Applications and Applied Mathematics: An International Journal (AAM)

Volume 12 | Issue 2 | Article 8

12-2017

Farlie-Gumbel-Morgenstern Family: Equivalence of Uncorrelation and Independence

G. Barmalzan
University of Zabol

F. Vali
Razi University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 12, Issue 2 (December 2017), pp. 766–775

Applications and Applied Mathematics:
An International Journal
(AAM)

Farlie-Gumbel-Morgenstern Family: Equivalence of Uncorrelation and Independence

[1]G. Barmalzan and [2]F. Vali

[1]Department of Statistics
University of Zabol
Sistan and Baluchestan, Iran
[2]Department of Statistics
Razi University
Kermanshah, Iran
[1]Ghobad.barmalzan@gmail.com; [2]Farzad-amar89@yahoo.com

Received: December 13, 2016; Accepted: August 14, 2017

Abstract

Considering the characteristics of the bivariate normal distribution, in which uncorrelation of two random variables is equivalent to their independence, it is interesting to verify this problem in other distributions. In other words, whether the multivariate normal distribution is the only distribution in which uncorrelation is equivalent to independence. In this paper, we answer to this question and establish generalized Farlie-Gumbel-Morgenstern (FGM) family is another family of distributions under which uncorrelation is equivalent to independence.

Keywords: Uncorrelation; Independence; Farlie-Gumbel-Morgenstern Family; Exchangeablity; Correlation Coefficient

MSC 2010 No.: 62F99, 47N30, 97K70

1. Introduction

Studying the dependence structure of bivariate distributions has an important role in statistics and probability. It is also important to test independence against quadrant dependence (QD). Many authors have investigated the dependence structure of bivariate distributions. Kochar and Gupta (1987, 1990) introduced a class of distribution-free tests for testing independence against QD and evaluated the empirical power for the bivariate exponential distribution of Block and Basu (1974) based on the sample sizes $n = 8$ and 12. Shetty and Pandit (2003) proposed a class of distribution-free tests to test independence against positive quadrant dependence (PQD), which is a generalization of Kochar and Gupta (1990). Amini et al. (2010) evaluated the empirical power of this class in FGM family for the sample sizes $n = 6, 8, 10, 12, 16$ and 20 based on the empirical distribution. Güven and Kotz (2008) introduced a new test statistic for testing independence against QD in generalized FGM family. Amini et al. (2011) obtained a dependence measure for generalized Farlie-Gumbel-Morgenstern (FGM) family in view of Kochar and Gupta (1987) and then compared this measure with Spearman's rho and Kendall's tau in FGM family. Moreover, these authors evaluated the empirical power of the class of distribution-free tests proposed by Kochar and Gupta (1987, 1990) based on exact distribution of a U-statistics.

Certain bivariate densities constructed from marginals have recently been suggested as models of hydrologic variates such as rainfall intensity and depth. It is pointed out that (i) these densities belong to the families of the Farlie-Gumbel-Morgenstern densities and the Farlie polynomial densities, which have been extensively studied in the statistical literature, and that (ii) these densities have a limited potential applicability in hydrology since they can model only weakly associated variates. Interested readers may refer to Long and Krzysztofowicz (1992) for more details.

The task of constructing a multivariate distribution having specified marginal distributions has challenged statisticians for decades. The problem of constructing a multivariate distribution is from interest on both theoretical and practical viewpoints. Several authors have worked on this problem; see, for example, Haight (1961), Mardia (1970), Singh and Singh (1991), Morgenstern (1956), Gumbel (1958) and Farlie (1960).

Joshi (1978) contracted a certain bivariate distribution which will illustrate the following situations which are frequently mentioned in literatures but handy examples of which are not obvious to come by.

(i) If X and Y are two random variables with the moment generating functions $M_1(t)$ and $M_2(t)$, respectively, then independence between X and Y implies that the moment generating function $M(t)$ of $X+Y$ is $M_1(t)\, M_2(t)$. However, $M(t) = M_1(t)\, M_2(t)$ does not imply independence between X and Y. An example of this is given in Cramér (1946), p. 317.
(ii) Univariate marginal distributions of a bivariate normal distribution are normal but there is a bivariate non-normal distribution which univariate marginals are normal.
(iii) The joint distribution of two non-independent random variables X and Y is such that X^2 and Y^2 are independent (see Parzen (1960), p. 297).

Let (X_1, X_2) denote a vector of continuous variates having joint density f and arbitrarily specified marginal functions: a density f_i and a distribution F_i of X_i, for $i = 1, 2$. The general form of the constructed bivariate density is

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \{1 + c\, \nu(F_1(x_1), F_2(x_2))\}, \tag{1}$$

where c is a scaler, and ν is a kernel which models the dependence structure. Scaler c may take any value in the range $-1 \leq c \leq 1$. Density (1) with kernel $\nu(u, w) = (2u - 1)(2w - 1)$ is well known in the statistical literature, generally under the name of its major developers, Farlie-Gumbel-

Morgenstern. It has been characterized further by Kotz (1975), Kotz and Johnson (1977), Schucany et al. (1978) and Marshall and Olkin (1988).

The rest of this paper is structured as follows. Section 2 presents some basic statistical concepts and some aspects of FGM family that will be used in the subsequent developments. In Section 3 equivalence of uncorrelation and independence in FGM family is established. In Section 4, distinction between FGM family and bivariate normal distribution is discussed. Finally, some concluding remarks are made in Section 5.

2. Some Aspects of This Family

In this section, we introduce a version of FGM family and discuss its statistical and probabilistic properties. Let (X, Y) be a pair of absolutely continuous random variables with the marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$. The FGM family of (X_1, X_2) for $k > 0$ and $r > 0$ is

$$f_\alpha(x_1, x_2) = f_1(x_1)f_2(x_2)\left\{1 + \alpha\left[\left(F_1^k(x_1) - \frac{1}{k+1}\right)\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right]\right\}, \tag{2}$$

where $-1 \leq \alpha \leq 1$. First, we show that $f_\alpha(x_1, x_2)$ is a bivariate density function with given marginal densities f_1 and f_2, for each α. It should be noted that

$$\left(F_1^k(x_1) - \frac{1}{k+1}\right)\left(F_2^r(x_2) - \frac{1}{r+1}\right) < \frac{kr}{(k+1)(r+1)} \leq 1, \tag{3}$$

and then for α , we observe that $1 + \alpha\left[\left(F_1^k(x_1) - \frac{1}{k+1}\right)\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right] \geq 0$. Also

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\alpha(x_1, x_2)\, dx_1\, dx_2 &= 1 + \alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x_1)f_2(x_2)(F_1^k(x_1) - \frac{1}{k+1}) \\
&\quad \times (F_2^r(x_2) - \frac{1}{r+1}) dx_1\, dx_2 \\
&= 1 + \int_{-\infty}^{\infty} f_2(x_2)\left(F_2^r(x_2) - \frac{1}{r+1}\right) \\
&\quad \times \left\{\left(\frac{F_1^{k+1}(x_1)}{k+1} - \frac{1}{k+1}\right)\Big|_{-\infty}^{+\infty}\right\} dx_2 \\
&= 1 + 0 = 1,
\end{aligned}$$

which it follows that $f_\alpha(x_1, x_2)$ is a joint density function.

Let (x_i, y_i), $i = 1, \cdots, n$ denote random samples from FGM family. Using the joint density function given in 2, the log likelihood function is obtained as,

$$l(\alpha) = \sum_{i=1}^{n} \ln(f_1(x_i)f_2(y_i)) + \sum_{i=1}^{n} \ln\left\{1 + \alpha\left[\left(F_1^k(x_i) - \frac{1}{k+1}\right)\left(F_2^r(y_i) - \frac{1}{r+1}\right)\right]\right\}.$$

Differentiating $l(\alpha)$ partially with respect to the α and equating it to zero, we get the following log likelihood equation,

$$\frac{\partial l(\alpha)}{\partial \alpha} = \sum_{i=1}^{n} \frac{\left(F_1^k(x_i) - \frac{1}{k+1}\right)\left(F_2^r(y_i) - \frac{1}{r+1}\right)}{1 + \alpha\left[\left(F_1^k(x_i) - \frac{1}{k+1}\right)\left(F_2^r(y_i) - \frac{1}{r+1}\right)\right]} = 0. \tag{4}$$

It can be seen that the above equation is non-linear with respect to α and hence obtaining closed form expression for the estimator is not possible. One may use Newton-Raphson method or any root finding algorithm to obtain solution to the system of non-linear equation given in Equation 4.

Definition 2.1.

Let $F(x, y)$ and $F_1(x), F_2(y)$, respectively, be the joint distribution function of (X, Y) and the marginal distribution functions of X and Y. Then, we say that X and Y are independent if and only if

$$F(x, y) = F_1(x)F_2(y), \qquad \text{for all } (x, y) \in \mathbb{R}^2. \tag{5}$$

It is important to note that in the FGM family X_1 and X_2 are not independent unless $\alpha = 0$.

Definition 2.2.

Let X and X be random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. If $E\{(X - EX)(Y - EY)\}$ exists, then $cov(X, Y) = E\{(X - EX)(Y - EY)\}$.

If X and Y are independent, then $cov(X, Y) = 0$ and we say that X and Y are uncorrelated. However, if $cov(X, Y) = 0$, then X and Y may be not necessarily independent. It is of important to note that independence is not a property of random variables but it is pertinent to probability space. On the other hand, if probability space switches, the independence between random variables may also change. So, we can say that independence is a property of probability space, not random variables. For example, suppose $X_1, \cdots, X_n$ are independent normal random variables, thus $\bar{X}$ and S^2 are independent, but if distribution of X_i's switch to Poisson with mean λ, then $cov(\bar{X}, S^2) = \lambda/n \neq 0$ and $\bar{X}$ and S^2 are not independent.

One of the key questions in judging the applicability of a bivariate density constructed from marginals concerns the flexibility of its dependence structure, in particular, the degree of association between variates that can be modeled. The following theorem presents correlation coefficient between X_1 and X_2 in the FGM family.

Theorem 2.1.

Let (X_1, X_2) be a random vector with FGM family. Then

$$\rho(X_1, X_2) \leq \alpha \, \frac{kr}{(k+1)(r+1)\sqrt{(2k+1)(2r+1)}}. \tag{6}$$

Proof:

The correlation coefficient between X_1 and X_2, $\rho(X_1, X_2)$, is given by

$$\rho(X_1, X_2) = \frac{cov(X_1, X_2)}{\sigma_1 \, \sigma_2}. \tag{7}$$

First, $cov(X_1, X_2) = \alpha J_1(k) J_2(r)$ readily observed where for $i = 1, 2$

$$J_i(m) = \int_{-\infty}^{+\infty} (x_i - \mu_i) \, f_i(x_i) \, \left(F_i^m(x_i) - \frac{1}{m+1} \right) dx_i. \tag{8}$$

By applying the Schwarz inequality to (8), we obtain

$$\begin{aligned} J_i^2(m) &\leq \left(\int_{-\infty}^{+\infty} (x_i-\mu_i)^2\, f_i(x_i)\; dx_i\right)\left(\int_{-\infty}^{+\infty}\left(F_i^m(x_i)-\frac{1}{m+1}\right)^2 f_i(x_i)\, dx_i\right) \\ &= \sigma_i^2 \int_0^1 \left(u_i^m - \frac{1}{m+1}\right)^2 du_i \\ &= \sigma_i^2\, \frac{m^2}{(m+1)^2(2m+1)}. \end{aligned}$$

Now, by replacing these observations in (7), we get

$$\rho(X_1, X_2) \leq \alpha\, \frac{kr}{(k+1)(r+1)\sqrt{(2k+1)(2r+1)}},$$

and, hence, the theorem. □

It is of important to note that if $k = r = 1$ then $\rho(X_1, X_2) \leq 1/3$. This point shows this family can have suitable fit for a set of observations that have weakly associated variates.

Definition 2.3.

The random variables $X_1, \cdots, X_n$ is said to be exchangeable if and only if

$$(X_1, \cdots, X_n) \stackrel{D}{=} (X_{i_1}, \cdots, X_{i_n}),$$

for all $n!$ permutations $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$, where $\stackrel{D}{=}$ means the same distribution on both sides of the equality.

Clearly if $X_1, \cdots, X_n$ are exchangeable, then X_i's are identically distributed but conversely may not be held. The following example further illustrates this point.

Example 2.1.

Let (X_1, X_2) be jointly distributed with density function

$$f(x_1, x_2) = \begin{cases} \frac{1}{4}(1 - x_1^3 x_2), & |x_1| \leq 1, |x_2| \leq 1, \\ \\ \quad 0, & \text{otherwise.} \end{cases}$$

It is easy to show that X_1 and X_2 are identically distributed and have the following density

$$g(x) = \begin{cases} \frac{1}{2}, & |x| \leq 1, \\ \\ 0, & \text{otherwise,} \end{cases}$$

but X_1 and X_2 are not exchangeable.

The following theorem shows in the FGM family, identically implies exchangeability and the converse is also true.

Theorem 2.2.

Let Y_1 and Y_2 have density functions f_1 and f_2, respectively, and (X_1, X_2) have the joint density function $f_\alpha(x_1, x_2)$. Then,

(i) X_1 identically distributed with Y_1 , and X_2 identically distributed with Y_2;
(ii) X_1 and X_2 are exchangeable if and only if Y_1 and Y_2 are identically distributed.

Proof:

(i) The proof of this part is straightforward.
(ii) Suppose X_1 and X_2 are exchangeable. So, $(X_1, X_2) \overset{D}{=} (X_2, X_1)$ and hence $X_1 \overset{D}{=} X_2$. Now, according to Part (i), the desired result is obtained. Conversely, suppose Y_1 and Y_2 are identically distributed. According to Part (i), we conclude $X_1 \overset{D}{=} X_2$ and then $f_\alpha(x_1, x_2) = f_\alpha(x_2, x_1)$. □

We recall that the joint distribution of a multiple random variables uniquely determines the marginal distributions of the component random variables, but in general, knowledge of marginal distributions is not enough to determine the joint distribution. Indeed, it is quite possible to have an infinite collection of joint densities f_α with given marginal densities.

3. Equivalence of Uncorrelation and Independence

In this section, we show that except bivariate normal distribution, FGM is another family of distributions, in which uncorrelation of two random variables is equivalent to their independence. Here, we first present bivariate normal distribution for its distinction with FGM family.

A two-dimensional random variable (X_1, X_2) is said to have a bivariate normal distribution (denoted $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$) if the joint density function is of the form

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}Q(x_1, x_2)\right\}, \qquad |x_1| < \infty, |x_2| < \infty,$$

where

$$Q(x_1, x_2) = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right),$$

and $|\mu_1| < \infty$, $|\mu_2| < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $|\rho| < 1$. The next theorem presents uncorrelation of two random variables is equivalent to their independence in the bivariate normal distribution.

Theorem 3.1. (Mardia et al. (1979))

Let (X_1, X_2) be a bivariate normal distribution. Then, X_1 and X_2 are independent if and only if they are uncorelated.

In the next theorem, we calculate $cov(X_1, X_2)$ in the FGM family.

Theorem 3.2.

Let (X_1, X_2) be a random vector with FGM family. Then,

$$cov(X_1, X_2) = \alpha E\left[X_2\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right] cov\left(X_1, F_1^k(x_1)\right).$$

Proof:

The condition density function and conditional expectation of X_2, given $X_1 = x_1$, respectively, are

$$f_{X_2|X_1=x_1}(x_2) = f_2(x_2)\left\{1+\alpha\Big(F_1^k(x_1)-\frac{1}{k+1}\Big)\Big(F_2^r(x_2)-\frac{1}{r+1}\Big)\right\},$$

and

$$E(X_2|X_1=x_1) = E(X_2)+\alpha\left(F_1^k(x_1)-\frac{1}{k+1}\right)E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right],$$

Then, the covariance between X_1 and X_2 is obtained as follows,

$$\begin{aligned}
cov(X_1,X_2) &= cov(X_1,E(X_2|X_1))\\
&= cov\left(X_1,E(X_2)+\alpha\left(F_1^k(x_1)-\frac{1}{k+1}\right)E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right]\right)\\
&= cov(X_1,E(X_2))+\alpha E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right]\\
&\quad \times cov\left(X_1,\left(F_1^k(x_1)-\frac{1}{k+1}\right)\right)\\
&= 0+\alpha E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right]cov\left(X_1,F_1^k(x_1)\right),
\end{aligned}$$

and hence the theorem. □

Theorem 3 shows that the covariance between X_1 and X_2 is a function of α and with knowledge of marginal distributions F_1 and F_2, we can obtain the value of $cov(X_1, X_2)$.

The following example further illustrates Theorem 3.2.

Example 3.1.

Let f_1 and f_2 be two density functions of $Uniform(0,1)$ and let (X_1, X_2) have the joint density function $f_\alpha(x_1, x_2)$. Then, we do get

$$\begin{aligned}
cov(X_1,X_2) &= \alpha E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right]cov\left(X_1,F_1^k(x_1)\right)\\
&= \alpha\,\frac{r}{2(r+1)(r+2)}\,\frac{k}{2(k+1)(k+2)}\\
&= \alpha\,\frac{rk}{4(r+1)(r+2)(k+1)(k+2)}.
\end{aligned}$$

The following theorem shows in the FGM family uncorrelation and independence are equivalent.

Theorem 3.3.

Let (X_1, X_2) be a random vector with FGM family. If $E\left[X_2\left(F_2^r(x_2)-\frac{1}{r+1}\right)\right]\neq 0$, then, X_1 and X_2 are independent if and only if they are uncorrelated.

Proof:

Let X_1 and X_2 be uncorrelated. So, $cov(X_1, X_2) = 0$ and then

$$\alpha E\left[X_2\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right] cov\left(X_1, F_1^k(x_1)\right) = 0, \tag{9}$$

It is evident that $cov\left(X_1, F_1^k(x_1)\right) \neq 0$. Then assumption theorem, Equation 9 is equivalent to $\alpha = 0$, which gives $f_\alpha(x_1, x_2) = f_1(x_1)f_2(x_2)$ and then X_1 and X_2 are independent. Conversely, if X_1 and X_2 are independent, then $cov(X_1, X_2) = 0$ and, X_1 and X_2 are uncorrelated. □

4. Distinction Between FGM family and Bivariate Normal Distribution

In this section, we establish FGM family as another family of distributions, in which uncorrelation of two random variables is equivalent to their independence, and it is different from bivariate normal distribution. On the other hand, bivariate normal distribution does not belong to FGM family. Let f_1 and f_2 be probability density functions of normal distribution with means $F_1^{-1}[(\frac{1}{k+1})^{1/k}]$, $F_2^{-1}[(\frac{1}{r+1})^{1/r}]$ and common variance 1. In this case, we show that the $f_\alpha(x_1, x_2)$ cannot belong to bivariate normal distribution. Let (X_1, X_2) be a bivariate normal distribution with parameters $(F_1^{-1}[(\frac{1}{k+1})^{1/k}], F_2^{-1}[(\frac{1}{r+1})^{1/r}], 1, 1, \rho)$. If $f_\alpha(x_1, x_2)$ belongs to bivariate normal distribution, then for all $(x_1, x_2) \in \mathbb{R}^2$, we must have

$$\begin{aligned} f_1(x_1)f_2(x_2)&\left\{1 + \alpha\left[\left(F_1^k(x_1) - \frac{1}{k+1}\right)\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right]\right\} \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \times \exp\left\{\frac{-Q(x_1, x_2)}{2(1-\rho^2)}\right\}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} Q(x_1, x_2) = &\left(x_1 - F_1^{-1}[(\frac{1}{k+1})^{1/k}]\right)^2 + \left(x_2 - F_2^{-1}[(\frac{1}{r+1})^{1/r}]\right)^2 \\ &- 2\rho\left(x_1 - F_1^{-1}[(\frac{1}{k+1})^{1/k}]\right)\left(x_2 - F_2^{-1}[(\frac{1}{r+1})^{1/r}]\right). \end{aligned} \tag{11}$$

Now, if we replace $(x_1, x_2) = (F_1^{-1}[(\frac{1}{k+1})^{1/k}], F_2^{-1}[(\frac{1}{r+1})^{1/r}])$ in (10), then we get

$$f_1\left(F_1^{-1}\left[\left(\frac{1}{k+1}\right)^{1/k}\right]\right)f_2(F_2^{-1}\left[\left(\frac{1}{r+1}\right)^{1/r}\right]) = \frac{1}{2\pi\sqrt{1-\rho^2}}, \tag{12}$$

which for holding relation (12), it is sufficient $\rho = 0$. But $\rho = 0$ implies that $f_\alpha(x_1, x_2) = f_1(x_1)f_2(x_2)$. Therefore, for all $(x_1, x_2) \in \mathbb{R}^2$ the following equation must hold

$$f_1(x_1)f_2(x_2)\left\{1 + \alpha\left[\left(F_1^k(x_1) - \frac{1}{k+1}\right)\left(F_2^r(x_2) - \frac{1}{r+1}\right)\right]\right\} = f_1(x_1)f_2(x_2), \tag{13}$$

which (13) dose not hold because the left side has a extra term. For example, for $(x_1, x_2) = (1, 1)$ then equation (13) does not hold. In FGM family, (X_1, X_2) is bivariate normal distribution if $\alpha = 0$. So, in this family $\rho = 0$ is equivalent to $\alpha = 0$. Therefore, bivariate normal distribution does not belong to the FGM family.

5. Conclusion

If random variables X and Y are independent, then $cov(X, Y) = 0$ and we say that X and Y are uncorrelated. However, if $cov(X, Y) = 0$, then X and Y may be not necessarily independent. It is of importance to note that independence is not a property of random variables and it is pertinent to probability space. On the other hand, if probability space changes, it may be the independence between random variables also changes. So, we can say that independence is a property of probability space, not random variables. For example, suppose $X_1, \cdots, X_n$ are independent normal random variables, thus $\bar{X}$ and S^2 are independent, but if distribution of X_i's switch to Poisson with mean λ, then $cov(\bar{X}, S^2) = \lambda/n \neq 0$ and $\bar{X}$ and S^2 are not independent. In this paper, we introduce a version of FGM family and its statistical and discuss its probabilistic properties. We also establish generalized Farlie-Gumbel-Morgenstern family is another family of distributions under which uncorrelation is equivalent to independence.

Acknowledgement:

We thank both referees and the Editor in Chief Professor Aliakbar Montazer Haghighi, for their valuable suggestions and constructive comments which helped us in revising the paper in the present form.

REFERENCES

Amini, M., Jabbari, H., Mohtashami Borzadaran, G. R. and Azadbakhsh, M. (2010). Power comparison of independence test for the Farlie-Gumbel-Moregenstern family, Communications of the Korean Statistical Society, Vol. 17, pp. 493–505.

Cramér, H. (1946). *Mathematical Methods of Statistics*, Princeton University Press.

Block, H. W. and Basu, A. P. (1974). A continuous bivariate exponential extension, Journal of the American Statistical Association, Vol. 69, pp. 1031–1037.

Farlie, D.J.G. (1960). The performance of some correlation coefficients for a general bivariate distribution, Biometrika , Vol. 47, pp. 307−323.

Gumbel, E.J. (1958). Distributions áphsieurs variables dont les marges sont données (with remarks by Fréchet, M.), Comptes Rendus de l'Académie des Science, Paris, Vol. 246, pp. 2717−2720.

Güven, B. and Kotz, S. (2008). Test of independence for generalized Farlie-Gumbel-Morgenstern distributions, Journal of Computational and Applied Mathematics, Vol. 212, pp. 102−111.

Kochar, S. G. and Gupta, R. P. (1987). Competitors of Kendall-tau test for testing independence against PQD, Biometrika, Vol. 74, pp. 664−669.

Kochar, S. G. and Gupta, R. P. (1990). Distribution-free tests based on sub-sample extrema for testing against positive dependence, Australian Journal of Statistics, Vol. 32, pp. 45−51.

Haight, F.A. (1961). Index to the distributions of mathematical statistics, Journal of Research of the National Bureau of Standards, Vol. 65, pp. 23−60.

Joshi, S.W. (1970). Construction of Certain Bivariate Distributions, The American Statistician, Vol. 24, pp. 32–32.

Kotz, S. (1975). Multivariate distributions at a cross road. In Patil, G. P.; Kotz, S.; Ord, J. K. (eds) Statistical distributions in scientific work 1, pp. 247–270, D. Reidel Publishing Company, Dordrecht, Holland.

Kotz, S. and Johnson, N. (1977). Propriétés de dépendance des distributions itérées, généralisées á deux variables Farlie-Gumbel-Morgenstern, Comptes Rendus de l'Académie des Science, Paris, Vol. 285, pp. 277–280.

Long, D. and Krzysztofowicz, R. (1992). Farlie-Gumbel-Morgenstern bivariate densities: Are they applicable in hydrology?, Stochastic Hydrology and Hydraulics, Vol. 6, pp. 47–54.

Mardia, K.V. (1970). *Families of bivariate distributions*, Hafner Publishing Company, Darien, Connecticut.

Mardia, K.V., Kent, J.T., Bibby, J.M. (1979). *Multivariate Analysis*, London: Academic Press.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions, Journal of the American Statistical Association, Vol. 83, pp. 834–841.

Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen, Mitteilungeblatt ffir mathematische statistik, Wfirzburg, Vol. 8, pp. 234–235.

Parzen, E. (1960). *Modern Probability Theory and Its Applications*, John Wiley & Sons.

Schucany, W. It., Parr, W. C. and Boyer, J. E. (1978). Correlation structure in Farlie-Gumbel-Morgenstern distributions, Biometrika, Vol. 65, pp. 650–653.

Shetty, I. D. and Pandit, P. V. (2003). Distribution-free tests for independence against positive quadrant dependence: a generalization, Statistical Methods and Applications, Vol. 12, pp. 5–17.

Singh, K. and Singh, V. P. (1991). Derivation of bivariate probability density functions with exponential marginals, Stochastic Hydrology and Hydraulics, Vol. 5, pp. 55–68.