Applications and Applied Mathematics: An International Journal (AAM)

Volume 12 | Issue 2 | Article 1

12-2017

Transient Solution of M[X1],M[X2]/G1,G2/1 with Priority Services, Modified Bernoulli Vacation, Bernoulli Feedback, Breakdown, Delaying Repair and Reneging

G. Ayyappan
Pondicherry Engineering College

J. Udayageetha
Perunthalaivar Kamarajar Arts College

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 12, Issue 2 (December 2017), pp. 633 – 657

Applications and Applied Mathematics:
An International Journal
(AAM)

Transient Solution of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ with Priority Services, Modified Bernoulli Vacation, Bernoulli Feedback, Breakdown, Delaying Repair and Reneging

[1]G. Ayyappan & [2]J. Udayageetha

[1]Department of Mathematics
Pondicherry Engineering College
Puducherry, India
[2]Department of Mathematics
Perunthalaivar Kamarajar Arts College
Puducherry, India
[1]ayyappanpec@hotmail.com, [2]udaya.shivani@pec.edu

Received: May 25, 2017; Accepted: October 19, 2017

Abstract

This paper considers a queuing system which facilitates a single server that serves two classes of units: high priority and low priority units. These two classes of units arrive at the system in two independent compound Poisson processes. It aims to decipher average queue size and average waiting time of the units. Under the pre-emptive priority rule, the server provides a general service to these arriving units. It is further assumed the server may take a vacation after serving the last high priority unit present in the system or at the service completion of each low priority unit present in the system. Otherwise, he may remain in the system. Also, if a high priority unit is not satisfied with the service given it may join the tail of the queue as a feedback unit or leave the system. The server may break down exponentially while serving the units. The repair process of the broken server is not immediate. There is a delay time to start the repair. The delay time to repair and repair time follow general distributions. We consider reneging to occur for the low priority units when the server is unavailable due to breakdown or vacation. We concentrate on deriving the transient solutions by using supplementary variable technique. Further, some special cases are also discussed and numerical examples are presented.

Keywords: Batch Arrival; Breakdown; Delay time to repair; Modified Server Vacation; Priority Queueing systems; Reneging; Transient Solution

MSC 2010 No.: 60K25, 68M30

1. Introduction

A priority mechanism in a queueing system differentiates customers based on their classes. Such differentiation appears in a number of situations of everyday life and in major engineering systems, notably, job scheduling in manufacturing, operating systems in computers and channel access protocols in communication networks. Correct assignment of priorities brings customer satisfaction while keeping the total workload unchanged. Extensive analysis and optimization in operation of queues with priority have been motivated by application to specific case study as well as for theoretical interest.

For priority queues, one must distinguish pre-emptive service from non-pre-emptive service. A service discipline is said to be non-pre-emptive if, once the service to a customer is started, it is not disrupted until the whole service requirement is completed. Thus, only at the end of each service time one of the waiting customers of the highest priority class is selected for the next service. Among the customers of the same class a tie is broken by usual rules for low priority queues, such as first-come-first-served (FCFS), last-come-first-served (LCFS), and random order for service (ROS). In a pre-emptive service queue, the service is given to one of the customers of the highest priority class present in the system at all times. The service of the low priority unit is immediately pre-empted by the arrival of a customer of higher priority class.

The pre-emptive discipline can be further broken down into three categories. Pre-emptive resume unit resumes service from the point where it was interrupted. Pre-emptive repeat-identical unit on its re-entry requires the same amount of service as it required on its earlier entry. Pre-emptive repeat-different unit on its re-entry requires a random service time independent of past pre-emptions and wasted service time.

The pre-emptive-repeat discipline is applicable whenever the technical considerations require the service to be repeated again. For example, if a computing machine breaks down, it may be necessary to re-run the program after it has been repaired, leading to pre-emptive repeat-identical discipline. However, if the program is rewritten in the interim, the pre-emptive repeat-different discipline is applicable. The pre-emptive repeat-different and repeat identical disciplines may also arise because of different reasons associated with the variability of the service times. In many real life situations, if a server is inoperative for some time if may increase the likelihood of unit losses due to balking and reneging. In these situations, the arriving unit may be discouraged due to long queue or other factors. Such queueing models involve the concept of balking and reneging and have been studied by several researchers.

We extend and develop this model by adding new assumptions such as feedback to the high priority units, reneging and system breakdowns. Customers may renege due to impatience during server breakdowns or during the time when the server takes vacation. This is a very realistic assumption and often we come across such queuing situations in real world phenomena.

Pre-emptive priority queues are basic models in queueing theory and have been studied by many researchers. The queueing systems was studied by Kleinrock (1976). Time-dependent solution of a priority queue with bulk arrival was studied by Hawkes (1965). The main concept of priority queue was integrated by Jaiswal (1968). The pre-emptive priority queueing discipline with exponential arrivals and service time has been discussed by several authors. The solution technique of difference equations in queueing system was elaborated by Cox (1955). The first published account of these discipline have been investigated by many, notable works include that of Cobham (1954). A paper of Takagi (1990) has generalized the time dependent analysis of $M/G/1$ model with vacation and exhaustive service.

The service facility becomes inoperative for a random period of time during which it is repaired. Thangaraj et al. (2010), Jain et al. (2014) and Murugan et al. (2015) used this similarity to study the breakdown models. The bulk arrival priority models with unreliable servers have been studied by Jain et al. (2008). Queues with impatient units have attracted the attention of many researchers and there is significant contribution by numerous researchers in this area. A lot of developments in the study of queues with impatient units is noticed in recent years.

Khalaf et al. (2011) have analysed the $M^X/G/1$ queueing model. Singh et al. (2014) derived the expressions for $M^X/G/1$ queueing model with balking and vacation. Baruah et al. (2013) have described a two stage batch arrival queue with reneging during vacation and breakdown periods. Haghighi et al.(2006) have discussed the parallel priority queueing system with finite buffers. N-policy for $M^X/G/1$ unreliable retrial G-queue with preemptive resume and multi-services was analysed by Bhagat et al. (2016). Gao (2015) have studied a preemptive priority retrial queue with two classes of customers and general retrial times. Madan (2011) has studied the non-pre-emptive queues with optional server vacation. Recently, Ayyappan et al. (2016) studied the balking behaviour of the non-pre-emptive priority queue with optional server vacation.

This paper considers an $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ priority queueing system. Under the pre-emptive priority rule the server provides the general service to the high priority and low priority units. It is assumed that the server may take vacations, but no vacation is allowed if there is even a single high priority unit present in the system. After completing the service to each low priority unit the server can take a vacation with probability θ or continue the next service with probability $1 - \theta$, if any. The server may breakdown while serving the units. Once there is a breakdown, the server will not be sent for repair immediately. There is a delay time to start the repair process, at the completion of which the server undergoes the repair process. After returning from repair, the server actively provides service to the high/low priority unit. If the high priority unit is not satisfied with the service, it may join the tail of the queue as a feedback unit. If the server is either breakdown or on vacation the low priority units may renege the queue. The rest of the paper organized as follows. In Section 1, we give the introduction about priority queueing discipline and literature review. Section 2 deals with notations used, mathematical formulation and governing equations of the model and outlines the transient solution. Section 3 gives the steady state solution of the system. Section 4 presents the various performance measures of the model. In Section 5 numerical results are computed. Finally, a conclusion is given.

2. Model description

The following assumptions are made about the queueing system.

(i) There are two types of units, the high priority and the low priority units which arrive in batches according to compound Poisson process with arrival rates λ_1 and λ_2 respectively and form two separate queues. Let $\lambda_1 c_{1,i}\ dt$ and $\lambda_2 c_{2,i}\ dt$ $(i = 1, 2, ...)$ be the probabilities that a batch of i units arrive at the system during a short interval of time $(t, t + dt)$, where for $0 \leq c_{1,i} \leq 1$,

$$\sum_{i=1}^{\infty} c_{1,i} = 1,$$

and for $0 \leq c_{2,i} \leq 1$,

$$\sum_{i=1}^{\infty} c_{2,i} = 1.$$

(ii) If a high priority unit arrives in a batch and finds a low priority unit in service, it pre-empts the low priority unit undergoing service; and the service of the pre-empted low priority unit begins only after the completion of service of all high priority units present in the system.

(iii) The service times for the high priority and low priority units are generally(arbitrary) distributed with distribution functions $B_i(s)$ and density functions $b_i(s)$, $i = 1, 2$ respectively. Let $\mu_i(x)dx$, $i = 1, 2$ be the conditional probability of service completion of high priority and low priority units during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},$$

and

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x) dx}.$$

(iv) If a high priority unit is not satisfied with the service given, it may join the tail of the queue as a feedback unit with probability p or permanently leaves the system with probability $1 - p$.

(v) If all the high priority units are served then the server can take a vacation with probability θ or continue the service to the low priority unit with probability $1 - \theta$. Also, every service completion to the low priority unit the server may take a vacation with probability θ or continue the service to the next unit with probability $1 - \theta$. If there are no units present in the system, the server remains idle and waiting for the new units to arrive. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$. Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\beta(x) = \frac{v(x)}{1 - V(x)},$$

and

$$v(s) = \beta(s) e^{-\int_0^s \beta(x) dx}.$$

(vi) If the server breaksdown while serving the unit, the service is interrupted and he is not sent for repair immediately. There is a delay time to start the repair process. Only at the completion of delay time the repair process starts. Immediately after returning from the repair, the server starts to serve high priority/low priority units.

(vii) The server may breakdown with breakdown rate α while serving the units. The delay time to repair and the repair process are generally distributed with distribution functions $D(s)$ and $R(s)$ and density functions $d(s)$ and $r(s)$ respectively. Let $\phi(x)dx$ and $\gamma(x)dx$ be the conditional probabilities of completion of a delay time and repair time respectively during the interval $(x, x+dx]$ given that the elapsed time is x, so that

$$\phi(x) = \frac{d(x)}{1-D(x)},$$

$$d(s) = \phi(s)e^{-\int_0^s \phi(x)dx},$$

and

$$\gamma(x) = \frac{r(x)}{1-R(x)},$$

$$r(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}.$$

(viii) Due to impatience the low priority unit may renege during the breakdown or vacation period with reneging rate δ, which is assumed to be exponentially distributed.

2.1. Definitions and notations

We define the following notations:

(i) $P^{(1)}_{m,n}(x,t)$ = Probability that at time t, the server is actively providing service and there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue excluding the one high priority unit in service with elapsed service time for this unit being x. $P^{(1)}_{m,n}(t) = \int_0^\infty P^{(1)}_{m,n}(x,t)dx$ denotes the probability that at time t there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue excluding one high priority unit in service without regard to the elapsed service time x.

(ii) $V_{m,n}(x,t)$ = Probability that at time t, the server is on vacation with elapsed vacation time x and there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue. $V_{m,n}(t) = \int_0^\infty V_{m,n}(x,t)dx$ denotes the probability that at time t there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue, without regard to the elapsed vacation time x .

(iii) $P^{(2)}_{0,n}(x,t)$ = Probability that at time t, the server is actively providing service and there are $n\ (\geq 0)$ low priority units in the queue excluding the one low priority unit in service with elapsed service time x. $P^{(2)}_{0,n}(t) = \int_0^\infty P^{(2)}_{0,n}(x,t)dx$ denotes the probability that at time t there are $n\ (\geq 0)$ low priority units in the queue excluding the one low priority unit in service without regard to the elapsed service time x.

(iv) $D_{m,n}(x,t)$ = Probability that at time t, the server is on breakdown with elapsed delay time to start repair x and there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue. $D_{m,n}(t) = \int_0^\infty D_{m,n}(x,t)dx$ denotes the probability that at time t there are $m\ (\geq 0)$ high priority units and $n\ (\geq 0)$ low priority units in the queue, without regard to the elapsed delay time to repair, x.

(v) $R_{m,n}(x,t)$ = Probability that at time t, the server is undergoing repair process with elapsed repair time x and there are $m\,(\geq 0)$ high priority units and $n\,(\geq 0)$ low priority units in the queue. $R_{m,n}(t) = \int_0^\infty R_{m,n}(x,t)dx$ denotes the probability that at time t there are $m(\geq 0)$ high priority units and $n\,(\geq 0)$ low priority units in the queue, without regard to the elapsed repair time x.

(vi) $Q(t)$ = Probability that at time t, there are no units present in the system and the server is available in the system, but idle.

2.2. Equations Governing the System

The Kolmogorov forward equations which govern the model are

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -\,(\lambda_1+\lambda_2+\mu_1(x)+\alpha)P^{(1)}_{m,n}(x,t) + \lambda_1\sum_{i=1}^{m}c_{1,i}P^{(1)}_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}c_{2,i}P^{(1)}_{m,n-i}(x,t);\ m\geq 1,\ n\geq 1, \tag{1}$$

$$\frac{\partial}{\partial t}P^{(1)}_{m,0}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,0}(x,t) = -\,(\lambda_1+\lambda_2+\mu_1(x)+\alpha)P^{(1)}_{m,0}(x,t) + \lambda_1\sum_{i=1}^{m}c_{1,i}P^{(1)}_{m-i,0}(x,t);\ m\geq 1, \tag{2}$$

$$\frac{\partial}{\partial t}P^{(1)}_{0,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{0,n}(x,t) = -\,(\lambda_1+\lambda_2+\mu_1(x)+\alpha)P^{(1)}_{0,n}(x,t) + \lambda_2\sum_{i=1}^{n}c_{2,i}P^{(1)}_{0,n-i}(x,t);\ n\geq 1, \tag{3}$$

$$\frac{\partial}{\partial t}P^{(1)}_{0,0}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{0,0}(x,t) = -(\lambda_1+\lambda_2+\mu_1(x)+\alpha)P^{(1)}_{0,0}(x,t), \tag{4}$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t) + \frac{\partial}{\partial x}V_{m,n}(x,t) = -\,(\lambda_1+\lambda_2+\beta(x)+\delta)V_{m,n}(x,t) + \lambda_1\sum_{i=1}^{m}c_{1,i}V_{m-i,n}(x,t) + \lambda_2\sum_{i=1}^{n}c_{2,i}V_{m,n-i}(x,t) + \delta V_{m,n+1}(x,t);\ m\geq 1,\ n\geq 1, \tag{5}$$

$$\frac{\partial}{\partial t}V_{m,0}(x,t) + \frac{\partial}{\partial x}V_{m,0}(x,t) = -\,(\lambda_1+\lambda_2+\beta(x))V_{m,0}(x,t) + \lambda_1\sum_{i=1}^{m}c_{1,i}V_{m-i,0}(x,t) + \delta V_{m,1}(x,t);\ m\geq 1, \tag{6}$$

$$\frac{\partial}{\partial t}V_{0,n}(x,t)+\frac{\partial}{\partial x}V_{0,n}(x,t)=-\,(\lambda_1+\lambda_2+\beta(x)+\delta)V_{0,n}(x,t)+\lambda_2\sum_{i=1}^{n}c_{2,i}V_{0,n-i}(x,t)$$
$$+\,\delta V_{0,n+1}(x,t);\;\; n\geq 1, \tag{7}$$

$$\frac{\partial}{\partial t}V_{0,0}(x,t)+\frac{\partial}{\partial x}V_{0,0}(x,t)=-\,(\lambda_1+\lambda_2+\beta(x))V_{0,0}(x,t)+\delta V_{0,1}(x,t), \tag{8}$$

$$\frac{\partial}{\partial t}P^{(2)}_{0,n}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{0,n}(x,t)=-\,(\lambda_1+\lambda_2+\mu_2(x)+\alpha)P^{(2)}_{0,n}(x,t)$$
$$+\,\lambda_2\sum_{i=1}^{n}c_{2,i}P^{(2)}_{0,n-i}(x,t); n\geq 1, \tag{9}$$

$$\frac{\partial}{\partial t}P^{(2)}_{0,0}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{0,0}(x,t)=-\,(\lambda_1+\lambda_2+\mu_2(x)+\alpha)P^{(2)}_{0,0}(x,t), \tag{10}$$

$$\frac{\partial}{\partial t}D_{m,n}(x,t)+\frac{\partial}{\partial x}D_{m,n}(x,t)=-\,(\lambda_1+\lambda_2+\phi(x)+\delta)D_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1,i}D_{m-i,n}(x,t)$$
$$+\,\lambda_2\sum_{i=1}^{n}c_{2,i}D_{m,n-i}(x,t)+\delta D_{m,n+1}(x,t);\;\; m\geq 1,\;\; n\geq 1, \tag{11}$$

$$\frac{\partial}{\partial t}D_{m,0}(x,t)+\frac{\partial}{\partial x}D_{m,0}(x,t)=-\,(\lambda_1+\lambda_2+\phi(x))D_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1,i}D_{m-i,0}(x,t)$$
$$+\,\delta D_{m,1}(x,t);\;\; m\geq 1, \tag{12}$$

$$\frac{\partial}{\partial t}D_{0,n}(x,t)+\frac{\partial}{\partial x}D_{0,n}(x,t)=-\,(\lambda_1+\lambda_2+\phi(x)+\delta)D_{0,n}(x,t)+\lambda_2\sum_{i=1}^{n}c_{2,i}D_{0,n-i}(x,t)$$
$$+\,\delta D_{0,n+1}(x,t);\;\; n\geq 1, \tag{13}$$

$$\frac{\partial}{\partial t}R_{m,n}(x,t)+\frac{\partial}{\partial x}R_{m,n}(x,t)=-\,(\lambda_1+\lambda_2+\gamma(x)+\delta)R_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1,i}R_{m-i,n}(x,t)$$
$$+\,\lambda_2\sum_{i=1}^{n}c_{2,i}R_{m,n-i}(x,t)+\delta R_{m,n+1}(x,t);\;\; m\geq 1,\;\; n\geq 1, \tag{14}$$

$$\frac{\partial}{\partial t}R_{m,0}(x,t)+\frac{\partial}{\partial x}R_{m,0}(x,t)=-\,(\lambda_1+\lambda_2+\gamma(x))R_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}c_{1,i}R_{m-i,0}(x,t)$$
$$+\,\delta R_{m,1}(x,t);\;\; m\geq 1, \tag{15}$$

$$\frac{\partial}{\partial t}R_{0,n}(x,t)+\frac{\partial}{\partial x}R_{0,n}(x,t)=-\,(\lambda_1+\lambda_2+\gamma(x)+\delta)R_{0,n}(x,t)+\lambda_2\sum_{i=1}^{n}c_{2,i}R_{0,n-i}(x,t)$$
$$+\,\delta R_{0,n+1}(x,t);\;\; n\geq 1, \tag{16}$$

$$\begin{aligned}\frac{d}{dt}Q(t) =& -(\lambda_1+\lambda_2)Q(t) + \int_0^\infty V_{0,0}(x,t)\beta(x)dx + (1-\theta)q\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx \\ &+ (1-\theta)\int_0^\infty P_{0,0}^{(2)}(x,t)\mu_2(x)dx. \qquad (17)\end{aligned}$$

The above set of equations are to be solved under the following boundary conditions.

$$\begin{aligned}P_{m,n}^{(1)}(0,t) =& q\int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx \\ &+ \lambda_1 c_{1,m+1}\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \int_0^\infty V_{m+1,n}(x,t)\beta(x)dx \\ &+ \int_0^\infty R_{m+1,n}(x,t)\gamma(x)dx;\ m\geq 1,\ n\geq 1, \qquad (18)\end{aligned}$$

$$\begin{aligned}P_{m,0}^{(1)}(0,t) =& \lambda_1 c_{1,m+1}Q(t) + q\int_0^\infty P_{m+1,0}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{m,0}^{(1)}(x,t)\mu_1(x)dx \\ &+ \int_0^\infty V_{m+1,0}(x,t)\beta(x)dx + \int_0^\infty R_{m+1,0}(x,t)\gamma(x)dx;\ m\geq 1, \qquad (19)\end{aligned}$$

$$\begin{aligned}P_{0,n}^{(1)}(0,t) =& q\int_0^\infty P_{1,n}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty V_{1,n}(x,t)\beta(x)dx \\ &+ \lambda_1 c_{1,1}\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \int_0^\infty R_{1,n}(x,t)\gamma(x)dx;\ n\geq 1, \qquad (20)\end{aligned}$$

$$\begin{aligned}P_{0,0}^{(1)}(0,t) =& \lambda_1 c_{1,1}Q(t) + q\int_0^\infty P_{1,0}^{(1)}(x,t)\mu_1(x)dx + p\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx \\ &+ \int_0^\infty V_{1,0}(x,t)\beta(x)dx + \int_0^\infty R_{1,0}(x,t)\gamma(x)dx, \qquad (21)\end{aligned}$$

$$V_{0,n}(0,t) = \theta q\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \theta\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx;\ n\geq 0, \qquad (22)$$

$$\begin{aligned}P_{0,0}^{(2)}(0,t) =& (1-\theta)q\int_0^\infty P_{0,1}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,1}^{(2)}(x,t)\mu_2(x)dx \\ &+ \int_0^\infty V_{0,1}(x,t)\beta(x)dx + \int_0^\infty R_{0,1}(x,t)\gamma(x)dx + \lambda_2 c_{2,1}Q(t), \qquad (23)\end{aligned}$$

$$\begin{aligned}P_{0,n}^{(2)}(0,t) =& (1-\theta)q\int_0^\infty P_{0,n+1}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,n+1}^{(2)}(x,t)\mu_2(x)dx \\ &+ \int_0^\infty V_{0,n+1}(x,t)\beta(x)dx + \int_0^\infty R_{0,n+1}(x,t)\gamma(x)dx + \lambda_2 c_{2,n+1}Q(t);\ n\geq 1, \qquad (24)\end{aligned}$$

$$D_{m,n}(0,t) = \alpha\int_0^\infty P_{m-1,n}^{(1)}(x,t)dx;\ m\geq 1,\ \ n\geq 0, \qquad (25)$$

$$D_{0,n}(0,t) = \alpha\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx;\ n\geq 1, \qquad (26)$$

$$R_{m,n}(0,t)=\int_0^\infty D_{m,n}(x,t)\phi(x)dx;\ m\geq 1,\ \ n\geq 0, \tag{27}$$

$$R_{0,n}(0,t)=\int_0^\infty D_{0,n}(x,t)\phi(x)dx;\ n\geq 1. \tag{28}$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$\left.\begin{array}{l} P^{(1)}_{m,n}(0)=P^{(1)}_{m,0}(0)=P^{(1)}_{0,n}(0)=P^{(1)}_{0,0}(0)=0,\\ P^{(2)}_{0,n}(0)=P^{(2)}_{0,0}(0)=0,\\ V_{m,n}(0)=V_{m,0}(0)=V_{0,n}(0)=V_{0,0}(0)=0,\\ D_{m,n}(0)=D_{m,0}(0)=D_{0,n}(0)=0,\\ R_{m,n}(0)=R_{m,0}(0)=R_{0,n}(0)=0,\ \text{and } Q(0)=1. \end{array}\right\} \tag{29}$$

The Probability Generating Functions (PGF) of this model are defined as

$$A(x,z_1,z_2,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(x,t), \tag{30}$$

where $A=P^{(i)},V,D,R.$ respectively, which are convergent inside the circle given by $|z_1|\leq 1,|z_2|\leq 1.$

The Laplace transform of function $f(t)$ is defined by,

$$\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt.$$

By using Laplace transforms and solving the equations from (1) to (17), we get,

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,s,z_1,z_2)+(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\mu_1(x)+\alpha)\overline{P}^{(1)}(x,s,z_1,z_2) \\ =0, \tag{31}$$

$$\frac{\partial}{\partial x}\overline{V}(x,s,z_1,z_2)+(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\beta(x)+\delta-\frac{\delta}{z_2})\overline{V}(x,s,z_1,z_2) \\ =0, \tag{32}$$

$$\frac{\partial}{\partial x}\overline{D}(x,s,z_1,z_2)+(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\phi(x)+\delta-\frac{\delta}{z_2})\overline{D}(x,s,z_1,z_2) \\ =0, \tag{33}$$

$$\frac{\partial}{\partial x}\overline{R}(x,s,z_1,z_2)+(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\gamma(x)+\delta-\frac{\delta}{z_2})\overline{R}(x,s,z_1,z_2) \\ =0. \tag{34}$$

Similarly for the boundary conditions from (18) to (28) we perform the similar operations as before and get,

$$\begin{aligned}\overline{P}_m^{(1)}(0,s,z_2) =&\lambda_1 c_{1,m+1}\overline{Q}(s) + q\int_0^\infty \overline{P}_{m+1}^{(1)}(x,s,z_2)\mu_1(x)dx \\ &+ p\int_0^\infty \overline{P}_m^{(1)}(x,s,z_2)\mu_1(x)dx + \lambda_1 c_{1,m+1} z_2\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)dx \\ &+ \int_0^\infty \overline{V}_{m+1}(x,s,z_2)\beta(x)dx + \int_0^\infty \overline{R}_{m+1}(x,s,z_2)\gamma(x)dx;\ m\geq 1,\end{aligned} \tag{35}$$

$$\begin{aligned}\overline{P}_0^{(1)}(0,s,z_2) =&\lambda_1 c_{1,1}\overline{Q}(s) + q\int_0^\infty \overline{P}_1^{(1)}(x,s,z_2)\mu_1(x)dx + p\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx \\ &+ \lambda_1 c_{1,1} z_2\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)dx + \int_0^\infty \overline{V}_1(x,s,z_2)\beta(x)dx \\ &+ \int_0^\infty \overline{R}_1(x,s,z_2)\gamma(x)dx,\end{aligned} \tag{36}$$

$$\overline{V}_0(0,s,z_2) =\theta q\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx + \theta\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx, \tag{37}$$

$$\overline{D}_m(0,s,z_2) =\alpha\int_0^\infty \overline{P}_{m-1}^{(1)}(x,s,z_2)dx;\ m\geq 1, \tag{38}$$

$$\overline{D}_0(0,s,z_2) =\alpha z_2\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)dx, \tag{39}$$

$$\overline{R}_m(0,s,z_2) =\int_0^\infty \overline{D}_m(x,s,z_2)\phi(x)dx;\ m\geq 1, \tag{40}$$

$$\overline{R}_0(0,s,z_2) =\int_0^\infty \overline{D}_0(x,s,z_2)\phi(x)dx, \tag{41}$$

$$\begin{aligned}z_2\overline{P}_0^{(2)}(0,s,z_2) =&(1-\theta)q\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx + (1-\theta)\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx \\ &+ \int_0^\infty \overline{V}_0(x,s,z_2)\beta(x)dx + \lambda_2 C_2(z_2)\overline{Q}(S) - (1-\theta)q\int_0^\infty \overline{P}_{0,0}^{(1)}(x,s)\mu_1(x)dx \\ &- (1-\theta)\int_0^\infty \overline{P}_{0,0}^{(2)}(x,s)\mu_2(x)dx - \int_0^\infty \overline{V}_{0,0}(x,s)\beta(x)dx \\ &+ \int_0^\infty \overline{R}_0(x,s,z_2)\gamma(x)dx.\end{aligned} \tag{42}$$

By combining (35) and (36) we get,

$$\begin{aligned}z_1\overline{P}^{(1)}(0,s,z_1,z_2) =&\lambda_1 C_1(z_1)\overline{Q}(s) + (q+pz_1)\int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx \\ &+ \lambda_1 C_1(z_1)z_2\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)dx - q\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx \\ &+ \int_0^\infty \overline{V}(x,s,z_1,z_2)\beta(x)dx - \int_0^\infty \overline{V}_0(x,s,z_2)\beta(x)dx \\ &+ \int_0^\infty \overline{R}(x,s,z_1,z_2)\gamma(x)dx - \int_0^\infty \overline{R}_0(x,s,z_2)\gamma(x)dx,\end{aligned} \tag{43}$$

and (42) becomes,

$$\begin{aligned} z_2\overline{P}_0^{(2)}(0,s,z_2) =& 1-(s+\lambda_1+\lambda_2[1-C_2(z_2)])\overline{Q}(s)+(1-\theta)q\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx \\ &+(1-\theta)\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx+\int_0^\infty \overline{V}_0(x,s,z_2)\beta(x)dx \\ &+\int_0^\infty \overline{R}_0(x,s,z_2)\gamma(x)dx. \end{aligned} \tag{44}$$

Integrating Equation (31) between 0 and x, we obtain

$$\begin{aligned} \overline{P}^{(1)}(x,s,z_1,z_2) =& \overline{P}^{(1)}(0,s,z_1,z_2) \\ &\times e^{\{-(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\alpha)x-\int_0^x \mu_1(t)dt\}}. \end{aligned} \tag{45}$$

Again, integrating (45) by parts with respect to x, we get

$$\overline{P}^{(1)}(s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B_1}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\alpha)}{(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\alpha)}\right]. \tag{46}$$

Multiplying (45) by $\mu_1(x)$ and integrating with respect to x, we get

$$\begin{aligned} \int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx =& \overline{P}^{(1)}(0,s,z_1,z_2) \\ &\times \overline{B_1}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\alpha). \end{aligned} \tag{47}$$

By the definition of vacation defined here, we have,

$$\overline{V}(0,s,z_1,z_2) = \overline{V}_0(0,s,z_2). \tag{48}$$

By repeating the procedure as before for Equations (32), (33), and (34) we get,

$$\begin{aligned} \overline{V}(x,s,z_1,z_2) =& \overline{V}(0,s,z_1,z_2) \\ &\times e^{\{-(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})x-\int_0^x \beta(t)dt\}}, \end{aligned} \tag{49}$$

$$\overline{V}(s,z_1,z_2) = \overline{V}(0,s,z_1,z_2)\left[\frac{1-\overline{V}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})}{s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}}\right], \tag{50}$$

$$\begin{aligned} &\int_0^\infty \overline{V}(x,s,z_1,z_2)\beta(x)dx \\ &= \overline{V}(0,s,z_1,z_2)\overline{V}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}), \end{aligned} \tag{51}$$

$$\begin{aligned} &\int_0^\infty \overline{V}(x,s,z_1,z_2)\beta(x)dx \\ &= \overline{V}_0(0,s,z_2)\overline{V}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}), \end{aligned} \tag{52}$$

$$\overline{P}_0^{(2)}(x,s,z_2) = \overline{P}_0^{(2)}(0,s,z_2)e^{\{-(s+\lambda_1+\lambda_2[1-C_2(z_2)]+\alpha)x-\int_0^x \mu_2(t)dt\}}, \tag{53}$$

$$\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx = \overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(s+\lambda_1+\lambda_2[1-C_2(z_2)]+\alpha), \tag{54}$$

$$\overline{D}(x,s,z_1,z_2) = \overline{D}(0,s,z_1,z_2) \times e^{\{-(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})x-\int_0^x \phi(t)dt\}}, \tag{55}$$

$$\overline{D}(s,z_1,z_2) = \overline{D}(0,s,z_1,z_2)\left[\frac{1-\overline{D}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})}{s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}}\right], \tag{56}$$

$$\int_0^\infty \overline{D}(x,s,z_1,z_2)\phi(x)dx = \overline{D}(0,s,z_1,z_2)\overline{D}\left(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}\right), \tag{57}$$

$$\overline{R}(x,s,z_1,z_2) = \overline{R}(0,s,z_1,z_2) \times e^{\{-(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})x-\int_0^x \gamma(t)dt\}}, \tag{58}$$

$$\overline{R}(s,z_1,z_2) = \overline{R}(0,s,z_1,z_2)\left[\frac{1-\overline{R}(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2})}{s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}}\right], \tag{59}$$

$$\int_0^\infty \overline{R}(x,s,z_1,z_2)\gamma(x)dx = \overline{R}(0,s,z_1,z_2)\overline{R}\left(s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2}\right). \tag{60}$$

By substituting the required values in (43) and (44), we get,

$$\begin{aligned}&\{\ \{z_1-(q+pz_1)\overline{B}_1(f_1(s,z_1,z_2))\}f_1(z_1,z_2)-\alpha z_1(1-\overline{B}_1(f_1(s,z_1,z_2)))\\ &\times \overline{D}(f_2(s,z_1,z_2))\ \overline{R}(f_2(s,z_1,z_2))\ \}\ \overline{P}^{(1)}(0,s,z_1,z_2) = \{(\lambda_1 C_1(z_1)\overline{Q}(s)\\ &-q\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(f_1(s,z_1,z_2))(1-\theta\overline{V}(f_2(s,z_1,z_2))+\theta\overline{V}(f_2(s,z_2))))f_1(s,z_2)\\ &+\overline{P}_0^{(2)}(0,s,z_2)\{\lambda_1 C_1(z_1)z_2(1-\overline{B_2})+\alpha z_2(1-\overline{B}_2(f_2(s,z_2)))(\overline{D}(f_2(s,z_1,z_2))\\ &\times \overline{R}(f_2(s,z_1,z_2))-\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2)))+\theta\overline{B}_2(f_2(s,z_2))(\overline{V}(f_2(s,z_1,z_2))\\ &-\overline{V}(f_2(s,z_2)))f_1(s,z_2)\}\}\left[\frac{f_1(s,z_1,z_2)}{(f_1(s,z_2))}\right],\end{aligned} \tag{61}$$

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left\{\begin{array}{l}\{1-(s+\lambda_1+\lambda_2[1-C_2(z_2)])\overline{Q}(s)f_1(s,z_2)\}+q\overline{P}_0^{(1)}(0,s,z_2)\\ \times \overline{B}_1(f_1(s,z_1,z_2))\{1-\theta+\theta\overline{V}(f_2(s,z_2))\}f_1(s,z_2)\end{array}\right\}}{\left\{\begin{array}{l}[z_2-(1-\theta)\overline{B}_2(f_2(s,z_2))-\theta\overline{B}_2(f_2(s,z_2))\overline{V}(f_2(s,z_1,z_2))]f_1(s,z_2)\\ -\alpha z_2(1-\overline{B}_2(f_2(s,z_2)))\overline{D}(f_2(s,z_2))\ \overline{R}(f_2(s,z_2))\end{array}\right\}}, \tag{62}$$

and,

$$
\begin{aligned}
f_1(s,z_1,z_2) &= s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\alpha,\\
f_2(s,z_1,z_2) &= s+\lambda_1[1-C_1(z_1)]+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2},\\
f_1(s,z_2) &= s+\lambda_1+\lambda_2[1-C_2(z_2)]+\alpha,\\
f_2(s,z_2) &= s+\lambda_1+\lambda_2[1-C_2(z_2)]+\delta-\frac{\delta}{z_2},\\
f_1(s,g(z_2)) &= s+\lambda_1[1-C_1(g(z_2))]+\lambda_2[1-C_2(z_2)]+\alpha.
\end{aligned}
$$

By applying Rouche's theorem on (61), we conclude that

$$\{z_1-(q+pz_1)\overline{B}_1(f_1(s,z_1,z_2))\}f_1(z_1,z_2)-\alpha z_1(1-\overline{B}_1(f_1(s,z_1,z_2)))\overline{D}(f_2(s,z_1,z_2))\times\overline{R}(f_2(s,z_1,z_2))$$

has one and only one zero inside the circle, $|z_1|=1$ for Re $(s)>0$, $|z_2|\leq 1$. Thus Equation (61) gives,

$$
\overline{P}_0^{(1)}(0,s,z_2)=\frac{\left\{\begin{array}{l}\lambda_1C_1(g(z_2))\overline{Q}(s)f_1(s,z_2)+\overline{P}_0^{(2)}(0,s,z_2)\{\lambda_1C_1(g(z_2))z_2(1-\overline{B_2}(f_1(s,z_2)))\\ \times\,\alpha z_2(1-\overline{B_2}(f_1(s,z_2)))(\overline{D}(f_2(s,C_1(g(z_2))))\,\overline{R}(f_2(s,C_1(g(z_2))))\overline{R}(f_2(s,z_2))\\ \times\,\overline{D}(f_2(s,z_2)))+\theta\overline{B_2}(f_2(s,z_2))(\overline{V}(f_2(s,z_1,z_2))-\overline{V}(f_2(s,z_2)))f_1(s,z_2)\}\end{array}\right\}}{q\overline{B}_1(f_1(s,z_2))(1-\theta\overline{V}(f_2(s,z_1,z_2))+\theta\overline{V}(f_2(s,z_2)))},
\tag{63}
$$

By substituting (63) in (61) and (62), we get

$$
\overline{P}_0^{(1)}(0,s,z_1,z_2)=
\frac{\left\{\begin{array}{l}\{\lambda_1C_1(z_1)\overline{Q}(s)(1-\theta\overline{V}(f_2(s,g(z_2)))\theta\overline{V}(f_2(s,z_2)))-\lambda_1C_1(g(z_2))(1-\theta\overline{V}(f_2(s,z_1,z_2))\\ +\,\theta\overline{V}(f_2(s,z_2)))\}f_1(s,z_2)f_1(s,z_1,z_2)+\overline{P}_0^{(2)}(0,s,z_2)\{\ \{\ \lambda_1C_1(z_1)z_2(1-\overline{B}_2(f_1(s,z_2)))\\ +\,\alpha z_2(1-\overline{B_2}(f_2(s,z_2)))(\overline{D}(f_2(s,z_1,z_2))\,\overline{R}(f_2(s,z_1,z_2))-\overline{D}(f_2(s,z_2))\,\overline{R}(f_2(s,z_2)))\\ +\,\theta\overline{B}_2(f_1(s,z_2))(\overline{V}(f_2(s,z_1,z_2))-\overline{V}(f_2(s,z_2)))f_1(s,z_2)\ \}\ (1-\theta\overline{V}(f_2(s,g(z_2)))\\ +\,\theta\overline{V}(f_2(s,z_2)))-\ \{\ \lambda_1C_1(g(z_2))z_2(1-\overline{B}_2(f_2(s,z_2)))+\alpha z_2(1-\overline{B}_2(f_2(s,z_2)))\\ \times\,(\overline{D}(f_2(s,g(z_2)))\,\overline{R}(f_2(s,g(z_2)))-\overline{D}(f_2(s,z_2))\,\overline{R}(f_2(s,z_2)))+\theta\overline{B}_2(f_1(s,z_2))\\ \times\,(\overline{V}(f_2(s,g(z_2)))-\overline{V})f_2(s,z_2)\ \}\ (1-\theta\overline{V}(f_2(s,z_1,z_2))+\theta\overline{V}(f_2(s,z_2)))\}f_1(s,z_1,z_2)\end{array}\right\}}{\left\{\begin{array}{l}f_1(s,z_2)(1-\theta\overline{V}(f_2(s,g(z_2)))+\theta\overline{V}(f_2(s,z_2)))\{(z_1-(q+pz_1)\overline{B}_1(f_1(s,z_1,z_2)))\\ \times\,f_1(s,z_1,z_2)-\alpha z_1(1-\overline{B}_1(f_1(s,z_1,z_2)))\overline{D}(f_2(s,z_1,z_2))\,\overline{R}(f_2(s,z_1,z_2))\}\end{array}\right\}}
\tag{64}
$$

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left\{\begin{array}{l}\{(1-(s+\lambda_1+\lambda_2[1-C_2(z_2)])\overline{Q}(s))(1-\theta\overline{V}(f_2(s,g(z_2)))+\theta\overline{V}(f_2(s,z_2)))\\ +\lambda_1 C_1(g(z_2))\overline{Q}(s)(1-\theta+\theta\overline{V}(f_2(s,z_2)))\}f_1(s,z_2)\end{array}\right\}}{\left\{\begin{array}{l}\{(z_2-(1-\theta+\theta\overline{V}(f_2(s,z_1,z_2))))\overline{B}_2(f_1(s,z_2))f_1(s,z_2)-\alpha z_2(1-\overline{B}_2(f_1(s,z_2)))\\ \times\overline{D}(f_2(s,z_2))\overline{R}(f_2(s,z_2))\}(1-\theta\overline{V}(f_2(g(z_2)))+\theta\overline{V}(f_2(z_2)))-(1-\theta+\theta\overline{V}(f_2))\\ \times\{\alpha z_2(1-\overline{B}_2(f_1(s,z_2)))(\overline{D}(f_2(s,g(z_2)))\,\overline{R}(f_2(s,g(z_2)))-\overline{D}(f_2(s,z_2))\,\overline{R}(f_2(s,z_2)))\\ +\lambda_1 C_1(g(z_2))z_2(1-\overline{B}_2)+\theta\overline{B}_2(\overline{V}(f_2(s,g(z_2)))-\overline{V}(f_1(s,z_2)))f_1(s,z_2)\}\end{array}\right\}} \tag{65}$$

The transient solutions of the model under consideration are obtained as,

$$\overline{P}^{(1)}(s,z_1,z_2) =\overline{P}^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}\right], \tag{66}$$

$$\begin{aligned}\overline{V}(s,z_1,z_2) =&\{\theta q\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(f_1(s,z_1,z_2))+\theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(f_1(s,z_2))\}\\ &\times\left[\frac{1-\overline{V}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}\right],\end{aligned} \tag{67}$$

$$\overline{P}_0^{(2)}(s,z_2) =\overline{P}_0^{(2)}(0,s,z_2)\left[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}\right], \tag{68}$$

$$\begin{aligned}\overline{D}(s,z_1,z_2) =&\Big\{\alpha z_1\overline{P}^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}\right]\\ &+\alpha z_2\overline{P}_0^{(2)}(0,s,z_2)\left[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}\right]\Big\}\times\left[\frac{1-\overline{D}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}\right],\end{aligned} \tag{69}$$

$$\begin{aligned}\overline{R}(s,z_1,z_2) =&\{\alpha z_1\overline{P}^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_1(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}\right]\\ &+\alpha z_2\overline{P}_0^{(2)}(0,s,z_2)\left[\frac{1-\overline{B}_2(f_1(s,z_2))}{f_1(s,z_2)}\right]\}\\ &\times\overline{D}(f_2(s,z_1,z_2))\left[\frac{1-\overline{R}(f_2(s,z_1,z_2))}{f_2(s,z_1,z_2)}\right].\end{aligned} \tag{70}$$

3. Steady-state Analysis: Limiting Behaviour

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t)$$

to the equations from (66) to (70), we obtain the steady-state solutions of this model,

$$P^{(1)}(z_1,z_2) =P^{(1)}(0,z_1,z_2)\left[\frac{1-\overline{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}\right], \tag{71}$$

$$V(z_1,z_2) =\{\theta q P_0^{(1)}(0,z_2)\overline{B}_1(f_1(z_1,z_2)) + \theta P_0^{(2)}(0,z_2)\overline{B}_2(f_1(z_2))\} \times \left[\frac{1-\overline{V}(f_2(z_1,z_2))}{f_2(z_1,z_2)}\right], \tag{72}$$

$$P_0^{(2)}(z_2) =P_0^{(2)}(0,z_2)\left[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}\right], \tag{73}$$

$$\begin{aligned} D(z_1,z_2) =&\{\alpha z_1 P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}] \\ &+ \alpha z_2 P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}]\} \times [\frac{1-\overline{D}(f_2(z_1,z_2))}{f_2(z_1,z_2)}], \end{aligned} \tag{74}$$

$$\begin{aligned} \overline{R}(z_1,z_2) =&\{\alpha z_1 P^{(1)}(0,z_1,z_2)\left[\frac{1-\overline{B}_1(f_1(z_1,z_2))}{f_1(z_1,z_2)}\right] \\ &+ \alpha z_2 P_0^{(2)}(0,z_2)\left[\frac{1-\overline{B}_2(f_1(z_2))}{f_1(z_2)}\right]\} \times \overline{D}(f_2(z_1,z_2))\left[\frac{1-\overline{R}(f_2(z_1,z_2))}{f_2(z_1,z_2)}\right]. \end{aligned} \tag{75}$$

In order to determine Q, we use the normalizing condition

$$P^{(1)}(1,1) + V(1,1) + P^{(2)}(0,1) + D(1,1) + R(1,1) + Q = 1.$$

For this, let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations from (71) to (75), we obtain,

$$P_q(z_1,z_2) =P^{(1)}(z_1,z_2) + V(z_1,z_2) + P^{(2)}(0,z_2) + D(z_1,z_2) + R(z_1,z_2),$$

$$W_q(z_1,z_2) =\frac{N_1(z_1,z_2)}{D_1(z_1,z_2)} + \frac{N_2(z_1,z_2)}{D_2(z_1,z_2)} + \frac{N_3(z_1,z_2)}{D_3(z_1,z_2)},$$

where

$$N_1(z_1,z_2) =\theta\lambda_1 C_1(g(z_2))(1-\overline{V}(f_2(z_1,z_2)))Q,$$

$$N_2(z_1,z_2) =P^{(1)}(0,z_1,z_2)(1-\overline{B}_1(f_1(z_1,z_2)))\{f_2(z_1,z_2) + \alpha z_1(1-\overline{R}(f_2(z_1,z_2)) \times \overline{D}(f_2(z_1,z_2)))\},$$

$$\begin{aligned} N_3(z_1,z_2) =&P_0^{(2)}(0,z_2)\{\theta\{\lambda_1 C_1(g(z_2))z_2(1-\overline{B}_2(f_2(z_2))) + \alpha z_2(1-\overline{B}_2(f_2(z_2))) \\ &\times (\overline{D}(f_2(g(z_2)))\overline{R}(f_2(g(z_2))) - \overline{R}(f_2(z_2))\overline{D}(f_2(z_2))) + \theta\overline{B}_2(f_1(z_2)) \\ &\times (\overline{V}(f_2(g(z_2))) - \overline{V}(f_2(z_2)))f_1(z_2)\}(1-\overline{V}(f_2(z_1,z_2))) + (1-\theta\overline{V}(f_2(g(z_2))) \\ &+ \theta\overline{V}(f_2(z_2)))\{(1-\overline{B}_2(f_2(z_2)))f_2(z_1,z_2) + \theta\overline{B}_2(f_2(z_2))(1-\overline{V}(f_2(z_1,z_2)))f_1(z_2) \\ &+ \alpha z_2(1-\overline{B}_2(f_1(z_2)))(1-\overline{R}(f_2(z_1,z_2))\overline{D}(f_2(z_1,z_2)))\}\}, \end{aligned}$$

$$D_1(z_1,z_2) =(1-\theta\overline{V}(f_2(g(z_2))) + \theta\overline{V}(f_2(z_2)))f_2(z_1,z_2),$$

$$D_2(z_1,z_2) =f_1(z_1,z_2)f_2(z_1,z_2),$$

$$D_3(z_1,z_2) =f_1(z_2)f_2(z_1,z_2)(1-\theta\overline{V}(f_2(g(z_2))) + \theta\overline{V}(f_2(z_2)))$$

From this we have,

$$P_0^{(2)}(0,1) = \frac{a}{b}, \tag{76}$$

where

$$\begin{aligned} a = &\Big\{ (1-\theta+\theta\overline{V}(\lambda_1))\lambda_2 E(I_2) + \lambda_1 E(I_1)E(I_3)(1-\theta+\theta\overline{V}(\lambda_1)) - \lambda_1\theta E(V) \\ &\times (-\lambda_1 E(I_3)E(I_1) - \lambda_2 E(I_2) + \delta) \Big\} (\lambda_1+\alpha)Q, \end{aligned}$$

$$\begin{aligned} b =&\Big\{ \Big\{ 1 + (1-\theta+\theta\overline{V}(\lambda_1))\overline{B}_2^{'}\lambda_2 E(I_2) - \theta\overline{V}^{'}(\lambda_1)(-\lambda_2 E(I_2)+\delta) \Big\} (\lambda_1+\alpha) \\ &+ [1-(1-\theta+\theta\overline{V}(\lambda_1))\overline{B_2}(\lambda_1+\alpha)](-\lambda_2 E(I_2)) - \alpha(1-\overline{B}_2)\overline{R}(\lambda_1)\overline{D}(\lambda_1) \\ &- \alpha\overline{B}_2^{'}(\lambda_1+\alpha)\overline{R}(\lambda_1)\overline{D}(\lambda_1)\lambda_2 E(I_2) - \alpha(1-\overline{B}_2(\lambda_1+\alpha))[-\lambda_2 E(I_2)+\delta] \\ &\times (\overline{R}^{'}(\lambda_1)\overline{D}(\lambda_1) + \overline{R}(\lambda_1)\overline{D}^{'}(\lambda_1))\Big\}[1-\theta+\theta\overline{V}(\lambda_1)] + \Big\{(1-(1-\theta+\theta\overline{V}(\lambda_1)) \\ &\times \overline{B}_2(\lambda_1+\alpha))[\lambda_1+\alpha] - \alpha(1-\overline{B}_2(\lambda_1+\alpha))\overline{R}(\lambda_1)\overline{D}(\lambda_1)\Big\} \Big\{ \theta E(V) \\ &\times [-\lambda_1 E(I_1)E(I_3) - \lambda_2 E(I_2) - \delta] + \theta\overline{V}^{'}(\lambda_1)[-\lambda_2 E(I_2)+\delta] \Big\} \\ &+ \Big\{ \alpha(1-\overline{B}_2(\lambda_1+\alpha))(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) + \lambda_1(1-\overline{B}_2) + \theta\overline{B}_2(\lambda_1+\alpha) \\ &\times (1-\overline{V}(\lambda_1))(\lambda_1+\alpha) \Big\} \times\theta\overline{V}^{'}(\lambda_1)[-\lambda_2 E(I_2)+\delta] + (1-\theta+\theta\overline{V}(\lambda_1)) \\ &\times \Big\{ \alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D})(\lambda_1)) + \alpha\overline{B}_2^{'}\lambda_2 E(I_2)(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) \\ &- \alpha(1-\overline{B}_2(\lambda_1+\alpha)) \Big\{ (E(R)+E(D))[-\lambda_1 E(I_1)E(I_3) - \lambda_2 E(I_2)+\delta] \\ &+ (\overline{R}^{'}(\lambda_1)\overline{D}(\lambda_1) + \overline{R}(\lambda_1)\overline{D}^{'}(\lambda_1))[-\lambda_2 E(I_2)+\delta] \Big\} + \lambda_1 E(I_1)E(I_3) \\ &\times (1-\overline{B}_2(\lambda_1+\alpha)) + \lambda_1(1-\overline{B}_2(\lambda_1+\alpha)) + \lambda_1 E(I_2)\lambda_2\overline{B}_2^{'}(\lambda_1+\alpha) \\ &- \theta\overline{B}_2^{'}(\lambda_1+\alpha)\lambda_2 E(I_2)(1-\overline{V}(\lambda_1))[\lambda_1+\alpha] + \theta\overline{B}_2(\lambda_1+\alpha) \Big\{ -E(V) \\ &\times [-\lambda_1 E(I_1)E(I_3) - \lambda_2 E(I_2)+\delta] - \overline{V}^{'}(\lambda_1)[-\lambda_2 E(I_2)+\delta] \Big\} (\lambda_1+\alpha) \\ &- \theta\lambda_2 E(I_2)\overline{B}_2(\lambda_1+\alpha)(1-\overline{V}(\lambda_1))\Big\} \end{aligned}$$

$$P^{(1)}(0,1,1) = \frac{\left\{ \begin{aligned} &\alpha\lambda_1 E(I_1)(1-E(I_3))Q + \alpha P_0^{(2)}(0,1) \Big\{ \theta E(V)[-\lambda_1 E(I_1)E(I_3) \\ &- \lambda_1 E(I_1)] \Big\{ \alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D})(\lambda_1) + \alpha\overline{B}_2(1-\overline{V}(\lambda_1)) \\ &+ \lambda_1(1-\overline{B}_2) \Big\} + (1-\theta+\theta\overline{V}(\lambda_1)) \Big\{ \lambda_1 E(I_1)(1-\overline{B}_2) \\ &\times (1-E(I_3)) + \alpha\lambda_1 E(I_1)(1-\overline{B}_2)[E(R)+E(D)](1-E(I_3)) \\ &+ \theta\lambda_1 E(I_1)E(V)\overline{B}_2(1-E(I_3)) \Big\}\Big\} \end{aligned} \right\}}{\left\{ \begin{aligned} &\alpha \Big\{ 1 + [\lambda_1 E(I_1) + \lambda_2 E(I_2)]\overline{B}_1^{'}(\alpha) - p\overline{B}_1(\alpha) \Big\} - [\lambda_1 E(I_1) \\ &+ \lambda_2 E(I_2)](1-\overline{B}_1) - \alpha(1-\overline{B}_1)\alpha\overline{B}_1^{'}(\alpha)[\lambda_1 E(I_1) + \lambda_2 E(I_2)] \\ &+ \alpha(1-\overline{B}_1(\alpha))[E(R)+E(D)][-\lambda_1 E(I_1) - \lambda_2 E(I_2)+\delta] \end{aligned} \right\}}. \tag{77}$$

We use the normalizing condition $P_q(1,1) + Q = 1$ to find Q, as

$$Q = \frac{\left\{\begin{array}{l} \alpha(\lambda_1+\alpha)(1-\theta+\theta\overline{V}(\lambda_1)) - (\lambda_1+\alpha)(1-\theta+\theta\overline{V}(\lambda_1)) \left\{ P^{(1)}(0,1,1) \right. \\ \left. \times (1-\overline{B}_1(\alpha))(1+\alpha(E(R)+E(D))) \right\} - \alpha P_0^{(2)}(0,1)\{\theta \left\{ \lambda_1(1-\overline{B}_2) \right. \\ +\alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1))(\lambda_1+\alpha)\theta\overline{B}_2(1-\overline{V}(\lambda_1)) \left. \right\} E(V) + \\ (1-\theta+\theta\overline{V}(\lambda_1)) \left\{ (1-\overline{B_2})(1+\alpha(E(R)+E(D))) - \theta[\lambda_1+\alpha] \right. \\ \left. \times \overline{B}_2 E(V)(\lambda_1+\alpha) \right\} \} \end{array}\right\}}{\left\{ \alpha(\lambda_1+\alpha)\{(1-\theta+\theta\overline{V}(\lambda_1)) + \theta\lambda_1 E(V)\} \right\}}. \tag{78}$$

And the utilization factor $\rho = 1 - Q$ is,

$$\rho = \frac{\left\{\begin{array}{l} \alpha\lambda_1(\lambda_1+\alpha)\theta E(V) + (\lambda_1+\alpha)(1-\theta+\theta\overline{V}(\lambda_1)) \left\{ P^{(1)}(0,1,1)(1-\overline{B_1}(\alpha)) \right. \\ \left. \times (1+\alpha(E(R)+E(D))) \right\} + \alpha P_0^{(2)}(0,1)\{\theta \left\{ \lambda_1(1-\overline{B}_2(\lambda_1+\alpha)) \right. \\ \left. +\alpha(1-\overline{B}_2(\lambda_1+\alpha))(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1))(\lambda_1+\alpha)\theta\overline{B}_2(\lambda_1+\alpha)(1-\overline{V}(\lambda_1)) \right\} \\ \times E(V) + (1-\theta+\theta\overline{V}(\lambda_1)) \left\{ (1-\overline{B}_2(\lambda_1+\alpha))(1+\alpha(E(R)+E(D))) \right. \\ \left. -\theta[\lambda_1+\alpha]\overline{B}_2(\lambda_1+\alpha)E(V)(\lambda_1+\alpha) \right\} \} \end{array}\right\}}{\left\{ \alpha(\lambda_1+\alpha) \left\{ (1-\theta+\theta\overline{V}(\lambda_1)) + \theta\lambda_1 E(V)\right\} \right\}}, \tag{79}$$

where $\rho < 1$ is the stability condition under which steady state exists, for the model studied.

4. The Average Queue Length

The mean number of units in high priority queue under the steady state is,

$$L_{q_1} = \frac{d}{dz_1} P_{q_1}(z_1, 1)|_{z_1=1}. \tag{80}$$

The mean number of units in the low priority queue under the steady state is,

$$L_{q_2} = \frac{d}{dz_2} P_{q_2}(1, z_2)|_{z_2=1}. \tag{81}$$

Thus,

$$\begin{aligned} L_{q_1} =& \frac{D_1'(1,1)N_1''(1,1) - D_1''(1,1)N_1'(1,1)}{2(D_1'(1,1))^2} + \frac{D_2'(1,1)N_2''(1,1) - D_2''(1,1)N_2'(1,1)}{2(D_2'(1,1))^2} \\ &+ \frac{D_3'(1,1)N_3''(1,1) - D_3''(1,1)N_3'(1,1)}{2(D_3'(1,1))^2}, \end{aligned}$$

$$\begin{aligned} L_{q_2} =& \frac{d_1'(1,1)n_1''(1,1) - d_1''(1,1)n_1'(1,1)}{2(d_1'(1,1))^2} + \frac{d_2'(1,1)n_2''(1,1) - d_2''(1,1)n_2'(1,1)}{2(d_2'(1,1))^2} \\ &+ \frac{d_3'(1,1)n_3''(1,1) - d_3''(1,1)n_3'(1,1)}{2(d_3'(1,1))^2}, \end{aligned}$$

where

$$N_1'(1,1) = -\theta\lambda_1^2 E(I_1)E(V)Q,$$

$$N_1''(1,1) = -\theta\lambda_1^3 E(I_1)^2 E(V^2)Q - \theta\lambda_1^2 E(V)E(I_1(I_1-1))Q,$$

$$N_2'(1,1) = -\lambda_1 E(I_1)P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha))[1+\alpha(E(R)+E(D))],$$

$$\begin{aligned} N_2''(1,1) = &-2\lambda_1 E(I_1)P'^{(1)}(0,1,1)(1-\overline{B}_1(\alpha))[1+\alpha(E(R)+E(D))] - 2[\lambda_1 E(I_1)]^2 \\ &\times \overline{B}_1'(\alpha)P^{(1)}(0,1,1)[1+\alpha(E(R)+E(D))] - P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha)) \\ &\times \left\{ \lambda_1 E(I_1(I_1-1))[1+\alpha(E(R)+E(D))] + 2\lambda_1 E(I_1)\alpha \left\{ E(R)+E(D) \right\} \right. \\ &\left. + \alpha \left\{ [\lambda_1 E(I_1)]^2 (E(R^2)+E(D^2)+2E(R)E(D)) \right\} \right\}, \end{aligned}$$

$$\begin{aligned} N_3'(1,1) = &P_0^{(2)}(0,1)\big\{\theta \left\{ \lambda_1(1-\overline{B}_2) + \alpha(1-\overline{B}_2(\lambda_1+\alpha))(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) \right. \\ &\left. + (\lambda_1+\alpha)\theta\overline{B}_2(\lambda_1+\alpha)(1-\overline{V}(\lambda_1)) \right\} E(V)(-\lambda_1 E(I_1)) \\ &+ (1-\theta+\theta\overline{V}(\lambda_1))\big\{(-\lambda_1 E(I_1)(1-\overline{B}_2)(\lambda_1+\alpha))[1+\alpha(E(R)+E(D))] \\ &- \lambda_1\theta E(I_1)\overline{B}_2(\lambda_1+\alpha)[\lambda_1+\alpha]E(V)\big\}\big\}, \end{aligned}$$

$$\begin{aligned} N_3''(1,1) = &P_0^{(2)}(0,1)\big\{\theta\big\{\lambda_1(1-\overline{B}_2(\lambda_1+\alpha)) + \alpha(1-\overline{B}_2(\lambda_1+\alpha))(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) \\ &+ (\lambda_1+\alpha)\theta\overline{B}_2(1-\overline{V}(\lambda_1))\big\}(-E(V^2)(\lambda_1 E(I_1)^2) - \lambda_1 E(I_1(I_1-1))E(V))\big\} \\ &+ (1-\theta+\theta\overline{V}(\lambda_1))\big\{\lambda_1 E(I_1(I_1-1))(1-\overline{B}_2(\lambda_1+\alpha)) \\ &+ \theta\overline{B}_2(\lambda_1+\alpha)[\lambda_1+\alpha][-E(V^2)(\lambda_1 E(I_1))^2 - E(V)\lambda_1 E(I_1(I_1-1))] \\ &+ \alpha(1-\overline{B}_2(\lambda_1+\alpha))\big\{(-(E(R^2)+2E(R)E(D)+E(D^2)))(\lambda_1 E I_1)^2 \\ &- \lambda_1 E(I_1(I_1-1))(E(R)+E(D))\big\}\big\}, \end{aligned}$$

$$D_1'(1,1) = -\lambda_1 E(I_1)(1-\theta+\theta\overline{V}(\lambda_1)),$$

$$D_1''(1,1) = -\lambda_1 E(I_1(I_1-1))(1-\theta+\theta\overline{V}(\lambda_1)),$$

$$D_2'(1,1) = -\alpha\lambda_1 E(I_1),$$

$$D_2''(1,1) = 2(\lambda_1 E(I_1))^2 - \alpha\lambda_1 E(I_1(I_1-1)),$$

$$D_3'(1,1) = -\lambda_1 E(I_1)(\lambda_1+\alpha)(1-\theta+\theta\overline{V}(\lambda_1)),$$

$$D_3''(1,1) = -\lambda_1 E(I_1(I_1-1))(\lambda_1+\alpha)(1-\theta+\theta\overline{V}(\lambda_1)),$$

$$n_1'(1,1) = \theta\lambda_1(-\lambda_2 E(I_2)+\delta)E(V)Q,$$

$$\begin{aligned} n_1''(1,1) = &-\theta\lambda_1(-\lambda_2 E(I_2)+\delta)^2 E(V^2)Q - \theta\lambda_1 E(V) \left\{ \lambda_2 E(I_2(I_2-1)) + 2\delta \right\} Q \\ &+ 2\theta\lambda_1 E(I_1)E(I_3)\big\{ -\lambda_2 E(I_2)+\delta\big\}E(V)Q, \end{aligned}$$

$$n_2'(1,1) = P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha))(-\lambda_2 E(I_2)+\delta)[1+\alpha(E(R)+E(D))],$$

$$\begin{aligned} n_2''(1,1) = &2P'^{(1)}(0,1,1)(1-\overline{B}_1(\alpha))(-\lambda_2 E(I_2)+\delta)[1+\alpha(E(R)+E(D))] \\ &+ 2P^{(1)}(0,1,1)\overline{B}_1'(\alpha)(\lambda_2 E(I_2))(-\lambda_2 E(I_2)+\delta)[1+\alpha(E(R)+E(D))] \\ &- P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha)) \left\{ (\lambda_2 E(I_2(I_2-1)) + 2\delta)[1+\alpha(E(R)+E(D))] \right. \\ &\left. + \alpha(-\lambda_2 E(I_2)+\delta)^2 (E(R^2)+E(D^2)+2E(R)E(D)) \right\}, \end{aligned}$$

$$
\begin{aligned}
n_3'(1,1) =& P_0^{(2)}(0,1)\big\{\theta\big\{\alpha(1-\overline{B}_2(\lambda_1+\alpha))(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) + \lambda_1(1-\overline{B}_2(\lambda_1+\alpha))\\
&+\theta(\lambda_1+\alpha)(1-\overline{V}(\lambda_1))\big\}E(V)(-\lambda_2 E(I_2)+\delta)\\
&+(1-\theta+\theta\overline{V}(\lambda_1))\big\{(1-\overline{B}_2)(-\lambda_2 E(I_2)+\delta)\\
&+\theta\overline{B}_2 E(V)(-\lambda_2 E(I_2)+\delta)(\lambda_1+\alpha)+\alpha(1-\overline{B}_2)[E(R)+E(D)](-\lambda_2+\delta)\big\}\big\},
\end{aligned}
$$

$$
\begin{aligned}
n_3''(1,1) =& 2P_0^{'(2)}(0,1)\big\{\theta\big\{\alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1)) + \lambda_1(1-\overline{B}_2)\\
&+\theta[\lambda_1+\alpha]\overline{B}_2(1-\overline{V}(\lambda_1))\big\}E(V)(-\lambda_2 E(I_2)+\delta)+(1-\theta+\theta\overline{V}(\lambda_1))\\
&\times\big\{(1-\overline{B}_2)(-\lambda_2 E(I_2)+\delta)+\theta\overline{B}_2 E(V)(-\lambda_2 E(I_2)+\delta)\\
&\times(\lambda_1+\alpha)+\alpha(1-\overline{B}_2)[E(R)+E(D)](-\lambda_2 E(I_2)+\delta)\big\}\big\}\\
&+P_0^{(2)}(0,1)\big\{2\theta\ \big\{\ \alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1))+\alpha\lambda_2 E(I_2)\overline{B}_2^{'})\\
&\times(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1))+\alpha(1-\overline{B}_2)\big\{[E(R)+E(D)](-\lambda_1 E(I_1)E(I_3)\\
&-\lambda_2 E(I_2)+\delta)-(\overline{R}^{'}(\lambda_1)\overline{D}(\lambda_1)+\overline{R}(\lambda_1)\overline{D}^{'}(\lambda_1))[-\lambda_2 E(I_2)+\delta]\big\}\\
&\times\lambda_1 E(I_1)E(I_3)(1-\overline{B}_2)+\lambda_1(1-\overline{B}_2)+\lambda_1\lambda_2 E(I_2)\overline{B}_2^{'}-\theta\lambda_2 E(I_2)\overline{B}_2\\
&\times(\lambda_1+\alpha)(1-\overline{V}(\lambda_1))-\theta\lambda_2 E(I_2)\overline{B}_2^{'}(1-\overline{V}(\lambda_1))(\lambda_1+\alpha)-\theta\overline{B}_2\big\}\\
&+E(V)[-\lambda_2+\delta]\theta\ \big\{\ \alpha(1-\overline{B}_2)(1-\overline{R}(\lambda_1)\overline{D}(\lambda_1))+\lambda_1(1-\overline{B}_2)\\
&+\theta\overline{B}_2[\lambda_1+\alpha](1-\overline{V}(\lambda_1))\ \big\}\ \big\{-E(V^2)[-\lambda_2 E(I_2)+\delta]^2\\
&+E(V)[-\lambda_2 E(I_2(I_2-1))-2\delta]\big\}+(1-\theta+\theta\overline{V}(\lambda_1))\ \big\{\ 2\lambda_2 E(I_2)\overline{B}_2^{'}\\
&\times(\lambda_2 E(I_2)+\delta)-(1-\overline{B}_2)(\lambda_2 E(I_2(I_2-1))+2\delta)+2\theta\overline{B}_2^{'}(\lambda_1+\alpha)\\
&\times(-\lambda_2 E(I_2))(-\lambda_2 E(I_2+\delta)E(V)[\lambda_1+\alpha]-\theta\overline{B}_2(-\lambda_2 E(I_2+\delta)^2\\
&\times E(V^2)[\lambda_1+\alpha]+\theta\lambda_2 E(I_2)\overline{B}_2(-\lambda_2 E(I_2(I_2-1))+\delta)E(V)[\lambda_1+\alpha]\\
&-2\theta\lambda_2 E(I_2)\overline{B}_2(-\lambda_2 E(I_2)+\delta)E(V)+2\alpha(1-\overline{B}_2)[E(R)+E(D)]\\
&\times(-\lambda_2 E(I_2)+\delta)+2\alpha\lambda_2 E(I_2)B_2^{'}[E(R)+E(D)](-\lambda_2 E(I_2)+\delta)\\
&-\alpha(1-\overline{B}_2)[E(R^2)+E(D^2)+2E(R)E(D)](-\lambda_2 E(I_2)+\delta)^2\\
&+\alpha(1-\overline{B}_2)[E(R)+E(D)](-\lambda_2 E(I_2(I_2-1))-2\delta)\ \big\}\\
&+2\ \big\{\ \theta E(V)(-\lambda_1 E(I_1)E(I_3)-\lambda_2 E(I_2)+\delta)+\theta\overline{V}^{'}(\lambda_1)(-\lambda_2 E(I_2)+\delta)\ \big\}\\
&\times\big\{(1-\overline{B}_2)(-\lambda_2 E(I_2)+\delta)+\theta\overline{B}_2 E(V)(-\lambda_2 E(I_2)+\delta)(\lambda_1+\alpha)\\
&+\alpha(1-\overline{B}_2)[E(R)+E(D)](-\lambda_2 E(I_2)+\delta)\big\}\big\},
\end{aligned}
$$

$$
d_1'(1,1) = (-\lambda_2 E(I_2)+\delta)(1-\theta+\theta\overline{V}(\lambda_1)),
$$

$$
\begin{aligned}
d_1''(1,1) =& -(1-\theta+\theta\overline{V}(\lambda_1))(\lambda_2 E(I_2(I_2-1))+2\delta)+(-\lambda_2 E(I_2)+\delta)\\
&\times 2\ \big\{\ \theta E(V)[-\lambda_1 E(I_1)E(I_3)-\lambda_2 E(I_2)+\delta]+\theta\overline{V}^{'}(\lambda_1)[-\lambda_2 E(I_2)+\delta]\ \big\},
\end{aligned}
$$

$$
d_2'(1,1) = -\alpha(-\lambda_2 E(I_2)+\delta),
$$

$$
d_2''(1,1) = -2(\lambda_2 E(I_2))(-\lambda_2 E(I_2)+\delta)-\alpha(\lambda_2 E(I_2(I_2-1))+2\delta),
$$

$$
d_3'(1,1) = (-\lambda_2 E(I_2)+\delta)(1-\theta+\theta\overline{V}(\lambda_1))(\lambda_1+\alpha),
$$

$$\begin{aligned}d_3''(1,1) =&2\left\{\ \theta E(V)[-\lambda_1 E(I_1)E(I_3)-\lambda_2 E(I_2)+\delta]+\theta\overline{V}^{'}(\lambda_1)[-\lambda_2 E(I_2)+\delta]\ \right\}\\ &\times(\lambda_1+\alpha)+[-\lambda_2 E(I_2(I_2-1))-2\delta](1-\theta+\theta\overline{V}(\lambda_1))(\lambda_1+\alpha)\\ &-2\lambda_2 E(I_2)[-\lambda_2 E(I_2)+\delta](1-\theta+\theta\overline{V}(\lambda_1)).\end{aligned}$$

4.1. The Average Waiting Time in the Queue

Average waiting time of a customer in the high priority queue is

$$W_{q_1}=\frac{L_{q_1}}{\lambda_1}, \tag{82}$$

Average Waiting time of a customer in the low priority queue is

$$W_{q_2}=\frac{L_{q_2}}{\lambda_2}. \tag{83}$$

where L_{q_1} and L_{q_2} have been found in eqs. (80) and (81).

4.2. Particular Cases

Case 1: $M/G/1$ Queueing model.

If there is no arrival of high priority units, no vacation, no breakdown, single arrival and the service time is exponential $\big($i.e $\lambda_1=0,\ \alpha=0,\ \theta=0,\ ,\ E(I_2)=1,\ E(I_2(I_2-1))=0,$ $E(I_3)=\frac{\lambda_2}{(\mu-\lambda_1)}$ and $E[I_3(I_3-1)]=\frac{2\lambda_2\mu}{(\mu-\lambda_1)^3}$), then,

$$\begin{aligned}Q&=1-\lambda_2E(B),\\ \rho&=\lambda_2E(B),\\ L_q&=\frac{D'(1)N''(1)-D''(1)N'(1)}{2(D'(1))^2},\end{aligned}$$

where

$$\begin{aligned}N'(1)&=\lambda_2E(B)Q,\\ N''(1)&=E(B^2)(\lambda_2)^2Q,\\ D_1'(1)&=1-E(B)\lambda_2,\\ D_1''(1)&=-E(B^2)(\lambda_2)^2.\end{aligned}$$

Case 2: $M^{[X]}/G/1$ Queueing model.

If there is no arrival of high priority units, no vacation, no breakdown, batch arrival and the service time is exponential, $\big($i.e $\lambda_1=0,\ \alpha=0,\theta=0,\ ,\ E(I_3)=\frac{\lambda_2}{(\mu-\lambda_1)},E[I_3(I_3-1)]=\frac{2\lambda_2\mu}{(\mu-\lambda_1)^3}$), then,

$$\begin{aligned}Q&=1-\lambda_2E(I_2)E(B),\\ \rho&=\lambda_2E(I_2)E(B),\\ L_q&=\frac{D'(1)N''(1)-D''(1)N'(1)}{2(D'(1))^2},\end{aligned}$$

where

$$N'(1) = \lambda_2 E(I_2)E(B)Q,$$
$$N''(1) = E(B^2)E(I_2^2)(\lambda_2)^2 Q + \lambda_2 E(B)E(I(I-1))Q,$$
$$D_1'(1) = 1 - E(I_2)E(B)\lambda_2,$$
$$D_1''(1) = -E(B^2)E(I_2^2)(\lambda_2)^2 - \lambda_2 E(B)E(I(I-1)).$$

The above two results coincide with the results of Gross et al. (1985).

5. Numerical Results

In order to see the effect of different parameters on the different states of the server, the utilization factor and proportion of idle time, we compute some numerical results. We consider the service time, vacation time and repair time to be exponentially distributed to numerically illustrate the feasibility of our results. Giving the suitable values to the parameters satisfy the stability condition in (128), we compute the following values.

Table 1: $(\lambda_2, \mu_1, \mu_2, \theta, \alpha, \phi, \gamma, \delta, p, \beta) = (0.4, 9, 9, 0.1, 0.9, 2, 5, 0.1, 0.7, 3)$.

Table 2: $(\lambda_1, \mu_1, \mu_2, \theta, \alpha, \phi, \gamma, \delta, p, \beta) = (0.2, 2, 2, 0.5, 0.1, 5, 8, 2, 0.1, 1.1)$.

Table 3: $(\lambda_1, \lambda_2, \mu_1, \mu_2, \theta, \phi, \gamma, \delta, p, \beta) = (0.1, 2.1, 1, 0.1, 1, 1, 1, 2, 0.7, 1)$.

Table 1. Effect of λ_1 on various queue characteristics

λ_1	Q	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
1.5	0.8186	0.1814	0.0139	0.0016	0.0093	0.0040
1.6	0.8063	0.1937	0.0506	0.0027	0.0316	0.0067
1.7	0.7939	0.2061	0.0977	0.0040	0.0575	0.0101
1.8	0.7814	0.2186	0.1571	0.0057	0.0873	0.0143
1.9	0.7688	0.2439	0.2304	0.0078	0.1212	0.0194
2.0	0.7561	0.2439	0.3194	0.0102	0.1597	0.0255
2.1	0.7433	0.2567	0.4263	0.0131	0.2030	0.0328
2.2	0.7305	0.2695	0.5532	0.0165	0.2515	0.0412
2.3	0.7175	0.2825	0.7025	0.0204	0.3054	0.0511
2.4	0.7044	0.2956	0.8766	0.0250	0.3652	0.0625

Table 1 clearly shows that as long as the arrival rate of high priority units increase the server's idle time decreases while the utilisation factor, average queue length for high priority units as well as their waiting time all increase.

Table 2 reveals that if the arrival rate of low priority units increases the server's idle time decreases while the utilisation factor, average queue length for low priority units their waiting time all increase.

Table 2. Effect of λ_2 on various queue characteristics

λ_2	Q	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
2.7	0.7314	0.2686	0.0028	0.0057	0.0140	0.0021
2.8	0.7241	0.2759	0.0028	0.0182	0.0140	0.0065
2.9	0.7172	0.2828	0.0028	0.0423	0.0140	0.0146
3.0	0.7106	0.2894	0.0028	0.0842	0.0140	0.0281
3.1	0.7042	0.2958	0.0028	0.1514	0.0140	0.0488
3.2	0.6980	0.3020	0.0028	0.2534	0.0140	0.0792
3.3	0.6920	0.3080	0.0028	0.4018	0.0140	0.1218
3.4	0.6862	0.3138	0.0028	0.6102	0.0140	0.1795
3.5	0.6805	0.3195	0.0028	0.8945	0.0140	0.2556
3.6	0.6750	0.3250	0.0028	1.2730	0.0140	0.3536

Table 3. Effect of α on various queue characteristics

α	Q	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
2.0	0.0398	0.9602	0.0048	0.3456	0.0483	0.1646
2.1	0.0334	0.9666	0.0202	0.5518	0.2022	0.2628
2.2	0.0278	0.9722	0.0410	0.8547	0.4097	0.4070
2.3	0.0229	0.9771	0.0686	1.2899	0.6857	0.6143
2.4	0.0185	0.9815	0.1049	1.9028	1.0490	0.9061
2.5	0.0147	0.9853	0.1522	2.7509	1.5225	1.3100
2.6	0.0113	0.9887	0.2134	3.9059	2.1340	1.8600
2.7	0.0082	0.9918	0.2918	5.4568	2.9176	2.5985
2.8	0.0055	0.9945	0.3915	7.5127	3.9149	3.5775
2.9	0.0032	0.9968	0.5176	10.2064	5.1761	4.8602

Table 3 shows that as long as breakdown rate increases, idle time decreases while the utilisation factor, queue size and waiting time of both high priority and low priority units all increase.

5.1. Graphical Study

We can plot the above data graphically to illustrate the feasibility of our results.

Figure 1 graphically represents the effect of high priority arrival rate over the idle period and queue length of the model. It is clear from the figure that if arrival rate increases while all other parameters are fixed the queue length of priority queue and busy period increase but idle time decreases.

Figure 2 graphically represents the effect of low priority arrival rate over the idle period and queue

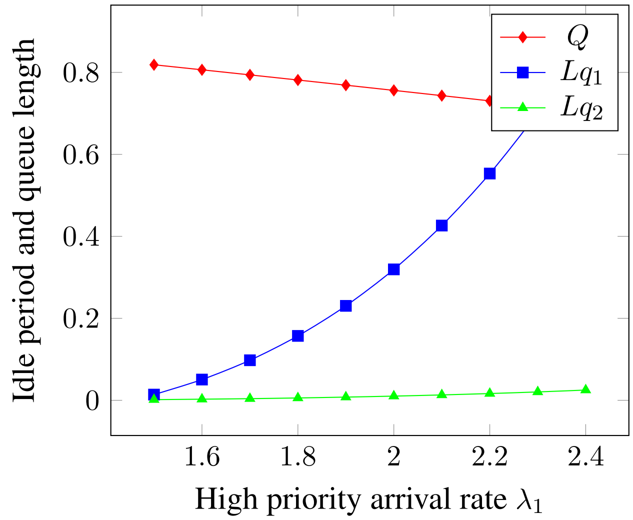

Figure 1. Average queue size vs. High priority arrival rate λ_1

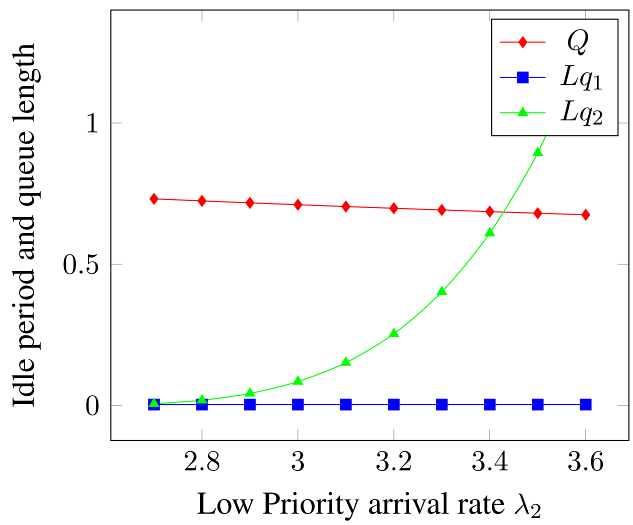

Figure 2. Average queue size vs. Low priority arrival rate λ_2

length of the model. It is clear from the figure that the queue length of low priority units increases rapidly, with the increase in λ_2.

Figure 3 graphically represents the effect of breakdown rate over the idle period and queue length of the model. Due to breakdown, the queue length of high and low priority units increase, the proportion of idle time of the server decreases and utilization factor or busy period increases.

6. Conclusion

In this paper a priority queueing system with modified Bernoulli vacation, Bernoulli feedback, breakdown, delay time to repair and reneging during breakdown and vacation period is analyzed. The server provides general service to two types of units namely high priority and low priority

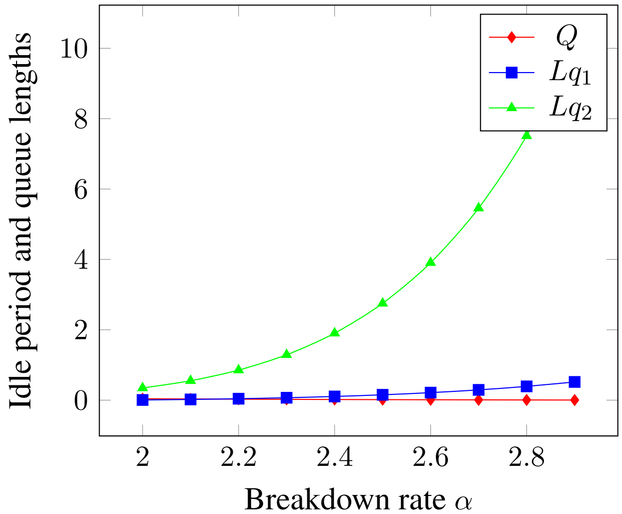

Figure 3. Average queue sizes vs. Breakdown rate α

units under preemptive priority rule. The probability generating functions of the number of customers in the high priority and low priority units are derived. By using the supplementary variable technique, average queue size, the average waiting time for the high priority and low priority units are computed. Under the stability condition, some numerical results and graphical study are also carried out.

Acknowledgement:

The authors are thankful to the anonymous referees and Professor Aliakbar Montazer Haghighi, for their valuable comments and suggestions for the improvement of the paper.

REFERENCES

Ayyappan, G. and Thamizhselvi, P. (2016). Priority Queueing System with a Single Server Serving Two Queues $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ with Balking and Optional Server Vacation, AAM: Intern. J., Vol. 11, Issue 1.

Bhagat, A. and Jain, M. (2016). N-policy for $M^X/G/1$ unreliable retrial G-queue with preemptive resume and multi-services, J. Oper. Res. Soc. China, Vol. 4, pp. 437–459.

Cobham, A. (1954). Priority assignment in waiting line problems, Operations Research, Vol. 2, pp. 70–76.

Cox, D.R. (1955). The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables, Proc. Cambridge Phil. Soc., Vol. 51, pp. 433–441.

Gao, S. (2015). A preemptive priority retrial queue with two classes of customers and general retrial times, Oper. Res. Int. J., Vol. 15, Issue 2, pp. 233–251.

Gross, D. and Harris, C.M. (1985). *Fundamentals of Queueing Theory: 2nd Edition*, Wiley, New York.

Haghighi, A.M. and Mishev, D. (2006). A parallel priority queueing system with finite buffers, Journal of Parallel and Distributed Computing, Vol. 66, pp. 379–392.

Hawkes, A.G. (1965). Time dependent solution of a priority queue with bulk arrivals, Operations Research, Vol. 13, No. 4, pp. 586–595.

Jain, M. and Bhargava, C. (2008). Bulk Arrival Retrial Queue with Unreliable Server and Priority Subscribers, International Journal of Operations Research, Vol. 5, No. 4, pp. 242–259.

Jaiswal, N.K. (1968). Priority Queues, Pacific Asian Journal of Mathematics.

Khalaf, R.F., Madan, K.C. and Lukas, C.A. (2011). A $M^X/G/1$ with Bernoulli Schedule General Vacation Times, General Extended Vacations, Random Breakdown, General Delay Time for Repairs to Start and General Repair Times, Journal of Mathematical Research, Vol. 3, No. 4.

Kleinrock, L. (1976). *Queueing Systems, Vol. 2: Computer Applications*, John Wiley & and Sons, New York.

Madan, K.C. (2011). A Non-Preemptive Priority Queueing System with a Single Server Serving Two Queues $M/G/1$ and $M/D/1$ with Optional Server Vacations Based on Exhaustive Service of the Priority Units, Applied Mathematics, Vol. 2, pp. 791–799.

Jain, M. and Jain, A. (2014). Batch Arrival Priority Queueing Model with Second Optional Service and Server Breakdown, International Journal of Operations Research, Vol. 11, No. 4, pp. 112–130

Baruah, M., Madan, K.C. and Eldabi, T. (2013). A Two Stage Batch Arrival Queue with Reneging during Vacation and Breakdown Periods, American Journal of Operations Research, Vol. 3, pp. 570–580.

Murugan, S.P.B. and Santhi, K. (2015). An $M/G/1$ queue with server breakdown and multiple working vacation, Appl. Appl. Math., Vol. 10, pp. 678–693.

Takagi, H. (1990). Time-dependent analysis of an $M/G/1$ vacation models with exhaustive service, Queueing Systems, Vol. 6, No. 1, pp. 369 – 390.

Thangaraj, V. and Vanitha, S. (2010). An $M/G/1$ queue with two-stage heterogeneous service compulsory server vacation and random breakdowns, Int. J. Contemp. Math. Sciences, Vol. 5, No. 7, pp. 307–322.

Singh, C.J., Jain, M. and Kumar, B. (2014). Analysis of $M^X/G/1$ queueing model with balking and vacation, Int. J. Oper. Res., Vol. 19, No. 2, pp. 154–73.