}}},$$

where

$$\Sigma = 27{a_0}^2{A_3}^2+4c^3A_3+4a_0{A_2}^3-A_2(c^2A_2+18a_0A_1A_3+18ca_0A_3) \\ -A_1{A_2}^2(A_1+2c)+4A_1A_3.$$

Thus, equation (25) for $m = 3$ will be as

$$u(x,t) = C_1e^{\lambda_1(k\ln(x)+ct)} + C_2e^{\lambda_2(k\ln(x)+ct)} + C_3e^{\lambda_3(k\ln(x)+ct)}.$$

Note

All the obtained results have been checked with Maple 13 by putting them back into the original equation and found to be correct.

4. Conclusion

In this paper, the Black-Sholes equation has been applied successfully with the Cauchy-Euler method and the method of separation of variables and new analytical solutions have been found. The linear partial differential equation (PDE) transformed to linear ordinary differential equation (ODE) as well. We acquired three types of solutions including hyperbolic, trigonometric and rational solutions. Descriptions of these methods are given and the obtained results reveal that three methods are tools for exploring partial differential models. Therefore, these methods can be applied to study many other linear and nonlinear partial differential equations which frequently arise in engineering and mathematical physics.

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