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Some relations on generalized Rice's matrix polynomials

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Abstract

The main aim of this paper is to obtain certain properties of generalized Rice's matrix polynomials such as their matrix differential equation, generating matrix functions, an expansion for them. We have also deduced the various families of bilinear and bilateral generating matrix functions for them with the help of the generating matrix functions developed in the paper and some of their applications have also been presented here.

Keywords: Hypergeometric matrix functions; Rice's matrix polynomials; Matrix differential equations; Generating matrix functions

MSC 2010 No.: 15A60, 33C20, 33C60, 33C70

1. Introduction and Preliminaries

An important generalization of special functions is special matrix functions. Special matrix polynomials appear in connection with matrix analogues of Hermite, Chebyshev and Legendre matrix differential equations and the corresponding polynomial families, see Aktaş et al. (2013), Altin et al. (2014), Çekim (2013), Çekim et al. (2011), Çekim et al. (2013), Defez and Jódar (2002), Erkus-Duman and Çekim (2014), Jódar et al. (1995), Jódar and Sastre (1998), Kargin and Kurt (2014a, 2014b), Kargin and Kurt (2015) (for a list of references). The author has earlier studied the Rice's matrix polynomials Shehata (2014) and the present paper carries those studies ahead motivated by the importance of special matrix polynomials of several recent works Aktaş (2014), Aktaş et al. (2012), Çekim and Aktaş (2015), Shehata (2014) and Tasdelen et al. (2011). We organize the present paper as follows: In Section 2, we give the matrix differential equation, generating matrix functions and an expansion of generalized Rice's matrix polynomials. The class of bilinear and bilateral generating matrix relations for generalized Rice's matrix polynomials have also been established in the Sections 3 and 4.

In this section, we give some basic facts or properties, definitions, lemmas, theorems and some notations and terminology, which has been used in the next sections.

Throughout this paper, the symbol $C^{N\times N}$ stands for the set of all square complex matrices of common order N and $\sigma(A)$ stands for the set of all the eigenvalues of $A \in \mathbb{C}^{N\times N}$. The matrices I and $\mathbf{0}$ will be denoted by the identity matrix and the null matrix (zero matrix) in $\mathbb{C}^{N\times N}$, respectively.

Definition 1.1.

If $A_0, A_1, \dots, A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq \mathbf{0}$, then by the matrix polynomial of degree n in x (x is a real variable or complex variable), we mean an expression of the form

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0.$$

Theorem 1.1. (Dunford and Schwartz (1957))

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, such that $AB = BA$, then

$$f(A)g(B) = g(B)f(A),$$

where $f(A)$ and $g(B)$ denote the images of functions $f(z)$ and $g(z)$ respectively, acting on the matrices A and B.

Definition 1.2. (Jódar and Cortés (1998a))

A matrix A in $\mathbb{C}^{N\times N}$ is said be a positive stable matrix if

$$Re(\mu) > 0, \text{ for all eigenvalues } \mu \in \sigma(A);\ \sigma(A) := \text{ spectrum of } A. \tag{1}$$

Fact 1.1. (Jódar and Cortés (1998b))

For $A \in \mathbb{C}^{N\times N}$, let us denote the real numbers $M(A)$ and $m(A)$ as in the following

$$M(A) = \max\{Re(z) : z \in \sigma(A)\};\ m(A) = \min\{Re(z) : z \in \sigma(A)\}. \tag{2}$$

Definition 1.3. (Jódar and Cortés (1998a))

If P is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix function $\Gamma(P)$ is defined by

$$\Gamma(P)=\int_0^\infty e^{-t}t^{P-I}dt;\quad t^{P-I}=\exp\Big((P-I)\ln t\Big). \tag{3}$$

Definition 1.4.

For $A\in\mathbb{C}^{N\times N}$, the matrix form of the Pochhammer symbol or shifted factorial is defined by (Jódar and Cortés (1998))

$$\begin{aligned}(A)_n&=A(A+I)(A+2I)\dots(A+(n-1)I)\\&=\Gamma(A+nI)\Gamma^{-1}(A)\ ;\ n\geq 1\ ,\ (A)_0=I.\end{aligned} \tag{4}$$

Definition 1.5. (Jódar and Cortés (1998b))

The hypergeometric matrix function ${}_2F_1(A,B;C;z)$ is defined in the form

$${}_2F_1\Big(A,B;C;z\Big)=\sum_{k=0}^{\infty}\frac{z^k}{k!}(A)_k(B)_k[(C)_k]^{-1}, \tag{5}$$

where A, B, and C are commutative matrices in $\mathbb{C}^{N\times N}$ such that $C+nI$ is an invertible matrix for every integer $n\geq 0$ and for $|z|<1$.

Lemma 1.1. (Lancaster (1969))

Let $\|\dots\|$ denotes any matrix norm for which $\|I\|=1$. If $\|A\|<1$ for a matrix A in $\mathbb{C}^{N\times N}$, then $\big(I-A\big)^{-c}$ exists and given by

$$\Big(I-A\Big)^{-c}=\sum_{k=0}^{\infty}\frac{(c)_k}{k!}A^k, \tag{6}$$

where c is a positive integer.

Fact 1.2. (Jódar and Cortés (1998b))

For any matrix A in $\mathbb{C}^{N\times N}$, we give the following relation

$$\Big(1-x\Big)^{-A}=\ {}_1F_0\Big(A;-;x\Big)=\sum_{n=0}^{\infty}\frac{1}{n!}(A)_n x^n\ ;\ |x|<1. \tag{7}$$

Notation 1.1.

For A is an arbitrary matrix in $\mathbb{C}^{N\times N}$ and using (4), we have the following relations

$$\begin{aligned} (A)_{n+k} &= (A)_n(A+nI)_k = (A)_k(A+kI)_n, \\ (A)_{2k} &= 2^{2k}\left(\frac{1}{2}A\right)_k\left(\frac{1}{2}(A+I)\right)_k, \\ (-nI)_k &= \begin{cases} \frac{(-1)^k\, n!}{(n-k)!}I, & 0\leq k\leq n; \\ \mathbf{0} \quad , & k>n, \end{cases} \\ (A)_{n-k} &= \begin{cases} (-1)^k(A)_n[(I-A-nI)_k]^{-1}, & 0\leq k\leq n; \\ \mathbf{0} \quad , & k>n. \end{cases} \end{aligned} \tag{8}$$

Definition 1.6. (Defez et al. (2004))

Let A and B be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions

$$Re(z) > -1,\ \forall\ z\in\sigma(A) \quad \text{and} \quad Re(w) > -1,\ \forall\ w\in\sigma(B). \tag{9}$$

For $n\geq 0$, the Jacobi matrix polynomials $\mathbf{P}_n^{(A,B)}(x)$ is defined by the hypergeometric matrix function

$$\mathbf{P}_n^{(A,B)}(x) = \frac{(B+I)_n}{n!}\ {}_2F_1\left(A+B+(n+1)I, -nI; B+I; \frac{1-x}{2}\right). \tag{10}$$

Lemma 1.2.

In (2002), Defez and Jódar have shown that for matrices $A(k,n)$ and $B(k,n)$ in $\mathbb{C}^{N\times N}$ when $n\geq 0$, $k\geq 0$, then the following relations are satisfied:

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n-mk)\ ;\ m\in N. \end{aligned} \tag{11}$$

Furthermore, we have the following relations

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk)\ ;\ m\in N, \end{aligned} \tag{12}$$

where $[x]$ denotes the greatest integer in x.

2. Generalized Rice's matrix polynomials

Definition 2.1.

Let P and Q be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (9), A and B are matrices in $\mathbb{C}^{N\times N}$, B satisfying the condition

$$B+nI \quad \text{is an invertible matrix for all integers}\, n\geq 0. \tag{13}$$

Let us consider the generalized Rice's matrix polynomials $H_n^{(P,Q)}(A,B,z)$ by hypergeometric matrix function

$$\begin{aligned}
&H_n^{(P,Q)}(A,B,z)=\frac{(P+I)_n}{n!}\ {}_3F_2(-nI,P+Q+(n+1)I,\\
&A;P+I,B;z)=\frac{(P+I)_n}{n!}\sum_{k=0}^{\infty}\frac{z^k}{k!}(-nI)_k(P+Q+(n+1)I)_k\\
&\times(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1}=(P+I)_n\sum_{k=0}^{n}\frac{(-1)^kz^k}{k!(n-k)!}\\
&\times(P+Q+(n+1)I)_k(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1},
\end{aligned} \tag{14}$$

where $0\leq k\leq n$ and all matrices are commutative.

Remark 2.1.

For the scalar case $N=1$, taking $P=Q=\mathbf{0}$, $A=\xi$, $B=p$ and $z=\nu$ in (14) gives the scalar Rice's polynomials $H_n(\xi,p,\nu)$ (see Rainville (1945) and Rice (1940))

$$H_n(\xi,p,\nu)=\ {}_3F_2(-n,n+1,\xi;1,p;\nu).$$

Corollary 2.1.

For the purpose of this work, with the help of Theorem 1 (see Shehata (2016b)), setting $p=3$ and $q=2$, we give the convergence properties for the generalized Rice's matrix polynomials:

(1) The power series (14) is convergent for with $|z|<1$ and diverges for $|z|>1$.
(2) The power series (14) is absolutely convergent for $|z|=1$ when

$$m(B)+m(P+I)>M(A)+M(P+Q+(n+1)I)+M(-nI).$$

(3) If the power series (14) is conditionally convergent for $|z|=1$ when

$$\begin{aligned}
M(A)+M(P+Q+(n+1)I)+M(-nI)-1&<m(B)+m(P+I)\\
&\leq M(A)+M(P+Q+(n+1)I)+M(-nI).
\end{aligned}$$

(4) The power series (14) is diverges for $|z|=1$ when

$$m(B)+m(P+I)\leq M(A)+M(P+Q+(n+1)I)+M(-nI)-1,$$

where $M(A)$ and $m(B)$ are defined in (2).

Remark 2.1.

We mention that for special case $P = Q = \mathbf{0}$, this reduces to Rice's matrix polynomials in Shehata (2014).

Remark 2.2.

We note that by taking $A = B$ in (14), we obtain the result

$$\begin{aligned} &H_n^{(P,Q)}(A, A, z) = \mathbf{P}_n^{(Q,P)}(1 - 2z) \\ &= \frac{(P+I)_n}{n!}\, {}_2F_1\Big(-nI, P + Q + (n+1)I; P + I; z\Big), \end{aligned} \tag{15}$$

where $\mathbf{P}_n^{(Q,P)}(x)$ is the Jacobi matrix polynomials defined in (10).

Theorem 2.1.

For $n > 0$, the generalized Rice's matrix polynomials satisfies the following matrix differential equation

$$\begin{aligned} &(1-z)z^2\, D^3 H_n^{(P,Q)}(A, B, z) + \Big[z(B + P + 2I) - \\ &z^2(A + P + Q + 4I)\Big] D^2 H_n^{(P,Q)}(A, B, z) + \Big[B(P + I) + n(n+1)zI \\ &- z(A + I)(P + Q + 2I) + nz(P + Q)\Big] D\, H_n^{(P,Q)}(A, B, z) \\ &+ nA(P + Q + (n+1)I)\, H_n^{(P,Q)}(A, B, z) = \mathbf{0}\ ;\ D = \frac{d}{dz}, \end{aligned} \tag{16}$$

where all matrices are commutative.

Proof:

Consider the differential operator $\theta = z\frac{d}{dz}$, $\theta z^k = kz^k$, yields that

$$\begin{aligned} &\theta\,(\theta\, I + P + I - I)(\theta\, I + B - I) H_n^{(P,Q)}(A, B, z) = \sum_{k=1}^{\infty} \frac{k\, z^k}{k!} \\ &\times (kI + P + I - I)(kI + B - I)(-nI)_k (P + Q + (n+1)I)_k (A)_k \\ &\times [(P+I)_k]^{-1}[(B)_k]^{-1} = \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!}(-nI)_k (P + Q + (n+1)I)_k \\ &\times (A)_k [(P+I)_{k-1}]^{-1}[(B)_{k-1}]^{-1}. \end{aligned}$$

Now, we replacing k by $k + 1$, we have

$$\begin{aligned}&\theta\ (\theta\ I+P)(\theta\ I+B-I)H_n^{(P,Q)}(A,B,z)\\&=\sum_{k=0}^{\infty}\frac{z^{k+1}}{k!}(-nI)_{k+1}(P+Q+(n+1)I)_{k+1}(A)_{k+1}[(P+I)_k]^{-1}[(B)_k]^{-1}\\&=z(\theta\ I-nI)(\theta\ I+P+Q+(n+1)I)(\theta\ I+A)H_n^{(P,Q)}(A,B,z).\end{aligned}$$

Thus, we show that $H_n^{(P,Q)}(A,B,z)$ is a solution of the following matrix differential equation

$$\begin{aligned}&\Big[\theta\ (\theta\ I+P)(\theta I+B-I)-z(\theta\ I-n\ I)\\&\times(\theta\ I+P+Q+(n+1)I)(\theta I+A)\Big]H_n^{(P,Q)}(A,B,z)=\mathbf{0}.\end{aligned}\tag{17}$$

Thus, $H_n^{(P,Q)}(A,B,z)$, given by (14), satisfies (16) in $|z|<1$. □

Theorem 2.2.

Let A, B, P and Q be matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions (9) and (13). Then, a generating matrix function for generalized Rice's matrix polynomials has the following form

$$\begin{aligned}&\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,z)t^n=(1-t)^{-P-Q-I}\\&\times{}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4zt}{(1-t)^2}\right),\end{aligned}\tag{18}$$

for $|t|<1$, $\left|\frac{4zt}{(1-t)^2}\right|<1$.

Proof:

Starting from (14) and using the result (12), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,z)t^n\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^kz^kt^n}{k!(n-k)!}(P+Q+I)_{n+k}(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^kz^kt^{n+k}}{k!n!}(P+Q+I)_{n+2k}(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1}.\end{aligned}$$

From (8) and (7), we can write

$$\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,z)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k z^k t^{n+k}}{k!n!}$$
$$\times (P+Q+I)_{2k}(P+Q+(2k+1)I)_n(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1}$$
$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}(P+Q+(2k+1)I)_n\frac{(-1)^k(4z)^k t^k}{k!}\left(\frac{1}{2}(P+Q+I)\right)_k$$
$$\times\left(\frac{1}{2}(P+Q+2I)\right)_k(A)_k[(P+I)_k]^{-1}[(B)_k]^{-1}=\sum_{k=0}^{\infty}(1-t)^{-P-Q-(2k+1)I}$$
$$\times\frac{(-1)^k(4z)^k t^k}{k!}\left(\frac{1}{2}(P+Q+I)\right)_k\left(\frac{1}{2}(P+Q+2I)\right)_k(A)_k$$
$$\times[(P+I)_k]^{-1}[(B)_k]^{-1}=(1-t)^{-P-Q-I}$$
$$\times\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4zt}{(1-t)^2}\right).$$

Hence the proof of Theorem 2.2 is completed. □

Now, we can use the series of generalized Rice's matrix polynomials together with their properties to prove the following result.

Theorem 2.3.

For a non-negative integer n, an expansion of generalized Rice's matrix polynomials is given as follows

$$\begin{aligned}z^n I=&(P+I)_n(B)_n[(A)_n]^{-1}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!}(P+Q+(2k+1)I)\\&\times(P+Q+I)_k\Big[(P+Q+I)_{n+k+1}\Big]^{-1}\Big[(P+I)_k\Big]^{-1}\times H_k^{(P,Q)}(A,B,z).\end{aligned}\tag{19}$$

Proof:

Equation (18) can be written in the form

$$\begin{aligned}&{}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4zt}{(1-t)^2}\right)\\&=(1-t)^{P+Q+I}\sum_{k=0}^{\infty}(P+Q+I)_k[(P+I)_k]^{-1}H_k^{(P,Q)}(A,B,z)t^k.\end{aligned}\tag{20}$$

In (20), we put that

$$\nu=-\frac{4t}{(1-t)^2}.$$

Then,

$$t=1-\frac{2}{1+\sqrt{1-\nu}}=-\frac{\nu}{(1+\sqrt{1-\nu})^2},$$

(20) can be written

$$\begin{aligned}&{}_3F_2\Big(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;\nu\, z\Big)\\&=\sum_{k=0}^{\infty}\frac{(-1)^n\nu^k}{2^{2k}}\Big(\frac{2}{1+\sqrt{1-\nu}}\Big)^{P+Q+(2k+1)I}(P+Q+I)_k\\&\Big[(P+I)_k\Big]^{-1}H_k^{(P,Q)}(A,B,z).\end{aligned}\tag{21}$$

Now, replacing

$$\begin{aligned}\Big(\frac{2}{1+\sqrt{1-\nu}}\Big)^{P+Q+(2k+1)I} &= {}_2F_1\Big(\frac{1}{2}(P+Q+(2k+1)I),\\&\frac{1}{2}(P+Q+(2k+2)I);P+Q+(2k+2)I;\nu\Big),\end{aligned}$$

we have

$$\begin{aligned}&{}_3F_2\Big(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;\nu\, z\Big)\\&=\sum_{k=0}^{\infty}\frac{(-1)^k\nu^k}{2^{2k}}\,{}_2F_1\Big(\frac{1}{2}(P+Q+(2k+1)I),\\&\frac{1}{2}(P+Q+(2k+2)I);P+Q+(2k+2)I;\nu\Big)\\&\times(P+Q+I)_k[(P+I)_k]^{-1}H_k^{(P,Q)}(A,B,z)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\nu^{n+k}}{n!2^{2k}}\Big(\frac{1}{2}(P+Q+(2k+1)I)\Big)_n\\&\times\Big(\frac{1}{2}(P+Q+(2k+2)I)\Big)_n\Big[(P+Q+(2k+2)I)_n\Big]^{-1}\\&\times(P+Q+I)_k[(P+I)_k]^{-1}H_k^{(P,Q)}(A,B,z)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{n!}\frac{\nu^{n+k}}{2^{2n+2k}}(P+Q+(2k+1)I)_{2n}\\&\times\Big[(P+Q+(2k+2)I)_n\Big]^{-1}(P+Q+I)_k\\&\times[(P+I)_k]^{-1}H_k^{(P,Q)}(A,B,z).\end{aligned}\tag{22}$$

Using (8), we get

$$(P+Q+(2k+1)I)_{2n}=(P+Q+I)_{2n+2k}\Big[(P+Q+I)_{2k}\Big]^{-1}.$$

Thus, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2n+2k}}\frac{\nu^{n+k}}{n!}(P+Q+I)_{2n+2k}(P+Q+(2k+1)I)$$
$$\times\Big[(P+Q+I)_{n+2k+1}\Big]^{-1}(P+Q+I)_k[(P+I)_k]^{-1}H_k^{(P,Q)}(A,B,z).$$

Therefore,

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{1}{2}(P+Q+I)\Big)_n\Big(\frac{1}{2}(P+Q+2I)\Big)_n(A)_n\\&\times\Big[(P+I)_n\Big]^{-1}\Big[(B)_n\Big]^{-1}\nu^n z^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{2^{2n}(n-k)!}\\&\times(P+Q+I)_{2n}(P+Q+(2k+1)I)\Big[(P+Q+I)_{n+k+1}\Big]^{-1}\\&\times(P+Q+I)_k\Big[(P+I)_k\Big]^{-1}H_k^{(P,Q)}(A,B,z)\nu^k,\end{aligned}\tag{23}$$

which yields equation (19) on equating coefficients of ν^n and using (8). Therefore, the proof of Theorem 2.3 is completed. □

Now, we mention some interesting special cases of our results of this section.

Corollary 2.2.

Substituting $P=Q=\mathbf{0}$ in (14), we obtain a known result in Shehata (2014)

$$\begin{aligned}z^nI=&(B)_n[(A)_n]^{-1}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!}((2k+1)I)(I)_n\\&\times\Big[(I)_{n+k+1}\Big]^{-1}H_k(A,B,z).\end{aligned}\tag{24}$$

Corollary 2.3.

Taking $A=B$ and replacing z by $\frac{1-x}{2}$ in (14), we have a known result in Defez et al. (2004)

$$\begin{aligned}(1-x)^n I = 2^n (P+I)_n \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} (P+Q+(1+2k)I) \\ \times (P+Q+I)_k [(P+Q+I)_{n+k+1}]^{-1} [(P+I)_k]^{-1} \mathbf{P}_k^{(Q,P)}(x).\end{aligned} \tag{25}$$

We now give a generating matrix function for the generalized Rice's matrix polynomials.

Theorem 2.4.

A generating matrix function for the generalized Rice's matrix polynomials is derived as

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} (C)_n H_m^{(P,Q)}(-nI, B, z) t^n = (1-t)^{-C} \\ \times H_m^{(P,Q)}\Big(C, B, -\frac{zt}{1-t}\Big); \ |t| < 1 \ , \ \left|\frac{zt}{1-t}\right| < 1,\end{aligned} \tag{26}$$

where $PC = CP$ and $QC = CQ$.

Proof:

By using (14), (7) and (12), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{1}{n!} (C)_n H_m^{(P,Q)}(-nI, B, z) t^n = \frac{1}{m!} (P+I)_m \sum_{n=0}^{\infty} \sum_{k=0}^{m} \frac{z^k t^{n+k}}{n! k!} \\ &\times (-mI)_k (C)_n (P+Q+(m+1)I)_k [(P+I)_k]^{-1} [(B)_k]^{-1} \\ &= \frac{1}{m!} (P+I)_m \sum_{k=0}^{m} \frac{1}{k!} (-mI)_k \sum_{n=0}^{\infty} \frac{1}{n!} (C+kI)_n t^n (C)_k \\ &\times (P+Q+(m+1)I)_k [(P+I)_k]^{-1} [(B)_k]^{-1} (-zt)^k = \frac{1}{m!} (P+I)_m \\ &\times \sum_{k=0}^{m} \frac{1}{k!} (-mI)_k (1-t)^{-C-kI} (C)_k (P+Q+(m+1)I)_k [(B)_k]^{-1} \\ &\times [(P+I)_k]^{-1} (-zt)^k = \frac{1}{m!} (1-t)^{-C} (P+I)_m \sum_{k=0}^{m} \frac{(-mI)_k}{k!} \\ &\times (C)_k (P+Q+(m+1)I)_k [(P+I)_k]^{-1} [(B)_k]^{-1} \Big(-\frac{zt}{1-t}\Big)^k \\ &= \frac{1}{m!} (1-t)^{-C} (P+I)_m \, {}_3F_2\Big(-mI, P+Q+(m+1)I, \\ &C; P+I, B; -\frac{zt}{1-t}\Big) = (1-t)^{-C} H_m^{(P,Q)}\Big(C, B, -\frac{zt}{1-t}\Big),\end{aligned}$$

which completes the proof of formula (26). □

Corollary 2.4.

For $C = B$, Equation (26) further reduces to following generating matrix function representation:

$$\sum_{n=0}^{\infty} \frac{1}{n!}(C)_n H_m^{(P,Q)}(-nI, C, z)t^n = (1-t)^{-C} \times \mathbf{P}_m^{(Q,P)}\left(1 + \frac{2zt}{1-t}\right) ;\ |t| < 1 ,\ \left|\frac{2zt}{1-t}\right| < 1. \tag{27}$$

3. Bilinear and bilateral generating matrix functionsfor the generalized Rice's matrix polynomials

The present section is a further attempt to establish a general theorem on a novel class of bilinear and bilateral generating matrix functions of various generalized Rice's matrix polynomials by using the similar method considered in Aktaş (2014), Aktaş et al. (2013), Aktaş et al. (2012), Altin et al. (2014), Çekim and Aktaş (2015), Tasdelen et al. (2011). In fact, we obtain the main results in the following theorem.

Theorem 3.1.

Corresponding to a non-vanishing matrix function $\Omega_\mu(y_1, y_2, \ldots, y_s)$ of s complex variables $y_1, y_2, \ldots, y_s$, $s \in N$ and involving a complex parameter μ, give a order, let us consider the following

$$\Lambda_{\mu,\nu}(y_1, y_2, \ldots, y_s; z) = \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s) z^k ;\quad a_k \neq 0 ,\ \mu, \nu \in \mathbb{C}, \tag{28}$$

where the coefficients a_k are assumed to non-vanishing in order for the matrix function on the left-hand side to be non-null. Suppose also that the matrix polynomials

$$\Psi_{n,m,\mu,\nu}(x; y_1, y_2, \ldots, y_s; \eta) = \sum_{k=0}^{[\frac{1}{m}n]} a_k (P + Q + I)_{n-mk}[(P + I)_{n-mk}]^{-1} \times H_{n-mk}^{(P,Q)}(A, B, x)\Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s)\eta^k ;\ n, m \in N, \tag{29}$$

where A, B, P and Q are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions (9) and (13), and (as usual) the notation $[\frac{n}{p}]$ means the greatest integer less than or equal to $\frac{n}{p}$ ($n \in N_0$, $p \in N$). Then, we have

$$\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \ldots, y_s; \frac{\eta}{t^m}\right)t^n = (1-t)^{-P-Q-I} \times {}_3F_2\left(\frac{1}{2}(P + Q + I), \frac{1}{2}(P + Q + 2I), A; P + I, B; -\frac{4xt}{(1-t)^2}\right) \times \Lambda_{\mu,\nu}(y_1, y_2, \ldots, y_s; \eta), \tag{30}$$

provided that each member of (30) exists.

Proof:

For the sake of convenience, let S denote the member of the assertion (30) of Theorem 3.1. Then, by substituting for the matrix polynomials $\Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \ldots, y_s; \frac{\eta}{t^m}\right)$ from the definition (29) into the left hand side of (30), we obtain the familiar generating matrix function

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \ldots, y_s; \frac{\eta}{t^m}\right) t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{1}{m}n]} a_k (P+Q+I)_{n-mk} [(P+I)_{n-mk}]^{-1} H_{n-mk}^{(P,Q)}(A, B, x) \\
&\quad \times \Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s) \eta^k t^{n-mk}.
\end{aligned} \tag{31}$$

Upon changing the order of summation in (31), if we replace n by $n + mk$, we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y_1, y_2, \ldots, y_s; \frac{\eta}{t^m}\right) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k (P+Q+I)_n \\
&\quad \times [(P+I)_n]^{-1} H_n^{(P,Q)}(A, B, x) \Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s) \eta^k t^n \\
&= \left[\sum_{n=0}^{\infty} (P+Q+I)_n [(P+I)_n]^{-1} H_n^{(P,Q)}(A, B, x) t^n\right] \\
&\quad \times \left[\sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s) \eta^k\right] = (1-t)^{-P-Q-I} \\
&\quad \times \ {}_3F_2\left(\frac{1}{2}(P+Q+I), \frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\right) \\
&\quad \times \Lambda_{\mu,\nu}(y_1, y_2, \ldots, y_s; \eta),
\end{aligned}$$

which completes the proof of Theorem 3.1. □

By expressing the multivariable matrix function $\Omega_{\mu+\nu k}(y_1, y_2, \ldots, y_s)$, $k \in N_0$ and $s \in N$ in terms of simpler matrix function of one and more variables, we can give further applications of Theorem 3.1. In the following, we obtain the results which provide a class of bilinear generating matrix functions for the generalized Rice's matrix polynomials.

Corollary 3.1.

Let

$$\Lambda_{\mu,\nu}(y; z) = \sum_{k=0}^{\infty} a_k (P+Q+I)_{\mu+\nu k} [(P+I)_{\mu+\nu k}]^{-1} H_{\mu+\nu k}^{(P,Q)}(A, B, y) z^k \ ;$$

$$a_k \neq 0\ ,\ \mu, \nu \in N_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)$$
$$\times(P'+Q'+I)_{\mu+\nu k}[(P'+I)_{\mu+\nu k}]^{-1}H_{\mu+\nu k}^{(P',Q')}(A',B',y)\eta^k\ ;\ n,m\in N,$$

where P, P', Q and Q' are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (9), and A, A', B and B' are matrices in $\mathbb{C}^{N\times N}$, B and B' satisfying the condition (13). Then, we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n=(1-t)^{-P-Q-I}\ {}_3F_2\Bigg(\frac{1}{2}(P+Q+I),$$
$$\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\Bigg)\Lambda_{\mu,\nu}(y;\eta), \tag{32}$$

provided that each member of (32) exists.

Proof:

Equation (32) can be proved using the same method as in proof of Theorem 3.1. □

Remark 3.1.

For the generalized Rice's matrix polynomials, by the generating matrix functions in (18) and taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)$$
$$\times(P'+Q'+I)_k[(P'+I)_k]^{-1}H_k^{(P',Q')}(A',B',y)\eta^k t^{n-mk}$$
$$=(1-t)^{-P-Q-I}\ {}_3F_2\Bigg(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;$$
$$P+I,B;-\frac{4xt}{(1-t)^2}\Bigg)(1-\eta)^{-P'-Q'-I}\ {}_3F_2\Bigg(\frac{1}{2}(P'+Q'+I),$$
$$\frac{1}{2}(P'+Q'+2I),A';P'+I,B';-\frac{4y\eta}{(1-\eta)^2}\Bigg).$$

In the next section, we proceed to discuss the various applications of the Theorem 3.1 for the case of certain special matrix functions including the generalized Rice's matrix polynomials.

4. Further Remarks and Applications

As we remarked above that the Theorem 3.1 provides a very generalization of certain classes of bilateral generating matrix functions for the Chebyshev, Gegenbauer, Jacobi, Legendre, Laguerre, modified Laguerre, Hermite and the generalized Rice's matrix polynomials, the same are being deduced now as known or new consequences of the Theorem 3.1.

Definition 4.1.

Let D be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1). Then, the Chebyshev matrix polynomials of the second kind are defined by means of the series (see Metwally et al. (2015))

$$U_n(x,D)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(n-k)!}{k!(n-2k)!}(x\sqrt{2D})^{n-2k},$$

and the generating matrix function

$$(I-xt\sqrt{2D}+t^2I)^{-1}=\sum_{n=0}^{\infty}U_n(x,D)t^n;\quad |t|<1,\quad |x|\leq 1, \tag{33}$$

where $I-xt\sqrt{2D}+t^2I$ and $xt\sqrt{2D}-t^2I$ are invertible matrices in $\mathbb{C}^{N\times N}$.

Corollary 4.1.

If

$$\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_kU_{\mu+\nu k}(y,D)z^k\ ;\ a_k\neq 0\ ,\ \mu,\nu\in N_0,$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}$$
$$\times H^{(P,Q)}_{n-mk}(A,B,x)U_{\mu+\nu k}(y,D)\eta^k\ ;\ n,m\in N,$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1), then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n=(1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\right.$$
$$\left.\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y;\eta), \tag{34}$$

provided that each member of (34) exists.

Remark 4.1.

Using the generating matrix function (33) for the $U_k(y,D)$ and by taking $a_k=1$, $\mu=0$ and

$\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)U_k(y,D)$$
$$\times\,\eta^k t^{n-mk} = \Big[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\Big]$$
$$\times\Big[\sum_{k=0}^{\infty}U_k(y,D)\eta^k\Big] = (1-t)^{-P-Q-I}\ {}_3F_2\Big(\frac{1}{2}(P+Q+I),$$
$$\frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\Big)(I-y\eta\sqrt{2D}+\eta^2 I)^{-1}$$

for $|t|<1$, $|\eta|<1$, $|x|<1$, $|y|\le 1$ and $\left|\frac{4xt}{(1-t)^2}\right|<1$.

Definition 4.2. (Jódar et al. (1995))

Let D be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$-\frac{1}{2}x \notin \sigma(D) \text{ for all } x\in Z^+\cup\{0\}. \tag{35}$$

Gegenbauer matrix polynomials are defined by

$$C_n^D(x) = \sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}(D)_{n-k}.$$

Notice that the Gegenbauer matrix polynomials are generated as follows (see Jódar et al. (1995) and Sayyed et al. (2004)):

$$F(x,t,D) = (1-2xt+t^2)^{-D} = \sum_{n=0}^{\infty}C_n^D(x)t^n. \tag{36}$$

If r_1 and r_2 are the roots of the quadratic equation $1-2xt+yt^2=0$ and r is the minimum of the set $\{r_1,r_2\}$, then the matrix function $F(x,t,D)$ regarded as a matrix function of t, is analytic in the disk $|t|<r$ for every real number in $|x|\le 1$.

Corollary 4.2.

Let

$$\Lambda_{\mu,\nu}(y;z) = \sum_{k=0}^{\infty}a_k C_{\mu+\nu k}^D(y)z^k\ ;\ a_k\neq 0\ ,\ \mu,\nu\in N_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;\eta) = \sum_{k=0}^{[\frac{1}{m}n]}a_k(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}$$
$$\times H_{n-mk}^{(P,Q)}(A,B,x)C_{\mu+\nu k}^D(y)\eta^k; n,m\in N,$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (35). Then, we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\Big(x;y;\frac{\eta}{t^m}\Big)t^n=(1-t)^{-P-Q-I}\ {}_3F_2\Big(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\Big)\Lambda_{\mu,\nu}(y;\eta), \tag{37}$$

provided that each member of (37) exists.

Remark 4.2.

Using the generating matrix function (36) for the $C_k^D(y)$ and taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)C_k^D(y)\eta^k t^{n-mk}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)C_k^D(y)\eta^k t^n\\&=\Big[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\Big]\Big[\sum_{k=0}^{\infty}C_k^D(y)\eta^k\Big]\\&=(1-t)^{-P-Q-I}\ {}_3F_2\Big(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;\\&-\frac{4xt}{(1-t)^2}\Big)(1-2y\eta+\eta^2)^{-D}.\end{aligned}$$

The Jacobi matrix polynomials are generated by (see Altin et al. (2014))

$$\sum_{n=0}^{\infty}(D+E+I)_n\mathbf{P}_n^{(D,E)}(x)[(D+I)_n]^{-1}t^n=(1-t)^{-D-E-I}\times\ {}_2F_1\Big(\frac{1}{2}(D+E+I),\frac{1}{2}(D+E+2I);D+I;\frac{2t(x-1)}{(1-t)^2}\Big), \tag{38}$$

where D and E are matrices in $\mathbb{C}^{N\times N}$ all of whose eigenvalues, z, satisfy the condition $Re(z)>-1$ with $|t|<1$, $|x|<1$ and $\left|\frac{2t(x-1)}{(1-t)^2}\right|<1$.

Corollary 4.3.

Let

$$\Lambda_{\mu,\nu}(y;z)=\sum_{k=0}^{\infty}a_k\mathbf{P}_{\mu+\nu k}^{(D,E)}(y)z^k\ ;\ a_k\neq 0\ ,\ \mu,\nu\in N_0,$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;\eta) = \sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1} \times H_{n-mk}^{(P,Q)}(A,B,x)a_k\mathbf{P}_{\mu+\nu k}^{(D,E)}(y)\eta^k \; ; \; n,m \in N,$$

where D and E are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (9). Then, we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;\frac{\eta}{t^m}\right)t^n = (1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I), \frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y;\eta), \tag{39}$$

provided that each member of (39) exists, where $DE = ED$.

Remark 4.3.

Using the generating matrix function (38) for the $\mathbf{P}_k^{(D,E)}(y)$ and taking $a_k = (D+E+I)_k[(D+I)_k]^{-1}$, $\mu = 0$ and $\nu = 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)(x) \\
&\times (D+E+I)_k\mathbf{P}_k^{(D,E)}(y)[(D+I)_k]^{-1}\eta^k t^{n-mk} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(P+Q+I)_n \\
&\times [(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)(D+E+I)_k\mathbf{P}_k^{(D,E)}(y)[(D+I)_k]^{-1}\eta^k t^n \\
&= \left[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\right] \\
&\times \left[\sum_{k=0}^{\infty}(D+E+I)_k\mathbf{P}_k^{(D,E)}(y)[(D+I)_k]^{-1}\eta^k\right] = (1-t)^{-P-Q-I} \\
&\times\ {}_3F_2\left(\frac{1}{2}(P+Q+I), \frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\right) \\
&\times (1-\eta)^{-D-E-I}\ {}_2F_1\left(\frac{1}{2}(D+E+I), \frac{1}{2}(D+E+2I); D+I; \frac{2\eta(y-1)}{(1-\eta)^2}\right).
\end{aligned}$$

Definition 4.3.

Let us consider the Legendre matrix polynomials $\mathbb{P}_n(x,D)$ defined in Shehata (2016a) as follows:

$$\mathbb{P}_n(x,D) = \sum_{k=0}^{n}\frac{(-1)^k(n+k)!}{k!(n-k)!}\left(\frac{1-x}{2}\right)^k\Gamma^{-1}(D+kI)\Gamma(D), n \geq 0$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$0 < Re(\lambda) < 1, \quad \text{for all } \lambda \in \sigma(D). \tag{40}$$

and $D + kI$ is an invertible matrix for all integers $k \geq 0$ and $\left|\frac{1-x}{2}\right| < 1$.

The Legendre matrix polynomials in (40) are generated by (see Shehata (2016a)):

$$\sum_{n=0}^{\infty} \mathbb{P}_n(x, D)t^n = (1-t)^{-1}\ {}_1F_1\left(\frac{1}{2}I; D; \frac{2t(x-1)}{(1-t)^2}\right); \quad |t| < 1,\ \left|\frac{2t(x-1)}{(1-t)^2}\right| < 1. \tag{41}$$

Now using the Theorem 3.1, we get the following result which provides a class of bilateral generating matrix relation for the Legendre matrix polynomials and the generalized Rice's matrix polynomials.

Corollary 4.4.

If

$$\Lambda_{\mu,\nu}(y; z) = \sum_{k=0}^{\infty} a_k \mathbb{P}_{\mu+\nu k}(y, D)z^k\ ;\ a_k \neq 0\ ,\ \mu, \nu \in N_0,$$

and

$$\Psi_{n,m,\mu,\nu}(x; y; \eta) = \sum_{k=0}^{[\frac{1}{m}n]} a_k (P + Q + I)_n[(P + I)_n]^{-1} \times H_n^{(P,Q)}(A, B, x)\mathbb{P}_{\mu+\nu k}(y, D)\eta^k;\ n, m \in N,$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (40), then we have

$$\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x; y; \frac{\eta}{t^m}\right)t^n = (1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P + Q + I), \frac{1}{2}(P + Q + 2I), A; P + I, B; -\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y; \eta), \tag{42}$$

provided that each member of (42) exists.

Remark 4.4.

For the case of Legendre matrix polynomials $\mathbb{P}_k(y, D)$, by the generating matrix function (41) and taking $a_k = 1$, $\mu = 0$ and $\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)\mathbb{P}_k(y,D)$$
$$\times\,\eta^k t^{n-mk} = \left[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\right]$$
$$\times\left[\sum_{k=0}^{\infty}\mathbb{P}_k(y,D)\eta^k\right] = (1-t)^{-P-Q-I}\;{}_3F_2\bigg(\frac{1}{2}(P+Q+I),$$
$$\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\bigg)(1-\eta)^{-1}$$
$$\times\;{}_1F_1\bigg(\frac{1}{2}I;D;\frac{2\eta(y-1)}{(1-\eta)^2}\bigg);|\eta|<1,\;\left|\frac{2\eta(y-1)}{(1-\eta)^2}\right|<1.$$

Next, if we set $s = 2$ and $\Lambda_{\mu,\nu}(y,w;z) = \sum_{k=0}^{\infty} a_k L_{\mu+\nu k}^{(D,\lambda)}(y,w)z^k$, $a_k \neq 0$, $\mu,\nu \in N_0$ in Theorem 3.1, then we get the following result which provides a class of bilateral generating matrix functions for Laguerre matrix polynomials of two variables and generalized Rice's matrix polynomials.

Definition 4.4.

Let D be a matrix in $\mathbb{C}^{N\times N}$ such that

$$-k \notin \sigma(D) \text{ for every integer } k > 0, \tag{43}$$

and λ is a complex number with $Re\,(\lambda) > 0$. Then, the Laguerre matrix polynomials are defined by (see Jódar and Sastre (1998), Shehata (2015a, 2015b, 2015c))

$$L_n^{(D,\lambda)}(x,y) = \sum_{k=0}^{n}\frac{(-1)^k(D+I)_n[(D+I)_k]^{-1}y^{n-k}(\lambda x)^k}{k!(n-k)!}.$$

According to Khan and Hassan (2010), Laguerre matrix polynomials of two variables are generated by

$$\sum_{n=0}^{\infty}L_n^{(D,\lambda)}(x,y)t^n = (1-yt)^{-(D+I)}\exp\left(\frac{-\lambda xt}{1-yt}\right), \tag{44}$$

where $t,x,y\in\mathbb{C}$ and $|yt|<1$.

Corollary 4.5.

Let

$$\Lambda_{\mu,\nu}(y,w;z) = \sum_{k=0}^{\infty}a_k L_{\mu+\nu k}^{(D,\lambda)}(y,w)z^k\;;\;a_k\neq 0\;,\;\mu,\nu\in N_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y;w;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}\times H_{n-mk}^{(P,Q)}(A,B,x)L_{\mu+\nu k}^{(D,\lambda)}(y,w)\eta^k;n,m\in N,$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (43), then we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y;w;\frac{\eta}{t^m}\right)t^n=(1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y,w;\eta), \tag{45}$$

provided that each member of (45) exists.

Remark 4.5.

For the Laguerre matrix polynomials $L_k^{(D,\lambda)}(y,w)$, by the generating matrix function (44) and taking $a_k=1$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)L_k^{(D,\lambda)}(y,w)\times\eta^k t^{n-mk}=\left[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\right]\times\left[\sum_{k=0}^{\infty}L_k^{(D,\lambda)}(y,w)\eta^k\right]=(1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\right)(1-w\eta)^{-(D+I)}\exp\left(\frac{-\lambda\, y\eta}{1-w\eta}\right).$$

Further, for the application of our Theorem 3.1, we get the following result on bilateral generating matrix relation involving modified Laguerre matrix polynomials.

Definition 4.5.

Let D be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (43) and λ is a complex parameter with $Re(\lambda)>0$, then the n^{th} modified Laguerre matrix polynomials $f_n^{(D,\lambda)}(x,y)$ is defined (see Khan and Hassan (2010) and Shehata (2015b))

$$f_n^{(D,\lambda)}(x,y)=\sum_{k=0}^{n}\frac{(D)_{n-k}(\lambda\,x)^k y^{n-k}}{k!(n-k)!}.$$

We recall that the modified Laguerre matrix polynomials of two variables are defined by the generating matrix function in Khan and Hassan (2010)

$$\sum_{n=0}^{\infty}f_n^{(D,\lambda)}(x,y)t^n=(1-yt)^{-D}e^{\lambda xt}\ ,\ t,x,y\in\mathbb{C}\ ,\ |yt|<1. \tag{46}$$

Corollary 4.6.

If we put $s = 2$ in the above theorem, we get the following result. Let

$$\Lambda_{\mu,\nu}(y,w;z) = \sum_{k=0}^{\infty} a_k f_{\mu+\nu k}^{(D,\lambda)}(y,w) z^k \ ; \ a_k \neq 0 \ , \ \mu,\nu \in N_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y,w;\eta) = \sum_{k=0}^{[\frac{1}{m}n]} a_k (P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1} \times H_{n-mk}^{(P,Q)}(A,B,x) f_{\mu+\nu k}^{(D,\lambda)}(y,w)\eta^k \ ; \ n,m \in N$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (43), then we have

$$\sum_{n=0}^{\infty} \Psi_{n,m,\mu,\nu}\left(x;y,w;\frac{\eta}{t^m}\right)t^n = (1-t)^{-P-Q-I} \ {}_3F_2\left(\frac{1}{2}(P+Q+I), \frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y,w;\eta), \tag{47}$$

provided that each member of (47) exists.

Remark 4.6.

Using the generating matrix function (46) for the modified Laguerre matrix polynomials $f_k^{(D,\lambda)}(y,w)$ and taking $a_k = 1$, $\mu = 0$ and $\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]} (P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1} H_{n-mk}^{(P,Q)}(A,B,x) f_{\mu+\nu\ k}^{(D,\lambda)}(y,w) \times \eta^k t^{n-mk} = \left[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\right] \times \left[\sum_{k=0}^{\infty} f_{\mu+\nu\ k}^{(D,\lambda)}(y,w)\eta^k\right] = (1-t)^{-P-Q-I} \ {}_3F_2\left(\frac{1}{2}(P+Q+I), \frac{1}{2}(P+Q+2I), A; P+I, B; -\frac{4xt}{(1-t)^2}\right)(1-w\eta)^{-D}e^{\lambda y\eta},$$

for $\left|\frac{4y\eta}{(1-\eta)^2}\right| < 1$, $|\eta| < 1$.

We now consider the Hermite matrix polynomials $H_n(y,w,D)$ of two variables satisfying the following generating matrix function in Shehata (2015c)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(y,w,D) = \exp\left(yt\sqrt{2D} - wt^2 I\right) \ ; \ |t| < \infty, \tag{48}$$

where D is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1), then we obtain the following result which provides a class of bilateral generating matrix functions for the matrix version of the Hermite matrix polynomials of two variables and generalized Rice's matrix polynomials.

Corollary 4.7.

Let

$$\Lambda_{\mu,\nu}(y,w;z)=\sum_{k=0}^{\infty}a_k H_{\mu+\nu k}(y,w,D)z^k \ ; \ a_k\neq 0 \ , \ \mu,\nu\in N_0$$

and

$$\Psi_{n,m,\mu,\nu}(x;y,w;\eta)=\sum_{k=0}^{[\frac{1}{m}n]}a_k(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1} \times H_{n-mk}^{(P,Q)}(A,B,x)H_{\mu+\nu k}(y,w,D)\eta^k \ ; \ n,m\in N,$$

where D is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1). Then, we have

$$\sum_{n=0}^{\infty}\Psi_{n,m,\mu,\nu}\left(x;y,w;\frac{\eta}{t^m}\right)t^n=(1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\right)\Lambda_{\mu,\nu}(y,w;\eta), \tag{49}$$

provided that each member of (49) exists.

Remark 4.7.

Using the generating matrix function (48) for the Hermite matrix polynomials $H_k(y,w,D)$ and taking $a_k=\frac{1}{k!}$, $\mu=0$ and $\nu=1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{m}n]}(P+Q+I)_{n-mk}[(P+I)_{n-mk}]^{-1}H_{n-mk}^{(P,Q)}(A,B,x)\frac{1}{k!}H_k(y,w,D) \times\eta^k t^{n-mk}=\left[\sum_{n=0}^{\infty}(P+Q+I)_n[(P+I)_n]^{-1}H_n^{(P,Q)}(A,B,x)t^n\right] \times\left[\sum_{k=0}^{\infty}\frac{1}{k!}H_k(y,w,D)\eta^k\right]=(1-t)^{-P-Q-I}\ {}_3F_2\left(\frac{1}{2}(P+Q+I),\frac{1}{2}(P+Q+2I),A;P+I,B;-\frac{4xt}{(1-t)^2}\right)\exp\left(y\eta\sqrt{2D}-w\eta^2 I\right).$$

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