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G. Kannadasan
Annamalai University

N. Sathiyamoorthi
Annamalai University

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The $FM^X/FM/1$ Queue with Multiple Working Vacation

[1]G. Kannadasan and [2]N. Sathiyamoorthi

Department of Mathematics
Annamalai University
Annamalainagar - 608 002, India
[1]klsk.g.21@gmail.com, [2]n.satyamurthi@gmail.com

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Abstract

This study investigates the batch arrival $FM^X/FM/1$ queue with multiple working vacation. For this fuzzy queuing model, this research obtains some performance measure of interest such as mean system length, mean system sojourn time, mean busy period for the server and working vacation period. Finally, numerical results are presented to show the effects of system parameters.

Keywords: $FM^X/FM/1$ queue; Multiple working vacation; Busy period; Length of system; Sojourn time

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1. Introduction

Vacation queues have been investigated for four decades as a useful tool for modeling and analyzing computer systems, communication networks, manufacturing and production systems. However, in these models, the server stops the original work in the vacation period and does not come back to the regular busy period until the vacation period ends.

Recently, Servi and Finn (2002) introduced working vacation policy, in which the server works at a different rate rather than completely stopping service during the vacation. In the working vacation queuing systems, the server can return back to the regular busy period even if the vacation period is not completed.

The basic queuing model of this research article is the $M^X/M/1$ queue with multiple working vacation customers arriving in batches according to a Poisson process. Service time in regular busy periods follow an exponential distribution. During the vacation period, arriving customers are served, and when a vacation ends, if there are no customers in the queue, another vacation is taken. Otherwise, the server switches service and a regular busy period starts. But, in many real situations, the system parameters are both possibilistic and probabilistic. Thus fuzzy analysis would be potentially much more useful and realistic than the commonly used crisp concepts.

Li and Lee has (1989) investigated analytical results for fuzzy queues using a general approach based on Zadeh's extension principle (1978). Negi and Lee's (1999) home inventory procedure use α-cut and two-variable simulation to analyze fuzzy queues. Using parametric programming Kao et al. (1999) constructed the membership functions of systems characteristic for fuzzy queues.

2. The Crisp Model

This study considers a queuing system in which the customers arrive in a group in a Poisson fashion with multiple working vacations. The arrive rate of a batch is specified by λ. It assumes that the arrival batch size X follows a geometric distribution with parameter q; that is,

$$P(X = k) = g_k,$$

where $0 \leq q \leq 1$, $k = 1, 2, ...,$ and $g = \dfrac{1}{q}$, $g^{(2)} = \dfrac{2-q}{q^2}$, then

$$G(z) = \textstyle\sum_{k=1}^{\infty} g_k z^k, \quad |z| \leq 1.$$

The customers are served using First-Come First-Service queuing discipline. Service time follows an exponential distribution with parameter μ. Upon completion of service, if there is no customer in the system, the server begins a vacation and the vacation duration follows an exponential distribution with parameter θ. During the working vacation period, arriving customers are served at a rate ν. When the vacation ends, if there are no customers in the queue, the server takes another vacation; otherwise the service switches service rate from ν to μ, and a regular busy period starts. The Laplace-Stieltjes transform of the service time distribution in a regular busy period and the service time in a working vacation time are

$$B^*(s) = \frac{\mu}{s+\mu}, \quad C^*(s) = \frac{v}{s+\nu},$$

respectively. Obviously, the numerator and the denominator of the expressions are both positive since $\nu < \mu$ and

$$\rho = \frac{\lambda g}{\mu} < 1.$$

Furthermore, we have

$$\pi_{00} = \frac{\mu(1-\rho)}{\delta}, \qquad \delta = \mu + \frac{\lambda}{\theta}\{1 - G(\alpha)\}(\mu - \nu).$$

Let $L(t)$ be the number of customers in the system at time t. Then,

$$J(t) = \begin{cases} 0, \text{ the system is in a working vacation period at time } t, \\ 1, \text{ the system is in a regular busy period at time } t. \end{cases}$$

Then the process $\{L(t), J(t)\}$ is a two-dimensional Markov chain with the state space

$$\Omega = \{(0,0)\} \cup \{(k,j)/k \geq 1, j = 0,1\}.$$

Using the lexicographical order for the states, the infinitesimal generator of the process is $\{L(t), J(t)\}$. Then,

$$Q = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \dots \\ C_0 & A_1 & A_2 & A_3 & \dots \\ & A_0 & A_1 & A_2 & \dots \\ & & A_0 & A_1 & \dots \\ & & \vdots & \vdots & \dots \end{bmatrix},$$

where

$$B_0 = -\lambda,\, B_i = (\lambda g_i, 0), i \geq 1,\, C_0 = (\nu, \mu)^T,$$

$$A_0 = \begin{bmatrix} \upsilon & 0 \\ 0 & \mu \end{bmatrix}, A_1 = \begin{bmatrix} -(\lambda+\nu+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{bmatrix},$$

$$A_i = \begin{bmatrix} \lambda g_{i-1} & 0 \\ 0 & \lambda g_{i-1} \end{bmatrix}, i \geq 2.$$

We derive the probability generating function stationary distribution for $\{L(t), J(t)\}$. Let (L, J) be the stationary limit of the process $\{L(t), J(t)\}$.

It assumes that:

$$\pi_k = (\pi_{k0}, \pi_{k1}), k \geq 1,$$

$$\begin{aligned} \pi_{kj} &= P(L = k, J = j) \\ &= \lim_{t\to\infty} P\{L(t) = k, J(t) = j\}, \quad (k,j) \in \Omega. \end{aligned}$$

The probability generating function of $\{\pi_{k0}\}$ and $\{\pi_{k1}\}$ are

$$Q_0(z) = \sum_{k=1}^{\infty} \pi_{k0} z^k, \|z\| \leq 1,$$

$$Q_1(z) = \sum_{k=1}^{\infty} \pi_{k1} z^k, \|z\| \leq 1.$$

Then the probability generating function of a stationary system of length L can be written as

$$L(z) = \pi_{00} + Q_0(z) + Q_1(z), \|z\| \leq 1,$$

$$L(z) = \frac{\pi(1-\rho)}{\delta} + \sum_{k=1}^{\infty} \pi_{k0} z^k + \sum_{k=1}^{\infty} \pi_{k1} z^k, \|z\| \leq 1,$$

$L = L_0 + L_d$, where L_0 is the stationary system length in the corresponding classical $M^X/M/1$ queue without vacation, and L_d is the additional system length due to vacation.

Now

$$L(z) = L_0(z) + L_d(z),$$

$$L_0(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda+\mu)z - \mu - \lambda z G(z)},$$

$$L_d(z) = \frac{\sigma(z)}{\delta\{(\lambda+\nu+\theta)z - \nu - \lambda z G(z)\}},$$

where

$$\sigma(z) = \mu(\lambda+\nu+\theta)z - \nu - \lambda z G(z) + \lambda z\{G(z) - G(\alpha)\}(\mu-\nu),$$

and

$$\delta = \mu + \tfrac{\lambda}{\theta}\{1 - G(\alpha)\}(\mu - \nu).$$

Thus, the mean length in the system is given by

$$L = \frac{\mu(1-\rho)(z-1)}{(\lambda+\mu)z - \mu - \lambda z G(z)} + \frac{\sigma(z)}{\delta\{(\lambda+\nu+\theta)z + \nu - \lambda z G(z)\}}.$$

Therefore,

$$E(L) = \frac{\lambda(g^{(2)} + g)}{2\mu(1-\rho)} + \frac{\lambda(\mu-\nu)[g\theta + (\lambda g - \nu)]\{1 - G(\alpha)\}}{\delta\theta^2}.$$

Hence,

$$E(L) = \frac{\lambda}{q}\left[\tfrac{2}{(\mu q - \lambda)} + \frac{(\mu-\nu)[\theta T_1 + (\mu - q\nu)T_2]}{\theta[\mu\theta T_1 + \lambda T_2(\mu-\nu)]}\right],$$

where $T_1 = 1 - \alpha + q\alpha$, $T_2 = 1 - \alpha$. The probability $P(J=0)$, that the server is in a working vacation period, and the probability $P(J=1)$, that the server is in a regular busy period, are given by

$$V = P(J = 0) = \frac{(\mu q - \lambda)[\theta T_1 + \lambda T_2]}{q[\theta \mu T_1 + \lambda T_2(\mu - \nu)]},$$

and

$$B = P(J = 1) = \frac{\lambda \theta T_1 + (\lambda - q\nu) T_2}{q[\mu \theta T_1 + \lambda(\mu - \nu) T_2]}.$$

We can obtain the Laplace-Stieltjes transform of the stationary sojourn time of an arbitrary customer. Let W and $W^*(s)$ denote the stationary sojourn time of an arbitrary customer and its LST, respectively.

If $\rho = \dfrac{\lambda g}{\mu} < 1$ and $\nu < \mu$, $W^*(s)$ is given by

$$\begin{aligned} W^*(s) = &\left[\frac{\pi_{00} + Q_0(B^*(s))\mu_\theta}{(\mu - \nu)s + \mu\theta} + Q_1(B^*(s))\right] \frac{\{1 - G(B^*(s))\}}{g\{1 - B^*(s)\}} \\ &+ \frac{(\mu - \nu)s\{\pi_{00} + Q_0(C^*(s + \theta))\}\{1 - G(C^*(s + \theta))\}}{g\{(\mu - \nu)s + \mu\theta\}\{1 - C^*(s + \theta)\}}. \end{aligned}$$

3. The Model in Fuzzy Environment

In this section, the arrival rate, the service rate, working vacation rate and mean busy period are assumed to be fuzzy number $\bar{\lambda}$, $\bar{\mu}$, $\bar{\theta}$, and $\bar{\beta}$, respectively. Now

$$\begin{aligned} \bar{\lambda} &= \left\{x, \mu_{\bar{\lambda}}(x); x \in S(\bar{\lambda})\right\}, \\ \bar{\mu} &= \left\{y, \mu_{\bar{\mu}}(y); y \in S(\bar{\mu})\right\}, \\ \bar{\theta} &= \left\{z, \mu_{\bar{\theta}}(z); z \in S(\bar{\theta})\right\}, \text{ and} \\ \bar{\beta} &= \left\{s, \mu_{\bar{\beta}}(s); s \in S(\bar{\beta})\right\}, \end{aligned}$$

where $S(\bar{\lambda})$, $S(\bar{\mu})$, $S(\bar{\theta})$ and $S(\bar{\beta})$ are the universal sets of the arrival rate, the service rate, busy period, and working vacation period, respectively. It defines $f(x, y, z, s)$ as the system performance measure related to the above defined fuzzy queuing model, which depends on the fuzzy membership function $\bar{f}(\bar{\lambda}, \bar{\mu}, \bar{\theta}, \bar{\beta})$. Applying Zadeh's extension principle (1978) the membership function of the performance measure $\bar{f}(\bar{\lambda}, \bar{\mu}, \bar{\theta}, \bar{\beta})$ can be defined as

$$\mu_{\bar{f}(\bar{\lambda},\bar{\mu},\bar{\theta},\bar{\beta})}(D) = \sup_{\substack{x \in S(\bar{\lambda}) \\ y \in S(\bar{\mu}) \\ z \in S(\bar{\theta}) \\ s \in S(\bar{\beta})}} \left\{\mu_{\bar{\lambda}}(x), \mu_{\bar{\mu}}(y), \mu_{\bar{\theta}}(z), \mu_{\bar{\beta}}(s) / D = f(x, y, z, s).\right\} \tag{A}$$

If the α-cuts of $\bar{f}(\bar{\lambda}, \bar{\mu}, \bar{\theta}, \bar{\beta})$ degenerates to some fixed value, then the system performance is a crisp number. Otherwise it is a fuzzy number.

$$\begin{aligned} E(W) &= \frac{E(L)}{\lambda g} - \frac{(1 - \pi_{00})}{\lambda g} \\ &= \left[\frac{2}{(\mu q - \lambda)} + \frac{(\mu - \nu)[\theta T_1 + (\mu - q\nu)T_2]}{\theta[\mu\theta T_1 + \lambda T_2(\mu - \nu)]}\right] - \frac{1}{\lambda}\left[\frac{q(\mu\theta T_1) + \lambda T_2(\mu - \nu) - \theta T_1(\mu q - \lambda)}{\lambda\theta\mu T_1 + \lambda T_2(\mu - \nu)}\right]. \end{aligned}$$

Under the stedy state condition $\rho = \dfrac{\lambda g}{\mu} < 1$.

We obtain the membership function some performance measures, namely the mean number of customer in the system $E(L)$, the mean sojourn time in the system $E(W)$, the busy period in the system B, and the working vacation period in the system V. For the system in terms of this membership function are

$$\mu_{\overline{E(L)}}(K) = \sup_{\substack{x\in S(\bar\lambda)\\ y\in S(\bar\mu)\\ z\in S(\bar\theta)\\ t\in S(\bar\beta)}} \{\min\{\mu_{\bar\lambda}(x), \mu_{\bar\mu}(y), \mu_{\bar\theta}(z), \mu_{\bar\beta}(s)/K\},$$

where

$$K = \frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right],$$

$$\mu_{\overline{E(W)}}(M) = \sup_{\substack{x\in S(\bar\lambda)\\ y\in S(\bar\mu)\\ z\in S(\bar\theta)\\ t\in S(\bar\beta)}} \{\min\{\mu_{\bar\lambda}(x), \mu_{\bar\mu}(y), \mu_{\bar\theta}(z), \mu_{\bar\theta}(s)/M\}\},$$

where

$$M = \left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right] - \frac{1}{x}\left[\frac{q(yzT_1) + xT_2(y-s) - zT_1(yq-x)}{zyT_1 + xT_2(y-s)}\right],$$

$$\mu_{\overline{B}}(N) = \sup_{\substack{x\in S(\bar\lambda)\\ y\in S(\bar\mu)\\ z\in S(\bar\theta)\\ t\in S(\bar\beta)}} \{\min\{\mu_{\bar\lambda}(x), \mu_{\bar\mu}(y), \mu_{\bar\theta}(z), \mu_{\bar\beta}(s)/N\}\},$$

where

$$N = \frac{(yq-x)[zT_1 + xT_2]}{q[zyT_1 + xT_2(y-s)]},$$

$$\mu_{\overline{V}}(O) = \sup_{\substack{x\in S(\bar\lambda)\\ y\in S(\bar\mu)\\ z\in S(\bar\theta)\\ t\in S(\bar\beta)}} \{\min\{\mu_{\bar\lambda}(x), \mu_{\bar\mu}(y), \mu_{\bar\theta}(z), \mu_{\bar\beta}(s)/O\}\},$$

where

$$O = \frac{xzT_1 + (x-qs)T_2}{q[yzT_1 + x(y-s)T_2]}.$$

Using the fuzzy analysis technique explain, we can find the membership of $\overline{E(L)}, \overline{E(W)}$ and the system is in a working vacation period and the probability that the system is in a regular busy period $\overline{V}$ and $\overline{B}$ as a function of the parameter α. Thus the α-cut approach can be used to develop the membership function of $\overline{E(L)}$, $\overline{E(W)}$, $\overline{V}$ and $\overline{B}$.

4. Performance Measure of Interest

The following performance measures are studied for this model in fuzzy environment.

The Membership Function of the Mean System Length

Based on Zadeh's extension principle, $\mu_{E(L)}(K)$ is the supermum of minimum over $\{\mu_{\bar{\lambda}}(x), \mu_{\bar{\mu}}(y), \mu_{\bar{\theta}}(z), \mu_{\bar{\beta}}(s)\}$,

$$K = \frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right], \tag{1}$$

to satisfying $\mu_{\overline{E(L)}}(K) = \alpha$, $0 < \alpha \leq 1$.

The following four cases arise:

Case(i): $\mu_{\bar{\lambda}}(x) = \alpha, \mu_{\bar{\mu}}(y) \geq \alpha, \mu_{\bar{\theta}}(z) \geq \alpha, \mu_{\bar{\beta}}(s) \geq \alpha$,
Case(ii): $\mu_{\bar{\lambda}}(x) \geq \alpha, \mu_{\bar{\mu}}(y) = \alpha, \mu_{\bar{\theta}}(z) \geq \alpha, \mu_{\bar{\beta}}(s) \geq \alpha$,
Case(iii): $\mu_{\bar{\lambda}}(x) \geq \alpha, \mu_{\bar{\mu}}(y) \geq \alpha, \mu_{\bar{\theta}}(z) = \alpha, \mu_{\bar{\beta}}(s) \geq \alpha$,
Case(iv): $\mu_{\bar{\lambda}}(x) \geq \alpha, \mu_{\bar{\mu}}(y) \geq \alpha, \mu_{\bar{\theta}}(z) \geq \alpha, \mu_{\bar{\beta}}(s) = \alpha$.

For Case (i), the lower and upper bound of α-cuts of $\overline{E(L)}$ can be obtained through the corresponding parametric non-linear programs,

$$[\overline{E(L)}]_\alpha^{L_1} = \min_{\Omega}\left\{\frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right]\right\},$$

$$[\overline{E(L)}]_\alpha^{U_1} = \max_{\Omega}\left\{\frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right]\right\}.$$

Similarly, we can calculate the lower and upper bounds of the α-cuts of $\overline{E(L)}$ for Cases (ii), (iii) and (iv). By considering the cases, simultaneously the lower and upper bounds of the α-cuts of $\overline{E(L)}$ can be written as

$$[\overline{E(L)}]_\alpha^{L} = \min_{\Omega}\left\{\frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right]\right\},$$

$$[\overline{E(L)}]_\alpha^{U} = \max_{\Omega}\left\{\frac{x}{q}\left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1 + (y-qs)T_2]}{z[yzT_1 + xT_2(y-s)]}\right]\right\},$$

such that

$$x_\alpha^L \leq x \leq x_\alpha^U, y_\alpha^L \leq y \leq y_\alpha^U, z_\alpha^L \leq z \leq z_\alpha^U, \text{ and } s_\alpha^L \leq s \leq s_\alpha^U.$$

If both $(\overline{E(L)})_\alpha^L$ and $(\overline{E(L)})_\alpha^U$ are invertible with respected to α, the left and right shape function,

$$L(K) = [(E(L))_\alpha^L]^{-1} \text{ and } R(K) = [(E(L))_\alpha^U]^{-1},$$

can be derived from which the membership function $\mu_{\overline{E(L)}}(K)$ can be considered as

$$\mu_{\overline{E(L)}}(K) = \begin{cases} L(K), & (E(L))^L_{\alpha=0} \leq K \leq (E(L))^U_{\alpha=0} \\ 1, & (E(L))^L_{\alpha=1} \leq K \leq (E(L))^U_{\alpha=1}. \\ R(D), & (E(L))^L_{\alpha=1} \leq K \leq (E(L))^U_{\alpha=0} \end{cases}$$

Membership Function of Mean System Sojourn Time

Similarly, we can calculate the lower and upper bounds of the α-cuts of $E(W)$ as $\mu_{\overline{E(W)}}(M)$ which is the supremum over $\{\mu_{\bar{\lambda}}(x),\mu_{\bar{\mu}}(y),\mu_{\bar{\theta}}(z),\mu_{\bar{\beta}}(t)\}$,

$$M = \left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1+(y-qs)T_2]}{z[yzT_1+xT_2(y-s)]}\right] - \frac{1}{x}\left[\frac{q(yzT_1)+xT_2(y-s)-zT_1(yq-x)}{zyT_1+xT_2(y-s)}\right], \tag{2}$$

$$E(W)t^L_\alpha = \min_\Omega \left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1+(y-qs)T_2]}{z[yzT_1+xT_2(y-s)]}\right] - \frac{1}{x}\left[\frac{q(yzT_1)+xT_2(y-s)-zT_1(yq-x)}{zyT_1+xT_2(y-s)}\right], \tag{3}$$

$$E(W)t^U_\alpha = \max_\Omega \left[\frac{2}{(yq-x)} + \frac{(y-s)[zT_1+(y-qs)T_2]}{z[yzT_1+xT_2(y-s)]}\right] - \frac{1}{x}\left[\frac{q(yzT_1)+xT_2(y-s)-zT_1(yq-x)}{zyT_1+xT_2(y-s)}\right], \tag{4}$$

such that

$$x^L_\alpha \leq x \leq x^U_\alpha,\ y^L_\alpha \leq y \leq y^U_\alpha,\ z^L_\alpha \leq z \leq z^U_\alpha,\ \text{and}\ s^L_\alpha \leq s \leq s^U_\alpha.$$

If both $(\overline{E(W)})^L_\alpha$ and $(\overline{E(W)})^U_\alpha$ are invertible with respected to α then the left and right shape function,

$$L(M) = \left[[E(W)]^L_\alpha\right]^{-1} \text{ and } R(M) = \left[[E(W)]^L_\alpha\right]^{-1},$$

can be derived from the membership function as

$$\mu_{\overline{E(W)}}(M) = \begin{cases} L(M), & (E(W))^L_{\alpha=0} \leq M \leq (E(W))^U_{\alpha=0} \\ 1, & (E(W))^L_{\alpha=1} \leq M \leq (E(W))^U_{\alpha=1} \\ R(D), & (E(W))^L_{\alpha=1} \leq M \leq (E(W))^U_{\alpha=0} \end{cases} \tag{5}$$

Membership Function of Mean Busy Period for the Server

The lower and upper bounds of the α cuts of $B(N)$ can be written as,

$$[\overline{B(N)}]_{\alpha}^{L} = \min_{\Omega}\left\{\frac{(yq-x)[zT_1+xT_2]}{q[zyT_1+xT_2(y-s)]}\right\}, \tag{6}$$

$$[\overline{B(N)}]_{\alpha}^{U} = \max_{\Omega}\left\{\frac{(yq-x)[zT_1+xT_2]}{q[zyT_1+xT_2(y-s)]}\right\}, \tag{7}$$

such that

$$x_{\alpha}^{L} \leq x \leq x_{\alpha}^{U},\ y_{\alpha}^{L} \leq y \leq y_{\alpha}^{U},\ z_{\alpha}^{L} \leq z \leq z_{\alpha}^{U},\ \text{and}\ s_{\alpha}^{L} \leq s \leq s_{\alpha}^{U}.$$

If both $(\overline{B(N)})_{\alpha}^{L}$ and $(\overline{B(N)})_{\alpha}^{U}$ are invertible with respect to α then the right and left shape function,

$$L(N) = \left[[B(N)]_{\alpha}^{L}\right]^{-1} \text{ and } R(N) = \left[[B(N)]_{\alpha}^{U}\right]^{-1}$$

can be derived from the membership function as

$$\mu_{\overline{B}}(O) = \begin{cases} L(N), & (B(N))_{\alpha=0}^{L} \leq N \leq (B(N))_{\alpha=0}^{U} \\ 1, & (B(N))_{\alpha=1}^{L} \leq N \leq (B(N))_{\alpha=1}^{U} \\ R(N), & (B(N))_{\alpha=1}^{L} \leq N \leq (B(N))_{\alpha=0}^{U}. \end{cases} \tag{8}$$

The membership function of working vacation period

The lower and upper bounds of the α cuts of $V(O)$ can be written as

$$\left[\overline{V(O)}\right]_{\alpha}^{L} = \min_{\Omega}\left\{\frac{xzT_1+(x-qs)T_2}{q[yzT_1+x(y-s)T_2]}\right\}, \tag{9}$$

$$\left[\overline{V(O)}\right]_{\alpha}^{U} = \max_{\Omega}\left\{\frac{xzT_1+(x-qs)T_2}{q[yzT_1+x(y-s)T_2]}\right\}, \tag{10}$$

such that

$$x_{\alpha}^{L} \leq x \leq x_{\alpha}^{U},\ y_{\alpha}^{L} \leq y \leq y_{\alpha}^{U},\ z_{\alpha}^{L} \leq z \leq z_{\alpha}^{U},\ \text{and}\ s_{\alpha}^{L} \leq s \leq s_{\alpha}^{U}.$$

If both $(\overline{V(O)})_{\alpha}^{L}$ and $(\overline{V(O)})_{\alpha}^{U}$ are invertible with respect to α then the left and right shape function,

$$L(O) = \left[[V(O)]_{\alpha}^{L}\right]^{-1} \text{ and } R(O) = \left[[V(O)]_{\alpha}^{U}\right]^{-1},$$

can be derived from which the membership function $\mu_{\overline{V}}(O)$ can be constructed as

$$\mu_{\overline{V}}(O) = \begin{cases} L(O), & (V(O))_{\alpha=0}^{L} \leq O \leq (V(O))_{\alpha=0}^{U} \\ 1, & (V(O))_{\alpha=1}^{L} \leq O \leq (V(O))_{\alpha=1}^{U} \\ R(N), & (V(O))_{\alpha=1}^{L} \leq O \leq (V(O))_{\alpha=0}^{U} \end{cases} \tag{11}$$

5. Numerical Study

The Mean System Length

Suppose the arrival rate $\overline{\lambda}$, the service rate $\overline{\mu}$, the vacation rate $\overline{\theta}$, and the busy period $\overline{\beta}$ are assumed to be trapezoidal fuzzy numbers described by:

$$\overline{\lambda} = [1, 2, 3, 4], \overline{\mu} = [11, 12, 13, 14], \overline{\theta} = [31, 32, 33, 34], \text{ and } \overline{\beta} = [51, 52, 53, 54]$$

per hour, respectively. Then

$$\lambda(\alpha) = \min_{x \in s(\bar{\lambda})} \{x \in s(\bar{\lambda}), \begin{cases} x - 1, & 1 \leq x \leq 2 \\ 1 \quad , & 2 \leq x \leq 3 \quad \geq \alpha\}, \\ 4 - x, & 3 \leq x \leq 4 \end{cases}$$

$$\max_{x \in s(\bar{\lambda})} \{x \in s(\bar{\lambda}), \begin{cases} x - 1, & 1 \leq x \leq 2 \\ 1 \quad , & 2 \leq x \leq 3 \quad \geq \alpha\}. \\ 4 - x, & 3 \leq x \leq 4 \end{cases}$$

That is,

$$\lambda(\alpha) = [1 + \alpha, 4 - \alpha], \qquad \mu(\alpha) = [11 + \alpha, 14 - \alpha]$$
$$\theta(\alpha) = [31 + \alpha, 34 - \alpha], \text{ and } \qquad \beta(\alpha) = [51 + \alpha, 54 - \alpha].$$

It is clear that, when $x = x_\alpha^U,\ y = y_\alpha^U,\ z = z_\alpha^U$ and $s = s_\alpha^U$, L attains its maximum value and, when $x = x_\alpha^L,\ y = y_\alpha^L,\ z = z_\alpha^L$ and $s = s_\alpha^L$, L attains its minimum value.

From the generated, for the given input value of $\overline{\lambda}$, $\overline{\mu}$, $\overline{\theta}$ and $\overline{\beta}$ with $P = 0.5$, we infer that,

i) For fixed values of x, y and z, K decreases as s increase.
ii) For fixed values of x, y and s, K decreases as z increase.
iii) For fixed values of x, z and s, K decreases as y increase.
iv) For fixed values of y, z and s, K decreases as x increase.

The smallest value occurs when x takes its lower bound. That is, $x = 1 + \alpha$ and y, z and s take their upper bounds given by $y = 14 - \alpha$, and $z = 34 - \alpha$, and $s = 54 - \alpha$, respectively. The maximum value of L occurs when $x = 4 - \alpha$, $y = 11 + \alpha$, $z = 31 + \alpha$, and $s = 51 + \alpha$.

If both $(L)_\alpha^L$ and $(L)_U^\alpha$ are invertible with respect to $'\alpha'$, then the left shape function $L(K) = [(L)]^{-1}$ and right shape function $R(K) = [(L)_\alpha^L]^{-1}$ can be obtained and from which the membership function $\mu_{\overline{L}(K)}$ can be constructed as:

$$\mu_{\overline{L}}(K) = \begin{cases} L(K), & K_1 \leq K \leq K_2 \\ 1, & K_2 \leq K \leq K_3 \\ R(K), & K_3 \leq K \leq K_4. \end{cases} \tag{12}$$

The values of K_1, K_2, K_3 and K_4 as obtained from (12) are

$$\mu_{\overline{L}}(K) = \begin{cases} L(K), & 0.4591 \le K \le 0.6893 \\ 1, & 0.6893 \le K \le 24.9267, \\ R(K), & 24.9267 \le K \le 36.2060 \end{cases}$$

The Mean System Sojourn Time

The smallest value of W occurs when x take its lower bound. That is, $x = 1 + \alpha$ and y, z, and s take their upper bounds given by $y = 14 - \alpha$, $z = 34 - \alpha$, and $s = 54 - \alpha$, respectively, and the maximum value of W occurs when $x = 4 - \alpha$, $y = 11 + \alpha$, $z = 31 + \alpha$, and $s = 51 + \alpha$. If both $(W)_\alpha^L$ and $(W)_\alpha^U$ are invertible with respect to α, then the left shape function $L(M) = [(W)_\alpha^L]^{-1}$ and right shape function $R(M) = [(W)_\alpha^U]^{-1}$ can be obtained, from which the membership function $\mu_{\overline{(W)}}(M)$ can be written as

$$\mu_{\overline{L}}(M) = \begin{cases} L(M), & M_1 \le M \le M_2 \\ 1, & M_2 \le M \le M_3, \\ R(M), & M_3 \le M \le M_4 \end{cases} \tag{13}$$

The values of M_1, M_2, M_3 and M_4 as obtained from (13) are

$$\mu_{\overline{L}}(M) = \begin{cases} L(M), & 0.0883 \le M \le 0.7999 \\ 1, & 0.7999 \le M \le 1.4024. \\ R(M), & 1.4024 \le M \le 1.7024. \end{cases}$$

The mean busy period of the server

The smallest values of B occurs when x takes its lower bound. That is, $x = 1 + \alpha$, and y, z, and s take their upper bounds given by $y = 14 - \alpha$, $z = 34 - \alpha$ and $s = 54 - \alpha$, respectively. The maximum value of B occurs when $x = 4 - \alpha$, $y = 11 + \alpha$, $z = 31 + \alpha$, and $s = 51 + \alpha$. If both $(B)_\alpha^L$ and $(B)_\alpha^U$ are invertible with respect to α, then the left shape function $L(N) = [(B)_\alpha^L]^{-1}$ and the right shape function $R(N) = [(B)_\alpha^U]^{-1}$ can be obtained, from which the membership function $\mu_{\overline{(B)}}(N)$ can be written as

$$\mu_{\overline{L}}(N) = \begin{cases} L(N), & N_1 \le N \le N_2 \\ 1, & N_2 \le N \le N_3 \\ R(N), & N_3 \le N \le N_4. \end{cases} \tag{14}$$

The values of N_1, N_2, N_3 and N_4, as obtained from (14), are

$$\mu_{\overline{L}}(N) = \begin{cases} L(N), & 0.4370 \le N \le 0.6690 \\ 1, & 0.6690 \le N \le 0.7655 \\ R(N), & 0.7655 \le N \le 0.8519. \end{cases}$$

The mean working vacation period

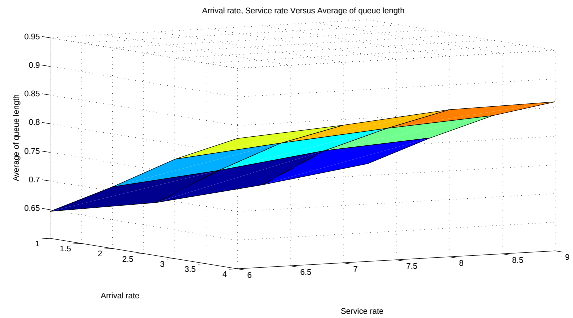

Figure 1. Arrival rate, service rate versus average of queue length is steady state.

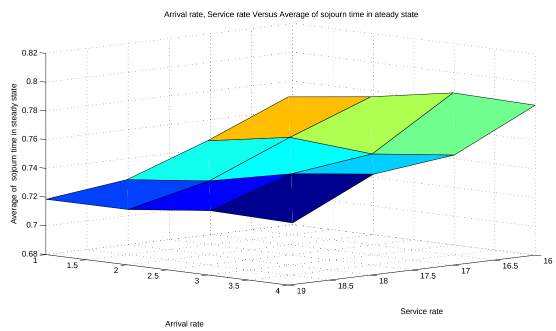

Figure 2. Arrival rate, service rate versus average of the sojourn time in steady state.

The smallest value of V occurs when x take its lower bound. That is, $x = 1+\alpha$, and y, z, and s take their upper bounds given by $y = 14-\alpha$, $z = 34-\alpha$ and $s = 54-\alpha$, respectively. The maximum value of V occurs when $x = 4-\alpha$, $y = 11+\alpha$, $z = 31+\alpha$ and $s = 51+\alpha$. If both $(V)_\alpha^L$ and $(V)_\alpha^U$ are invertible with respect to α, then the left shape function $L(O) = [(V)_\alpha^L]^{-1}$ and the right shape function $R(O) = [(V)_\alpha^U]^{-1}$ can be obtained, from which the membership function $\mu_{\overline{(V)}}(O)$ can be written as

$$\mu_{\overline{L}}(O) = \begin{cases} L(O), & O_1 \leq O \leq O_2 \\ 1, & O_2 \leq O \leq O_3 \\ R(O), & O_3 \leq O \leq O_4, \end{cases} \tag{15}$$

The values of O_1, O_2, O_3 and O_4, as obtained from (15), are

$$\mu_{\overline{L}}(O) = \begin{cases} L(O), & 0.2059 \leq O \leq 0.2755 \\ 1, & 0.2755 \leq O \leq 0.3469 \\ R(O), & 0.3469 \leq O \leq 0.4429. \end{cases}$$

Further, by fixing the vacation rate by a crisp value $\theta = 31.40$ and taking the arrival rate $\overline{\lambda} =$

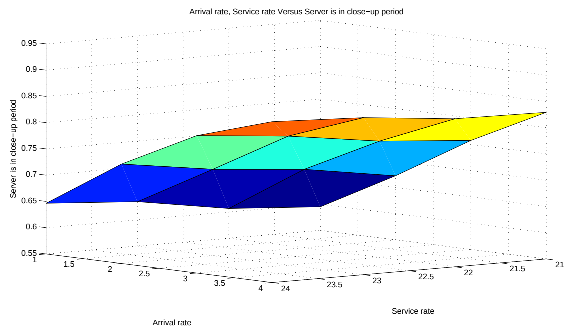

Figure 3. Arrival rate, service rate versus server is in close-up period.

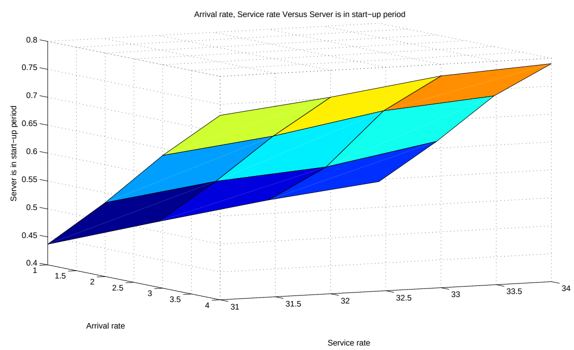

Figure 4. Arrival rate, service rate versus server is in start-up period.

$[1, 2, 3, 4]$ and the service rate $\overline{\mu} = [21, 22, 23, 24]$, both trapezoidal fuzzy numbers, the values of the mean system length are generated and are given from the graph; it is observed that, as $\overline{\lambda}$ increases, the mean system length increases for the fixed value of the service rate, whereas for fixed value of arrival rate, the mean system length decreases as service rate increases. Similar conclusions can be obtained for the case $\theta = 33.60$.

Again, for fixed values of $\theta = 31.40$ and taking $\overline{\lambda} = [1, 2, 3, 4]$ and $\overline{\mu} = [11, 12, 13, 14]$, the graphs of mean sojourn time are drawn in Figure 2. This figure shows that as the arrival rate increases then the sojourn time also increases, while the sojourn time decreases as the service rate increases in both the case.

The graph of the busy period for $\theta = 31.40$ are presented in Figure 3. As the arrival rate increases, the busy period also increases for fixed values of the service rate, whereas for fixed values of arrival rate, the busy period decreases as the service rate increases on the expected time.

It is also observed from the data generated that the membership value of the mean system length is 1, when the ranges of the arrival rate, the service rate, and the vacation rate lie in the intervals $(2, 3.4)$, $(13, 13.6)$, and $(21.8, 22.4)$, respectively.

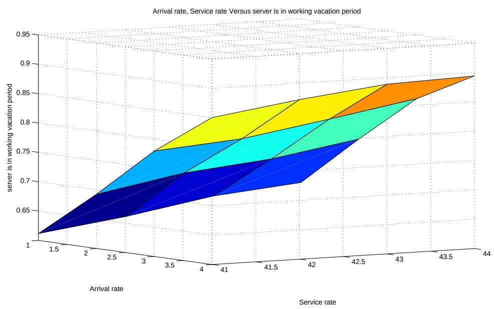

Figure 5. Arrival rate, service rate versus server is in working vacation period.

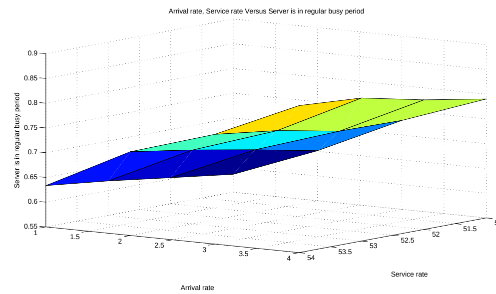

Figure 6. Arrival rate, service rate versus server is in regular busy period.

6. Conclusion

In this research paper, we have studied the $FM^X/FM/1$ queue with multiple working vacations. We have obtained the performance measures such as mean system length, mean system waiting time, mean busy period of the server, and mean working vacation period. We have obtained numerical results to all the performance measures for this fuzzy queue. For the application of this fuzzy queue, there are situations, particularly in transportation systems (bus service, trains, and express elevators), where the service provided is a group that a group of customers can be served simultaneously with batch servicing in this process.

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