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Numerical solution of fractional elliptic PDE's by the collocation method

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Abstract

In this presentation a numerical solution for the solution of fractional order of elliptic partial differential equation in R^2 is proposed. In this method we use the Radial basis functions (RBFs) method to benefit the desired properties of mesh free techniques such as no need to generate any mesh and easily applied to multi dimensions. In the numerical solution approach the RBF collocation method is used to discrete fractional derivative terms with the Gaussian basis function. Two dimensional numerical examples are presented and discussed, which conform well with the corresponding exact solutions.

Keywords: Conformable fractional calculus; radial basis function; collocation method

MSC 2010 No.: 65L60, 26A33

1. Introduction

In a number of practices, data is produced with no knowledge of a function from which it was derived. Therefore, an approximation model is needed. However, many physical systems could only be modelled by using the non-integer order of derivatives and integrals. For instance, non-integer order of models are studied in control theory, computational analysis and engineering Kilbas (2016), Samko (1993). Thus, a number of new definitions have been introduced in academia to provide the best method for fractional calculus. Here, all fractional derivatives do not provide some properties such as Product Rule, Quotient Rule, Chain Rule, Roll's Theorem and Mean Value Theorem. To overcome some of these and other difficulties, Abdeljawad (2015), Khalil (2014), Katugampola (2014), came up with an interesting idea that extends the familiar limit definition of the derivative of a function. In this work, we focus on numerical solution of partial differential equations which are modelled with Katugampola derivatives Katugampola (2014).

Poisson equation is one of the most popular elliptic differential equations with broad utility in theoretical physics, mechanical engineering and electrostatics. However, a number of physical systems could only be modelled by using the non-integer order of derivatives and integrals. A lot

of analytical and numerical methods of such systems have been proposed in academia such as variation iteration method Khan (2011), fractional finite difference method Borhanifar (2012), Meerschaert (2016), homotopy perturbation method Li (2009), Song (2007), Adomian decomposition method Yang (2010), Grag (2011) and any other areas Eslami (2016a), Eslami (2016b), Ekici (2016).

In addition to this, radial basis functions method is one of the more practical ways of solving fractional order of models. The most important property of an RBF technique is that there is no need to generate any mesh and so it is called the mesh-free method. One only requires the pairwise distance between points for an RBF approximation Buhmann (2003), Cheney (1999). This method is easy to implement in multi dimensional cases due to the nature of RBF. On the other hand in order to solve partial differential equations (PDEs) Kansa proposed RBF collocation method which is mesh-free and easy-to-handle in comparison with the other methods Kansa (1990a), Kansa (1990b), Franke (1998).

This prospective study was designed to investigate the use of RBF methods to solve the conformable fractional elliptic partial differential equations via Kansa's collocation method. The remainder of this work is organized as follows: In Section 2, the related definitions of RBFs is summarised. Then, non integer order of Poisson equation is summarized in Section 3. In Section 4, the conformable fractional derivative and integrals have been reviewed. The computational scheme is given in Section 5, while some numerical experiments are presented in Section 6. Finally, we have summarised the current study in Section 7.

2. Radial Basis Function Interpolation

In this part, the fundamental concept of the mesh-free radial basis function interpolation are explained. Consider a function $u : \mathrm{R}^d \to \mathrm{R}$ a real valued function with d variables, which is to be approximated by $\mathtt{I}_{\mathbf{x}} : \mathrm{R}^d \to \mathrm{R}$, for given values $u(\mathbf{x}_i) : i = 1, 2, \cdots, n$, where $\mathbf{x}_i : i = 1, 2, \cdots, n$ is a set of distinct points in R^d, named the center set $\mathbf{x}$.

Then, the approximation to the function u is of the form:

$$\mathtt{I}_{\mathbf{x}}(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k \psi_k(\|\mathbf{x} - \mathbf{x}_k\|),$$

where $\psi_k : \mathrm{R}^d \to \mathrm{R}$ is a univariate radial basis function. Now the interpolation condition can be constructed as $\mathtt{I}_{\mathbf{x}}(\mathbf{x}_m) = u(\mathbf{x}_m)$, $m = 1, 2, \cdots, N$. Namely, the interpolation condition is

$$\sum_{k=1}^{N} \lambda_k \psi_k(\|\mathbf{x}_m - \mathbf{x}_k\|) = u(\mathbf{x}_m), \quad m = 1, 2, \cdots, N.$$

In other words, the system of matrix for interpolation condition can be written as $[A]\{\lambda\} = \{u\}$, where the entries of the matrix A are $A_{m,k} = \psi_k(\|\mathbf{x}_m - \mathbf{x}_k\|)$ such that $m, k = 1, 2, \cdots, N$, $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}^T$ and $u = \{u_1, u_2, \cdots, u_N\}^T$. This scheme is also called RBF collocation method. The interpolant of $u(\mathbf{x})$ is unique if and only if the matrix X is non-singular. It has been discussed about sufficient conditions for $\psi(r)$ to guarantee non-singularity of the a matrix

Buhmann (2003), Cheney (1999).

Commonly used radial basis functions are

$$\psi(r)=\begin{cases} e^{-r^2}, & \text{Gaussian,}\\ \sqrt{1+r^2}, & \text{Multiquadric,}\\ \frac{1}{\sqrt{1+r^2}}, & \text{Inverse Multiquadric,}\\ \frac{1}{1+r^2}, & \text{Inverse Quadratic,}\\ r^2\log(r), & \text{Thin - Plate Spline,}\\ r, & \text{Linear Spline.}\end{cases}$$

In addition to these, RBFs can be stated with the help of a scaling parameter called the shape parameter ε. This can be done in the manner that $\psi(r)$ is replaced by $\psi(\varepsilon r)$. In general, shape parameters have been chosen arbitrarily since there are no exact results about how to choose the best shape parameter, and so it can be decided by the user.

3. Poisson Equations

The general form of Poisson equation on a finite domain $\Omega=\{(x,y)\,|\,(x,y)\in[0,1]\times[0,1]\}$ is

$$\nabla^2 u(x,y)=f(x,y),$$

where ∇^2 is the Laplace operator. In two dimensional Cartesian coordinates the Poisson equation takes the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=f(x,y).$$

In the case of $f(x,y)=0$, Poisson equation convert to Laplace's equation. Here, we begin by briefly reviewing the fractional Poisson equation. The fractional order of Poisson equation can be given as follows:

$$\nabla^{\alpha,\beta}u(x,y)=\left(\frac{\partial^\alpha}{\partial x^\alpha}+\frac{\partial^\beta}{\partial y^\beta}\right)u(x,y)=f(x,y),\quad 1<\alpha,\quad \beta\le 2, \tag{1}$$

with Dirichlet boundary conditions. In order to provide mesh-free numerical solution of Equation (1), we will use the radial basis function method which will be summarized below.

4. Conformable Fractional Calculus

In this paper, we will present and test conformable fractional version of ordinary differential equations with the help of the Katugampola conformable fractional calculus. In detail,

Katugampola conformable derivatives, or α-derivatives, for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(u)(t) = \lim_{\varepsilon \to 0} \frac{u\left(te^{\varepsilon t^{-\alpha}}\right) - u(t)}{\varepsilon}, \; D^{\alpha}(u)(0) = \lim_{t \to 0} D^{\alpha}(u)(t), \tag{2}$$

provided the limits exist (for detail see, Katugampola (2014)). If u is fully differentiable at t, then,

$$D^{\alpha}(u)(t) = t^{1-\alpha} \frac{du}{dt}(t).$$

A function u is α-differentiable at a point $t \geq 0$ if the limit in (2) exists and is finite. This definition yields the following results;

Theorem 1.

Let $\alpha \in (0,1]$ and u, v be $\alpha -$differentiable at a point $t > 0$. Then,

i. $D^{\alpha}(au + bv) = aD^{\alpha}(u) + bD^{\alpha}(v)$, for all $a, b \in \mathrm{R}$,

ii. $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$,

iii. $D^{\alpha}(uv) = uD^{\alpha}(v) + vD^{\alpha}(u)$,

iv. $D^{\alpha}\left(\frac{u}{v}\right) = \frac{uD^{\alpha}(v) - vD^{\alpha}(u)}{v^2}$,

v. $D^{\alpha}(t^n) = nt^{n-\alpha}$ for all $n \in \mathrm{R}$,

vi. $D^{\alpha}(u \circ v)(t) = u'(v(t))D^{\alpha}(v)(t)$ for u is differentiable at $v(t)$.

Definition 1.

Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $u : [a,b] \to \mathrm{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b u(x) d_{\alpha} x := \int_a^b u(x) x^{\alpha - 1} dx,$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$.

Remark 1.

$$I_{\alpha}^{a}(u)(t) = I_{1}^{a}\left(t^{\alpha-1}u\right) = \int_a^t \frac{u(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1.

Let the conformable differential operator D^α be given as in (1), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions u and v are α-differentiable as needed. Then,

i. $D^\alpha(\ln t) = t^{-\alpha}$ for $t > 0$,

ii. $D^\alpha\left[\int_a^t u(t,s)d_\alpha s\right] = u(t,t) + \int_a^t D^\alpha\left[u(t,s)\right]d_\alpha s$,

iii. $\int_a^b u(x)D^\alpha(v)(x)d_\alpha x = uv\Big|_a^b - \int_a^b v(x)D^\alpha(u)(x)d_\alpha x.$

In this study, we introduced numerical solution of Katugampola type conformable fractional ordinary differential equation via radial basis function collocation method.

5. Computational Scheme

In this section, we present a numerical scheme to solve fractional elliptic partial differential equation via non-symmetric method with radial basis functions. Let take the Poisson equation of the form

$$\begin{aligned} \nabla^{\alpha,\beta}u(x,y) &= f(x,y), \quad (x,y) \quad \text{in} \quad \Omega, \\ u(x,y) &= g(x,y), \quad (x,y) \quad \text{on} \quad \partial\Omega, \end{aligned}$$

with Dirichlet boundary conditions where $\Omega \in \mathsf{R}^2$. Thus, we are trying to compute u while f and g are fixed. We can now use Kansa's RBF collocation method Kansa (1990a), Kansa (1990b). We build in a simple one-dimensional model. Let us propose an approximation solution $\mathbf{u}$ of the form

$$\mathbf{u} = \sum_{i=1}^{N} a_i \psi(\|x - x_i\|_2), \tag{3}$$

where $\mathsf{X} = x_1, x_2, \ldots, x_N$ are the set of nodes in Ω. Then, the collocation matrix which constructed by using Poisson equation and boundary condition to the collocation points X will be of the form

$$[A] = \begin{pmatrix} \nabla^{\alpha,\beta}[\psi] \\ \psi \end{pmatrix},$$

where the two blocks are constituted of entries:

$$\nabla^{\alpha,\beta}[\psi]_{i,j} = \nabla^{\alpha,\beta}\psi(\|x_i - x_i\|_2), \quad x_i \in \mathsf{I}, \quad x_j \in \mathsf{X},$$
$$\psi i, j = \psi(\|x_i - x_i\|_2), \qquad x_i \in \mathsf{B}, \quad x_j \in \mathsf{X},$$

where I and B represent a set of interior and a set of boundary points of the set of X collocation points, respectively (i.e., $\mathsf{X} = \mathsf{I} \cup \mathsf{B}$). The problem described above is called well-posed (or correctly-set) if the linear matrix system $A\boldsymbol{\lambda} = \mathbf{F}$, where $\mathbf{F}$ is composed of $\mathbf{f} = [f(x_i)]$, $x_i \in \mathsf{I}$, and $\mathbf{g} = [g(x_i)]$, $x_i \in \mathsf{B}$, has a unique solution. The outstanding properties of multiquadrics in terms of certainty and complexity, made Kansa to particularly suggest its use.

The main difference between numerical solution of integer and non-integer order of elliptic PDE's is calculation of RBF derivatives. In other words, one need to compute conformable fractional derivatives of any radial basis functions, say multiquadric. An example is Katugampola fractional derivative of multiquadric which is given below

$$D^{\alpha}\sqrt{1+t^2} = t^{1-\alpha}\frac{d}{dt}\sqrt{1+t^2}\ .$$

These results are used straight-forwardly in the collocation radial basis functions for solving fractional PDEs. Although there appears infinite sums in the previous formulas, one can truncate the terms once they are smaller than the machine precision.

Consequently, we have the following matrix system:

$$\begin{pmatrix} \nabla^{\alpha,\beta}\psi_{1,1} & \cdots & \cdots & \nabla^{\alpha,\beta}\psi_{1,N} \\ \nabla^{\alpha,\beta}\psi_{2,1} & \cdots & \cdots & \nabla^{\alpha,\beta}\psi_{2,N} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \nabla^{\alpha,\beta}\psi_{N-1,1} & \cdots & \cdots & \nabla^{\alpha,\beta}\psi_{N-1,N} \\ \psi_{N,1} & \cdots & \cdots & \psi_{N,N} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \\ \lambda_{N-1} \\ \lambda_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_{N-1} \\ g_N \end{pmatrix}.$$

As a result, the matrix system of N equations with N unknowns is available. Then we must solve this system to calculate the unknown coefficients. Hence, we have used the Gauss elimination method with total pivoting to solve such a system. Consequently, $\mathbf{u}(\mathbf{x})$ given in equation (3) can be computed.

6. Numerical Experiments

In order to verify the proposed method in the previous section we will give some numerical experiments results of some fractional Poisson equations. In all our numerical experiments, the numerical solution of PDEs are evaluated at 100×100 equally spaced points (these are uniformly distributed random points) in the domain $[0,1] \in \mathsf{R}^2$. This range can be generalized to a wider range of possible solutions. In these experiments we use the multiquadric basis function and take the $\varepsilon = 4$. The implementation of the method in all our experiments has been done in *Matlab*. Finally, in order to show the applicability of the proposed technique under all circumstances we

have used different non-integer order of PDEs.

Experiment 1.

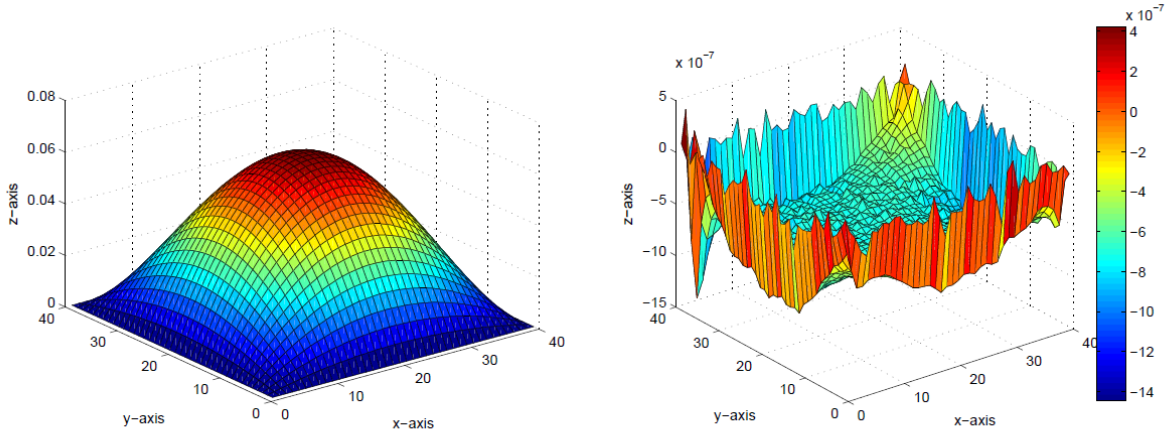
Let us consider the following conformable fractional Poisson equation

$$\left(\frac{\partial^{4/3}}{\partial x^{4/3}}+\frac{\partial^{3/2}}{\partial y^{3/2}}\right)u(x,y)=\frac{8x^{7/3}y-8x^{4/3}y+8x(y-1)y^{4/3}+x^{1/3}+y^{1/3}}{3x^{1/3}y^{1/3}},$$

on a finite domain $(x,y)\in\Omega=[0,1]^2$ with the boundary condition

$$u(x,y)=0 \quad (x,y)\in\partial\Omega.$$

The exact solution is given by $u(x,y)=x(1-x)y(1-y)$.



(A) Numerical solution (B) Maximum error for RBF solution

Figure 1. Approximate solution of target function $u(x,y)$ (left) and maximum error for RBF solution (right)

Experiment 2.

Let us consider the Conformable fractional Poisson equation

$$\left(\frac{\partial^{3/2}}{\partial x^{3/2}}+\frac{\partial^{4/3}}{\partial y^{4/3}}\right)u(x,y)=\pi\cos\left(\pi y/2\right)\left(\frac{\cos(\pi x)}{4\sqrt{x}}+\pi\sqrt[4]{x}\sin(\pi x)\right)$$

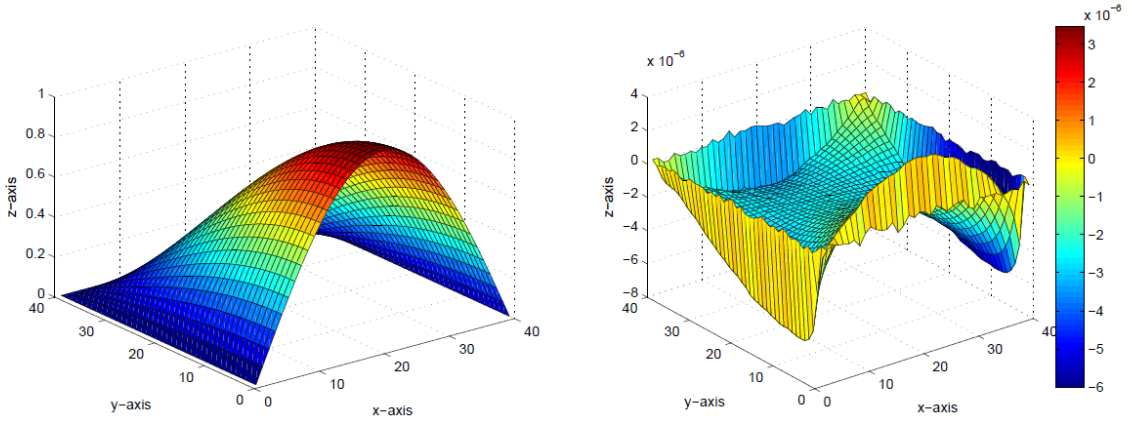
$$-\frac{\pi\sin(\pi x)}{2}\left(\frac{\sin(\pi y/2)}{3\sqrt[3]{y}}+\frac{\pi\sqrt[3]{y^2}\cos(\pi y/2)}{2}\right),$$

on a finite domain $(x,y)\in\Omega=[0,1]^2$ with the boundary condition

$$u(x,y)=\sin(\pi x) \quad (x,y)\in\Omega_1,$$
$$u(x,y)=0 \quad (x,y)\in\Omega_2,$$

where $\Omega_1=\{(x,y)\,|\,x\in[0,1],y=0\}$ and $\Omega_2=\partial\Omega\setminus\Omega_1$. The exact solution is given by

$$u(x,y)=\sin(\pi x)\cos(\pi y/2).$$



(A) Numerical solution (B) Maximum error for RBF solution

Figure 2. Approximate solution of target function $u(x,y)$ (left) and maximum error for RBF solution (right)

In Figure 1 and Figure 2, we present the multiquadric solution conformable fractional Poisson equations along with its maximum error, respectively. These figures show that the RBF method has been successfully applied to the numerical solution problem of fractional order Poisson equation in R^2 with encouraging performance. These results confirm the superior performance of RBF methods for numerical solution of fractional PDEs.

7. Conclusion

The purpose of the current study was to propose a numerical scheme to solve conformable fractional ordinary differential equation with the help of radial basis function collocation technique. The contribution of this study has been to confirm by numerically. The experiments verified that the numerical solutions are compatible with the exact solutions.

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