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P. Manoharan
Annamalai University

Shakir Majid
Annamalai University

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Stationary Analysis of a Multiserver queue with multiple working vacation and impatient customers

[1]P. Manoharan & [2]Shakir Majid

Department of Mathematics
Annamalai University
Annamalainagar - 608 002, India
[1]drmanomaths.hari@gmail.com, [2]shakirku16754@gmail.com

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Abstract

We consider an $M/M/c$ queue with multiple working vacation and impatient customers. The server serves the customers at a lower rate rather than completely halts the service during this working vacation period. The impatience of the customer's arises when they arrive during the working vacation period, where the service rate of the customer's is lower than the normal busy period. The queue is analyzed for multiple working vacation policies. The policy of a MWV demands the server to keep taking vacation until it finds at least a single customer waiting in the system at an instant vacation completion. On returning of the server from his vacation along with finding at least one customer in the system, the server changes its service rate, thereby giving rise to a non-vacation period; otherwise the server immediately goes for another WV. We formulate the probability generating function for the number of customers present when the server is both in a service period as well as in a working vacation period. We further derive a closed-form solution for various performance measures such as the mean queue length and the mean waiting time. The stochastic decomposition properties are verified for the model.

Keywords: $M/M/c$ queue; Working vacation; Impatient customers; Stochastic decomposition; Generating function

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1. Introduction

The main motive of this presentation is to study the stationary behavior of $M/M/c$ queue with multiple working vacation and impatient customers in which the server operates with variation in service rate but does not completely halt the service during this vacation period. Therefore such a vacation is called a working vacation. Queuing models with server vacations or working vacations have been studied by various researchers during the last two decades. The literature may be divided into two groups: (i) when the server is on a vacation and (ii) when the server is on a working vacation. When the server is on a vacation, the survey paper by Doshi (1986) and the monograph of Takagi (1991) act as the reference for the readers. The research work done by Takagi (1991) and Doshi (1986) stresses on a single server. As far as the case of multiple-server system with vacations is concerned, Levy and Yechiali (1976) first studied the $M/M/c$ queue with asynchronous vacation policy. Later, Chao and Zhao (1998) analyzed the $M/M/c$ with both synchronous and asynchronous vacation policies and computed the stationary probability distribution by providing some algorithms. Zhang and Tian (2003a, 2003b) recently contributed enough analysis of $M/M/c$ with synchronous multiple/single vacations of partial servers.

$M/M/1$ queuing models with multiple working vacations were first studied by Servi and Finn (2002), where inter-arrival times, service times during service period, service times during vacation period, and vacation times are all exponentially distributed. (Such a model is denoted by $M/M/1/WV$ queue.) Later the $M/M/1/WV$ model was also studied by Liu et al. (2007) to obtain explicit expressions of the performance measures and their stochastic decomposition by using the quasi-birth death and matrix-geometric method. The Servi and Finn (2002) queue was extended to a $M/G/1/WV$ queue by Wu and Takagi (2006). They assumed that service times during service period, service times during vacation period, as well as vacation times are all generally distributed. Further, they assumed that at the end of the working vacation, if the system has customers, the server switches to another service rate, where a different distribution is followed by the service times. The $GI/M/1/WV$ queue is the advanced form of Servi and Finn (2002), extended by Baba (2005). Not only assuming general independent arrival, they also considered service times during service period, service times during vacation period, as well as vacation times following exponential distribution. The analysis of single working vacation in $GI/M/1/N$ and $GI/M/1/1$ queuing systems was done by Banik (2010) by assuming that the service time and vacation time was distributed exponentially. Laxmi and Yesuf (2011) recently presented an $GI/M/1$ batch service queue with a policy of exponential working vacation. Lin and Ke (2009) analyzed the $M/M/c$ queue with single working vacation. A short survey on recent developments in vacation queuing models has been also given by Ke et al. (2010).

In the past, queuing models with customers' impatience have been studied by various authors such as Altman and Yechiali (2006, 2008), Boxma and de Waal (1994), Yechiali (2007), Baccelli et al. (1984), Daley (1965), Van Houdt et al. (2003), Yue et al. (2011) and Yue et al. (2014), where the cause of impatience was either a long wait already experienced in the queue or a long wait anticipated by a customer upon arrival. Situations arise where the absence of the server becomes the cause of the customers' impatience, more precisely, because of the server vacation, and this situation is not dependent of the customers in system. The impatience of customers, studied by Altman and Yechiali (2006) in vacation systems of $M/M/1$, $M/G/1$ and also $M/M/c$ queues, in which each customer arriving finds no server on duty, ultimately activates an independent random impatient timer. By the time the timer expires, if the server fails to return, the customers exit from the queue and never come back. On the other hand, if the server comes back in due time, the customer stays in the system until the completion of his service.

Recently, the $M/M/1$ model with WV, where the impatience of the customers is due to WV, was studied by Yue et al. (2011). They have studied the model in which during WV all the customers in the system become impatient and if the service of the customers has not been completed before the impatient timer of the customers expires, the customer exits from the queue. Here, we assume a more constrained model in which only those customers becomes impatient during the WV, which are waiting in queue and not the ones which are under service.

The model is presented as follows. In Section 2, we provide the description of the model. In Section 3, we formulate the model as a quasi-birth-death process and find the partial probability generating functions for the distribution of queue sizes, when the server is in WV period and when in regular busy period. By solving the arising differential equations, we find the closed-form solutions. Many performance measures such as the mean queue length and the mean waiting time are obtained in Section 4. In Section 5, stochastic decomposition properties are verified. Finally the conclusion is given in Section 6.

2. Model Description

We consider a system in which the arrivals occur according to a Poisson process with parameter λ. The service times during the normal busy period has an exponential distribution with mean $\frac{1}{\mu_b}$, where we consider the stability condition that $\rho = \frac{\lambda}{c\mu_b} < 1$. The server holds on a WV as soon as the system becomes empty. In the due course of vacation period, the customers are served at an exponential rate μ_v, where $\mu_v < \mu_b$. The time of a vacation period is exponentially distributed with parameter θ. The policy of a MWV demands the server to keep taking vacation until it finds at least a single customer waiting in the system at an instant vacation completion. Upon the return of the server from his vacation along with finding at least a single customer in the system, the service rate is switched from μ_v to μ_b by the server, thereby giving rise to a non-vacation period, otherwise the server immediately goes for another WV. On the other hand, if the termination of vacation in the middle of an ongoing service occurs, the service rate is shifted and the server continues the service at the higher rate until completion of the service.

The impatience of the customers in the system is explained as follows. If the arriving customer wishes to join a queue and notices that the system is in WV, he activates an impatient time T, which is exponentially distributed with parameter ξ alongside being independent of the number of customers present in the system at a very specific moment. If the vacation of the server ends before the time T expires, the customer remains in the system until the completion of his service or else the customer evacuates himself from the system and never comes back. Thus it is concluded that only those customers whose arrival occurs during a WV of the server are impatient. On the other hand, a customer, if he arrives during a WV period and finds the server free, will get served immediately and there holds no reason for him to be impatient. This type of impatient policy is different from that of the impatient policy studied in Altman and Yechiali (2006), in which all arriving customers become impatient during the vacation period, since a pure vacation policy is considered. The server gives service to the customers on the basis of first come first served (FCFS). The inter-arrival times, service times, vacation duration times, and the impatient times all are taken to be mutually independent.

Let $\{N(t), t \geq 0\}$ be the number of customers in the system at time t and $J(t)$ be the state of system at time t, where $J(t)$ is defined as follows:

$$J(t) = \begin{cases} 1, & \text{when the servers are a non-vacation period at time } t, \\ 0, & \text{when the servers are in WV period at time } t. \end{cases}$$

Then, $\Delta = \{(N(t), J(t)), t \geq 0\}$ is a two dimensional continuous time discrete state Markov chain with state space

$$S = \left\{ \{(0,0)\} \bigcup \{(i,j)\}, i = 1, 2, ..., j = 0, 1 \right\}.$$

3. The Stationary Distribution

Let us define the stationary probabilities for a Markov chain Δ as

$$P_{ij} = P\{N(t) = i, J(t) = j\}, \quad i = 0, 1, 2, ..., j = 0, 1.$$

Then, the stationary equations are

$$\lambda P_{00} = (\mu_v + \xi) P_{1,0} + \mu_b P_{11}, \tag{1}$$

$$[\lambda + \theta + n(\mu_v + \xi)] P_{n,0} = \lambda P_{n-1,0} + (n+1)(\mu_v + \xi) P_{n+1,0}, \text{ if } 1 \leq n \leq c-1, \tag{2}$$

$$[\lambda + \theta + c\mu_v + n\xi] P_{n,0} = \lambda P_{n-1,0} + [c\mu_v + (n+1)\xi] P_{n+1,0}, \text{ if } n \geq c, \tag{3}$$

$$(\lambda + \mu_b) P_{11} = \theta p_{1,0} + 2\mu_b P_{2,1}, \tag{4}$$

$$(\lambda + n\mu_b) P_{11} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}, \text{ if } 2 \leq n \leq c-1, \tag{5}$$

$$(\lambda + c\mu_b) P_{n,1} = \lambda P_{n-1,1} + c\mu_b P_{n+1,1} + \theta P_{n,0}, \text{ if } n \geq c. \tag{6}$$

Define the (partial) probability generating functions (PGF), for $0 < z < 1$,

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$P_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1},$$

with $P_0(1) + P_1(1) = 1$ and $P'_0(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,0}$.

Multiplying (2) and (3) with z^n and summing over n, we get the following differential equation after using (1) and rearrangement of terms.

$$\begin{aligned} 0 =& \xi z(1-z) P'_0(z) - [(1-z)(\lambda z - c\mu_v) + z\theta] P_0(z) + [z\theta - c\mu_v(1-z)] P_{0,0} \\ &+ z\mu_b P_{1,1} - \mu_v(1-z) \sum_{n=1}^{c} (n-c) P_{n,0} z^n, \end{aligned} \tag{7}$$

or

$$0 = P_0'(z) - \left[\frac{\lambda z - c\mu_v}{z\xi} + \frac{\theta}{\xi(1-z)}\right] P_0(z) + \left[\frac{\theta}{\xi(1-z)} - \frac{c\mu_v}{z\xi)}\right] P_{0,0} + \frac{\mu_b}{\xi(1-z)} P_{1,1} - \frac{\mu_v}{\xi z} \sum_{n=1}^{(n-c)} z^n P_{n,0}. \tag{8}$$

This is an ordinary linear differential equation with constant coefficients. To solve it, an integrating factor can be found as

$$I.F = e^{-\int\left[\frac{\lambda z - c\mu_v}{\xi z} + \frac{\theta}{\xi(1-z)}\right]dz} = e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}}.$$

Hence, the general solution to the differential equation (8) is given by

$$\frac{d}{dz}\left[e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}} P_0(z)\right] = \left\{-\left(\frac{\theta}{\xi(1-z)} - \frac{c\mu_v}{z\xi}\right) P_{0,0} - \frac{\mu_b}{\xi(1-z)} P_{1,1} + \frac{\mu_v}{\xi z} \sum_{n=1}^{(n-c)} z^n P_{n,0}\right\} e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}}.$$

Integrating from 0 to z, we get

$$P_0(z) = e^{\frac{\lambda z}{\xi}} z^{-\frac{c\mu_v}{\xi}} (1-z)^{-\frac{\theta}{\xi}} \left\{\frac{c\mu_v}{\xi} P_{0,0} \int_0^z e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}-1} (1-z)^{\frac{\theta}{\xi}} dz - \left[\frac{\theta}{\xi} P_{0,0} + \frac{\mu_b}{\xi} P_{1,1}\right] \int_0^z e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}} (1-z)^{\frac{\theta}{\xi}-1} dz + \frac{\mu_v}{\xi} \int_0^z Q_1(z) e^{-\frac{\lambda z}{\xi}} z^{\frac{c\mu_v}{\xi}-1} (1-z)^{\frac{\theta}{\xi}} dz\right\}, \tag{9}$$

where

$$Q_1(z) = \sum_{n=1}^{(n-c)} z^n P_{n,0}. \tag{10}$$

$$P_0(z) = z^{-\frac{c\mu_v}{\xi}} (1-z)^{-\frac{\theta}{\xi}} \left[\frac{c\mu_v}{\xi} P_{0,0} A(z) - \left(\frac{\theta}{\xi} P_{0,0} + \frac{\mu_b}{\xi} P_{1,1}\right) B(z) + \frac{\mu_v}{\xi} C(z)\right], \tag{11}$$

where

$$A(z) = \int_0^z e^{\frac{\lambda}{\xi}(z-X)} X^{\frac{c\mu_v}{\xi}-1} (1-X)^{\frac{\theta}{\xi}} dX, \tag{12}$$

$$B(z) = \int_0^z e^{\frac{\lambda}{\xi}(z-X)} X^{\frac{c\mu_v}{\xi}} (1-X)^{\frac{\theta}{\xi}-1} dX, \tag{13}$$

$$C(z) = \int_0^z Q_1(X) e^{\frac{\lambda}{\xi}(z-X)} X^{\frac{c\mu_v}{\xi}-1} (1-X)^{\frac{\theta}{\xi}} dX. \tag{14}$$

Now, determine $P_0(z)$ for $\lim_{z\to1}$ which gives

$$\lim_{z\to1} P_0(z) = P_0(1) = \left[\frac{c\mu_v}{\xi}P_{0,0}A(1) - \left(\frac{\theta}{\xi}P_{0,0} + \frac{\mu_b}{\xi}P_{1,1}\right)B(1) + \frac{\mu_v}{\xi}C(1)\right]\lim_{z\to1}(1-z)^{\frac{-\theta}{\xi}}.$$

Since

$$0 \le P_0(1) = \sum_{n=0}^{\infty} P_{n,0} \le 1 \text{ and } \lim_{z\to1}(1-z)^{-\frac{\theta}{\xi}} = \infty,$$

we must have the term

$$\frac{c\mu_v}{\xi}P_{0,0}A(1) - \left(\frac{\theta}{\xi}P_{0,0} + \frac{\mu_b}{\xi}P_{1,1}\right)B(1) + \frac{\mu_v}{\xi}C(1) = 0,$$

or

$$P_{1,1} = \frac{\mu_v C(1)}{\mu_b B(1)} + \left[\frac{\mu_v}{\mu_b}\frac{A(1)}{B(1)} - \frac{\theta}{\mu_b}\right]P_{0,0}. \tag{15}$$

Hence, we have $P_{1,1}$ in terms of $P_{0,0}$.

Using (15) in (11), we get

$$\begin{aligned} P_0(z) = &\left\{\frac{c\mu_v}{\xi}\left[A(z) - \frac{A(1)}{cB(1)}B(z)\right]P_{0,0}\right\} z^{\frac{-c\mu_v}{\xi}}(1-z)^{-\frac{\theta}{\xi}} \\ &- \left\{\frac{\mu_v}{\xi}\left[\frac{C(1)}{B(1)}B(z) - C(z)\right]\right\} z^{\frac{-c\mu_v}{\xi}}(1-z)^{-\frac{\theta}{\xi}}. \end{aligned} \tag{16}$$

Now, we will find the generating function $P_1(z)$. Multiplying (5) and (6) by z^n and summing over n, we get after using (4),

$$(1-z)(\lambda z - c\mu_b)P_1(z) = z\theta P_0(z) - (\theta P_{0,0} + \mu_b P_{1,1})z + \mu_b(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \tag{17}$$

Using (14) in (16), we get

$$P_1(z) = \left[\frac{z\theta}{c\mu_b}P_0(z) - \frac{\mu_v}{c\mu_b}\frac{C(1)}{B(1)}z - \frac{\mu_v}{c\mu_b}\frac{A(1)}{B(1)}zP_{0,0} + \frac{(1-z)Q(z)}{c}\right](z-1)^{-1}(1-\rho z)^{-1}, \tag{18}$$

where

$$Q(z) = \sum_{n=1}^{(n-c)} P_{n,1}z^n. \tag{19}$$

Theorem 3.1.

If $\rho < 1$ and $0 < \xi < \mu_v$, then the probability $P_{0,0}$ is given by

$$\begin{aligned} P_{0,0} = &\frac{(\theta+\xi)\left[(\mu_b c - \lambda) - \mu_b Q(1)\right] - \mu_v\theta Q_1(1)}{c\mu_v\theta + \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}\left[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)\right]} \\ &- \frac{\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}\left[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)\right]}{c\mu_v\theta + \frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}\left[\theta(\lambda - c\mu_v) + (\theta+\xi)(\mu_b c - \lambda)\right]}. \end{aligned} \tag{20}$$

Proof:

Applying L'Hopital's rule in (11), we get

$$\begin{aligned}
\lim_{z\to 1} P_0(z) &= \lim_{z\to 1}\left\{\frac{\frac{c\mu_v}{\xi}P_{0,0}A(z)-\left(\frac{\theta}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right)B(z)+\frac{\mu_v}{\xi}C(z)}{z^{\frac{c\mu_v}{\xi}}(1-z)^{\frac{\theta}{\xi}}}\right\}\\
&= \lim_{z\to 1}\left\{\frac{\frac{c\mu_v}{\xi}P_{0,0}z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}-\left[\frac{\theta}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right]z^{\frac{c\mu_v}{\xi}}(1-z)^{\frac{\theta}{\xi}-1}}{\frac{c\mu_v}{\xi}z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}-\frac{\theta}{\xi}z^{\frac{c\mu_v}{\xi}}(1-z)^{\frac{\theta}{\xi}-1}}\right\}\\
&+ \lim_{z\to 1}\left\{\frac{\frac{\mu_v}{\xi}Q_1(z)z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}}{\frac{c\mu_v}{\xi}z^{\frac{c\mu_v}{\xi}-1}(1-z)^{\frac{\theta}{\xi}}-\frac{\theta}{\xi}z^{\frac{c\mu_v}{\xi}}(1-z)^{\frac{\theta}{\xi}-1}}\right\}\\
&= \lim_{z\to 1}\left\{\frac{\frac{c\mu_v}{\xi}P_{0,0}z^{-1}-\left[\frac{\theta}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right](1-z)^{-1}+\frac{\mu_v}{\xi}Q_1(z)z^{-1}}{\frac{c\mu_v}{\xi}z^{-1}-\frac{\theta}{\xi}(1-z)^{-1}}\right\}\\
&= \lim_{z\to 1}\left\{\frac{\frac{c\mu_v}{\xi}P_{0,0}(1-z)-\left[\frac{\theta}{\xi}P_{0,0}+\frac{\mu_b}{\xi}P_{1,1}\right]z+\frac{\mu_v}{\xi}Q_1(z)(1-z)}{\frac{c\mu_v}{\xi}(1-z)-\frac{\theta}{\xi}z}\right\}, \qquad (21)
\end{aligned}$$

which gives

$$\theta P_0(1) = \theta P_{0,0} + \mu_b P_{1,1}. \qquad (22)$$

Similarly from (16), we get

$$\begin{aligned}
\lim_{z\to 1} P_1(z) &= \lim_{z\to 1}\left[\frac{\theta P_0(z)+z\theta P_0'(z)-(\theta P_{0,0}+\mu_b P_{1,1})}{\lambda(1-z)-(\lambda z-c\mu_b)}\right]-\frac{\mu_b}{\lambda-c\mu_b Q}(1)\\
&= \frac{\theta P'_0(1)}{c\mu_b-\lambda}+\frac{\mu_b}{c\mu_b-\lambda}Q(1),
\end{aligned}$$

which implies that

$$P'_0(1) = \left(\frac{c\mu_b-\lambda}{\theta}\right)P_1(1)-\frac{\mu_b}{\theta}Q(1). \qquad (23)$$

From (7), we get

$$P'_0(z) = \frac{[(1-z)(\lambda z-c\mu_v+z\theta]P_0(z)-[z\theta-c\mu_v(1-z)]-z\mu_b P_{1,1}}{\xi z(1-z)}+\frac{\mu_v}{\xi z}Q_1(z).$$

Applying L'Hopital's rule, we get

$$\begin{aligned}
P'_0(1) &= \lim_{z\to 1}\left\{\frac{[\lambda(1-\lambda z)+c\mu_v+\theta]\,P_0(z)+P_0'(z)\,[(1-z)\lambda z-c\mu_v+z\theta]-(\theta+c\mu_v)P_{0,0}-\mu_b P_{1,1}}{\xi(1-\lambda z)}\right\}\\
&+\frac{\mu_v}{\xi}Q_1(1).
\end{aligned}$$

Therefore, using (21), we get

$$P_0'(1) = \frac{(\lambda-c\mu_v)P_0(1)+c\mu_v P_{0,0}+\mu_v Q_1(1)}{\theta+\xi}. \qquad (24)$$

From (22) and (23), we have

$$\left(\frac{c\mu_b-\lambda}{\theta}\right)P_1(1)-\frac{\mu_b}{\theta}Q(1)=\frac{(\lambda-c\mu_v)P_0(1)+c\mu_v P_{0,0}+\mu_v Q_1(1)}{\theta+\xi},$$

which gives

$$[\theta(\lambda-c\mu_v)+(\theta+\xi)(\mu_b c-\lambda)]\left[P_{0,0}+\frac{\mu_b}{\theta}P_{1,1}\right]+c\theta\mu_v P_{0,0}$$
$$=(\theta+\xi)\left[(\mu_b c-\lambda)-\mu_b Q(1)\right]-\mu_v\theta Q_1(1).$$

Putting $P_{1,1}$ in terms of $P_{0,0}$, we get the expression for $P_{0,0}$ as

$$\begin{aligned}P_{0,0}=&\frac{(\theta+\xi)\left[(\mu_b c-\lambda)-\mu_b Q(1)\right]-\mu_v\theta Q_1(1)}{c\mu_v\theta+\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}\left[\theta(\lambda-c\mu_v)+(\theta+\xi)(\mu_b c-\lambda)\right]}\\&-\frac{\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}\left[\theta(\lambda-c\mu_v)+(\theta+\xi)(\mu_b c-\lambda)\right]}{c\mu_v\theta+\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}\left[\theta(\lambda-c\mu_v)+(\theta+\xi)(\mu_b c-\lambda)\right]}.\end{aligned}\tag{25}$$

This completes the proof. ■

4. Performance Measures

From (21), the equilibrium probability that the system is in working vacation is

$$P_0(1)=\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}+\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0},\tag{26}$$

and the probability that the system is in nonvacation period is

$$P_1(1)=1-\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}-\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}.\tag{27}$$

The mean number of customers when the system is on WV vacation period is

$$E(N_0)=P_0(1)=\left(\frac{c\mu_b-\lambda}{\theta}\right)\left[1-\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}-\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right]-\frac{\mu_b}{\theta}Q(1),\tag{28}$$

and the expected number of customers when the server is on non vacation period is

$$\begin{aligned}E(N_1)&=P'_1(1)=\frac{\theta}{\mu_b}E(N_0)\\&=\left(\frac{c\mu_b-\lambda}{\mu_b}\right)\left[1-\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}-\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right]-Q(1).\end{aligned}\tag{29}$$

Hence, the steady state mean number of customers in the system is

$$\begin{aligned}E(N)&=E(N_0)+E(N_1)\\&=(c\mu_b-\lambda)\left[\frac{1}{\theta}+\frac{1}{\mu_b}\right]\left[1-\frac{\mu_v}{\theta}\frac{C(1)}{B(1)}-\frac{c\mu_v}{\theta}\frac{A(1)}{B(1)}P_{0,0}\right]-\left(1+\frac{\mu_b}{\theta}\right)Q(1).\end{aligned}\tag{30}$$

The mean waiting time in the system can be derived by using the Little formula as

$$E(W) = \frac{E(N)}{\lambda}.$$

From the moment of arrival of a customer in the system until departure, the waiting time W of a customer is measured, either after completion of service or due to abandonment. However, a more important measure of system performance is the total waiting time of a customer who actually completes his service before leaving the system. Let us denote this by W_{served}.

Let $W_{n,j}$ denote the conditional waiting time of a customer who does not abandon the system, given that the system upon his arrival is (n, j). Clearly, we have

$$E(W_{n,1}) = \frac{n+1}{\mu_b}, \quad n = 1, 2, 3, ... \tag{31}$$

We derive $E(W_{n,0})$ by using the method of Altman and Yechiali (2006).

When $j = 0$ and $n \geq 1$,

$$\begin{aligned} E(W_{n,0}) = & \frac{\theta}{\theta+\lambda+(n+1)(\mu_v+\xi)}\left[\frac{1}{\theta+\lambda+(n+1)(\mu_v+\xi)} + E(W_{n,1})\right] \\ & + \frac{\lambda}{\theta+\lambda+(n+1)(\mu_v+\xi)}\left[\frac{1}{\theta+\lambda+(n+1)(\mu_v+\xi)} + E(W_{n,0})\right] \\ & + \frac{n\xi}{\theta+\lambda+(n+1)(\mu_v+\xi)}\left[\frac{1}{\theta+\lambda+(n+1)(\mu_v+\xi)} + E(W_{n-1,0})\right] \\ & + \frac{n\mu_v}{\theta+\lambda+(n+1)(\mu_v+\xi)}\left[\frac{1}{\theta+\lambda+(n+1)(\mu_v+\xi)} + E(W_{n-1,0})\right]. \end{aligned}$$

The second term above follows from the fact that a new arrival does not change the waiting time of a customer present in the system, while the third term takes into consideration that only n customers can abandon the system as our customer is not impatient.

$$\begin{aligned} E(W_{n,0}) = & \frac{1}{\theta+(n+1)(\mu_v+\xi)} \times \\ & \left\{\frac{\theta+\lambda+n(\mu_v+\xi)}{\theta+\lambda+(n+1)(\mu_v+\xi)} + \frac{(n+1)\theta}{\mu_b} + n(\mu_v+\xi)E(W_{n-1,0})\right\}. \end{aligned} \tag{32}$$

For $j = 0$ and $n = 0$,

$$\begin{aligned} E(W_{0,0}) = & \frac{\theta}{\theta+\lambda+\mu_v+\xi}\left[\frac{1}{\theta+\lambda+\mu_v+\xi} + \frac{1}{\mu_b}\right] \\ & + \frac{\lambda}{\theta+\lambda+\mu_v+\xi}\left[\frac{1}{\theta+\lambda+\mu_v+\xi} + E(W_{0,0})\right], \end{aligned}$$

which can be simplified to

$$E(W_{0,0}) = \frac{1}{\theta+\mu_v+\xi}\left[\frac{\theta+\lambda}{\theta+\lambda+\mu_v+\xi} + \frac{\theta}{\mu_b}\right]. \tag{33}$$

Using (32) and iterating (31), we obtain for $n \geq 0$

$$E(W_{n,0}) = \frac{1}{\theta + (n+1)(\mu_v + \xi)} \left\{ \frac{\theta + \lambda + n(\mu_v + \xi)}{\theta + \lambda + (n+1)(\mu_v + \xi)} + \frac{(n+1)\theta}{\mu_b} + \sum_{k=1}^{n} \left[\frac{\theta + \lambda + (k-1)(\mu_v + \xi)}{\theta + \lambda + k(\mu_v + \xi)} + \frac{k\theta}{\mu_b} \prod_{i=k}^{n} \frac{i(\mu_v + \xi)}{\theta + i(\mu_v + \xi)} \right] \right\}.$$

Finally, we get the mean waiting time of customers served by the system as

$$E(W_{served}) = \sum_{n=0}^{\infty} P_{n,0} E(W_{n,0}) + \sum_{n=1}^{\infty} P_{n,1} E(W_{n,1}),$$

which after using (30) becomes

$$E(W_{served}) = \sum_{n=0}^{\infty} P_{n,0} E(W_{n,0}) + \frac{E(N_1) + P_1(1)}{\mu_b}. \tag{34}$$

5. Stochastic Decompositions in the MWV Model

In order to have a better comparison with the already existing models, we often try to decompose the quantities of interest into various factors. A stochastic decomposition property plays a vital role in vacation queuing models and points out the effects of system vacation on its performance measures like queue length and waiting times. We can try to do the same for the system under consideration as this is an $M/M/c$ model with MWVs. We have derived the following two results that this impatient WV model under consideration also satisfies such decomposition for stationary queue length and stationary waiting times into one that corresponds to a pure $M/M/1$ system plus a quantity that is due to the WV.

Theorem 5.1.

For $\rho < 1$, the stationary queue length N can be decomposed into a sum of two independent random variables as $N = N_c + N_d$, where N_c is the queue length of a classical M/M/c queue with vacation and N_c is the additional queue length due to effect of MMV with its PGF

$$N_d(z) = \frac{P_{0,0}}{(c\mu_b - \lambda)(1-z)} \times \left\{ \left[\frac{(c\mu_b - z\lambda)(1-z) - z\theta}{P_{0,0}} \right] P_0(z) + z \left[\frac{\mu_v C(1)}{P_{0,0} B(1)} - \frac{c\mu_v A(1)}{B(1)} \right] + \frac{\mu_b (1-z)}{P_{0,0}} Q(z) \right\}.$$

Proof:

$$\begin{aligned}
N(z) &= P_0(z) + P_1(z) \\
&= \left[1 + \frac{z\theta}{(1-z)(\lambda z - \mu_b)}\right] P_0(z) - \frac{z\left[\frac{\mu_v C(1)}{B(1)} + C\mu_v \frac{A(1)}{B(1)} P_{0,0}\right] - \mu_b(1-z)Q(z)}{(1-z)(\lambda z - c\mu_b)} \\
&= \left(\frac{c\mu_b - \lambda}{c\mu_b - \lambda z}\right)\left\{\left[\frac{c\mu_b - \lambda z}{c\mu_b - \lambda} - \frac{\theta z}{(c\mu_b - \lambda)(1-z)}\right] P_0(z)\right. \\
&\qquad \left. + \frac{z}{(1-z)(c\mu_b - \lambda)}\left[\frac{\mu_v c(1)}{B(1)} + c\mu_v \frac{A(1)}{B(1)} P_{0,0}\right] + \frac{\mu_b}{c\mu_b - \lambda} Q(z)\right\}.
\end{aligned}$$

or

$$N(z) = \frac{1-\rho}{1-\rho z} \times N_d(z).$$

$P_0(z)$ and $P_1(z)$ are positive, as they are PGFs, and so $P_0(z)+P_1(z) > 0$, and for $0 < z < 1$ and $\rho < 1$, $\left(\frac{1-\rho}{1-\rho z}\right) > 0$. Therefore, $N_d(z)$ is positive. Also, for $z = 1$, then $N_d(1) = 1$. Hence, $N_d(z)$ is a PGF. ∎

Theorem 5.2.

If $\rho < 1$, the stationary waiting time can be decomposed into a sum of two independent random variable as $W = W_c + W_d$, Where W_c is the waiting time of a customer corresponding to classical $M/M/c$ queue and has exponential distribution with parameter $\mu_b(1-\rho)$ and W_d is the additional delay due to the effect of MWV with its Laplace Stieltjes transform (LST).

$$\begin{aligned}
W_d^*(s) = \frac{P_{0,0}}{(c\mu_b - \lambda)s}&\left\{\left[\frac{(c\mu_b + s - \lambda)s - \theta(\lambda - s)}{P_{0,0}}\right] P_0\left(1 - \frac{s}{\lambda}\right)\right. \\
&\left. + (\lambda - s)\left[\frac{\mu_v}{P_{0,0}}\frac{C(1)}{B(1)} - \frac{c\mu_v A(1)}{B(1)}\right] - \frac{\mu_b s P_{0,0}}{Q}\left(1 - \frac{s}{\lambda}\right)\right\}.
\end{aligned} \tag{35}$$

Proof:

From the distributional form of Little's law, in Keilson and Servi (1988), we have the relation

$$N(z) = W^*(\lambda(1-z)).$$

Let $S = \lambda(1-z)$, which gives $z = (1 - \frac{S}{\lambda})$ and $1 - z = \frac{S}{\lambda}$. Putting these values in (36), we get the desired expression. ∎

6. Conclusion

Many authors in the past have studied queuing models with impatient customers, where the cause of impatience was considered to be either a heavy load of the server leading to a long wait that will be experienced by the customer already in a queue or due to anticipation of impatience by a customer upon arrival. We analyzed in this paper the impatient behavior of customers in an $M/M/c$ with multiple working vacation, where customers' impatience is due to slow service rate during a working vacation. We have derived the explicit expressions for the probability generating functions of the number of customers in the system, when the system is in a regular busy period and in a working vacation period. We have calculated the values of important performance measures like the mean queue length and mean waiting time and stochastic decomposition properties are also verified.

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