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Abstract

In this paper, a numerical method is developed to approximate the soliton solutions of some nonlinear wave equations in terms of the Jacobi polynomials. Wave are very important phenomena in dispersion, dissipation, diffusion, reaction, and convection. Using the wave variable converts these nonlinear equations to the nonlinear ODE equations. Then, the operational Collocation method based on Jacobi polynomials as bases is applied to approximate the solution of ODE equation resulted. In addition, the intervals of the solution will be extended using an rational exponential approximation (REA). The KdV, Boussinesq, and Kaup–Kuperschmidt equations are studied as the test examples. Finally, numerical computation of the conservation values shows the effectiveness and stability of the proposed method.

Keywords: KdV Equation; Boussinesq Equation; Kaup–Kuperschmidt; Equation; Jacobi Polynomials; Rational Exponential Approximation; Conservation Values

MSC 2010 No.: 33C45, 33Q51, 33Q53, 65M12, 65M70, 65Z05

1. Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, and biology. For this reason, finding the exact or approximate solutions of these equations is very important. One of the basic physical problems for these models is to obtain their traveling solutions. In recent years, a variety of effective analytical and semi analytical methods have been been used for nonlinear PDEs, such as tanh–coth method (cf. Wazwaz (2007), Wazwaz (2008a), Wazwaz (2008b)), Exp–function method (cf. Ravi et al. (2017), Ekici et al. (2017)), Dai et al. (2009), Hirota bilinear method (cf. Wazwaz (2010), Wazwaz (2008c), Wazwaz (2008d)), $G'/G-$expansion method (cf. Manafian et al. (2017), Mohyud–Din and Bibi (2017), Kabir et al. (2011), Ekici (2017), Shehata (2010), Zheng et al. (2015)), Adomian decomposition method (cf. Achouri et al. (2009)), Variational iteration method (cf. Lu (2011)), Differential transform method (cf. Mohamed and Gepreel (2017), Yu et al. (2016), Yang et al. (2016)), and many others. The specific cases of the Jacobi polynomials have been applied to solve the nonlinear equations. For example, Bhrawy et al. have presented a new Legendre spectral collocation method for fractional Burgers equations (cf. Behrawy et al. (2015)). In Doha (2014), the authors used Jacobi–Gauss–Lobatto collocation method for the numerical solution of $(1+1)-$ nonlinear Schrödinger equation. Also, Bhrawy (2014) has presented an pseudospectral approximation based on Jacobi polynomials for generalized Zakharov system. Two spectral tau algorithms based on Jacobi polynomials have been applied to solve multi–term time–space fractional partial differential equations and time fractional diffusion–Wave equations (cf. Bhrawy et al. (2014), Bhrawy and Zaky (2015)).

In this current paper, we try to handle the Jacobi polynomials in a different way for achieving more desirable results. Since the existing analytical methods are restricted for obtaining exact solutions of nonlinear wave equations, we interested in the numerical solution of this kind of nonlinear problems. The numerical methods are of great importance for approximating the solutions of nonlinear PDEs, especially when the nonlinear differential equation under consideration faces difficulties in obtaining its exact solution. In this case, we usually use an appropriate and efficient numerical method. The aim of this paper is to use the wave variable and convert the nonlinear PDE to the corresponding nonlinear ODE. The resulting ODE will be solved by operational Jacobi Collocation method. For this purpose, integral operational matrices based on Jacobi polynomials will be constructed. Thus, the principle nonlinear equation is transformed by a system of nonlinear algebraic equations which is solved by well known Newton iteration method and determined unknown coefficients. It should be mentioned that the $(N+1)$ roots of Jacobi polynomials $P_{N+1}^{(\alpha,\beta)}(x)$ are considered as the collocation nodes. Since the approximate solutions are in the form of polynomials, they are unable to express the soliton nature of the exact solutions. For this reason, the approximate solution obtained will be improved using a rational exponential approximation (REA). Consequently, the improved solutions coincide with the soliton solutions and the interval of convergence is extended. The proposed scheme is applied for special cases of KdV, Boussinesq, and Kaup–Kuperschmidt equations as follows Wazwaz (2009):

$i)$ KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

$ii)$ Boussinesq equation:

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \tag{2}$$

$iii)$ Kaup–Kuperschmidt equation:

$$u_t - u_{xxxxx} + 20uu_{xxx} + 50u_x u_{xx} - 80u^2 = 0. \tag{3}$$

The remainder of this paper is organized as follows: The Jacobi polynomials, some of their properties, and the operational matrices of integration and product are introduced in Section 2. Afterwards, Section 3 is devoted to describe the proposed scheme. The proposed method for solving the nonlinear wave equations will be applied in Section 4. For this purpose, the Equations (1)–(3) are considered as the illustrative examples. A conclusion is presented in Section 5.

2. Jacobi polynomials and their operational matrices

The well–known Jacobi polynomials are defined on the interval $[-1, 1]$, constitute an orthogonal system with respect to the weight function $w^{(\alpha,\beta)}(x) = (1-x)^\alpha\ (1+x)^\beta$, and can be determined with the following recurrence formula:

$$\begin{aligned} P_{i+\ 1}^{(\alpha,\beta)}(x) = A(\alpha,\beta,i)\ P_i^{(\alpha,\beta)}(x) + x\ B(\alpha,\beta,i)\ P_i^{(\alpha,\beta)}(z) \\ - D(\alpha,\beta,i)\ P_{i-\ 1}^{(\alpha,\beta)}(x), \quad i = 1, 2, ..., \end{aligned} \tag{4}$$

where

$$A(\alpha,\beta,i) = \frac{(2\ i+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(i+1)(i+\alpha+\beta+1)(2\ i+\alpha+\beta)},$$
$$B(\alpha,\beta,i) = \frac{(2\ i+\alpha+\beta+2)(2\ i+\alpha+\beta+1)}{2(i+1)(i+\alpha+\beta+1)},$$
$$D(\alpha,\beta,i) = \frac{(i+\alpha)(i+\beta)(2\ i+\alpha+\beta+2)}{(i+1)(i+\alpha+\beta+1)(2\ i+\alpha+\beta)},$$

and

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2}.$$

The orthogonality condition of Jacobi polynomials is

$$\int_{-1}^{1} P_j^{\ (\alpha,\beta)}(x)\ P_k^{\ (\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx = h_k\ \delta_{jk},$$

where

$$h_k = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\ \Gamma(k+\alpha+\beta+1)}.$$

The analytic form of Jacobi polynomials is given by Szego (1939),

$$\begin{aligned} P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\ (i-k)!\ k!} \left(\frac{1+x}{2}\right)^k, \\ i = 0, 1, \dots\ . \end{aligned} \tag{5}$$

A continuous function $u(x)$, square integrable in $[-1,1]$, can be expressed in terms of the Jacobi polynomials as,

$$u(x)=\sum_{j=0}^{\infty} C_j\ P_j^{(\alpha,\beta)}(x), \tag{6}$$

where the coefficients C_j are given by

$$C_j=\frac{1}{\theta_j}\int_{-1}^{1} u(x)\ P_j^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx, \quad j=0,1,\dots\ .$$

In practice, only the first $(N+1)$ terms the Jacobi polynomials are considered. Therefore, one has

$$u_N(x)=\sum_{j=0}^{N} C_j\ P_j^{(\alpha,\beta)}(x)=\Phi^T(x)\ C=C^T\ \Phi(x), \tag{7}$$

where the vectors C and $\Phi(x)$ are given by

$$C=[C_0,C_1,...,C_N]^T, \quad \Phi(x)=[P_0^{(\alpha,\beta)}(x),P_1^{(\alpha,\beta)}(x),...,P_N^{(\alpha,\beta)}(x)]^T. \tag{8}$$

Some other properties of the Jacobi polynomials are presented as follows:

$$(1)\ P_i^{\ (\alpha,\beta)}(0)=(-1)^i\frac{\Gamma(i+\beta+1)}{\Gamma(i+\alpha+\beta+1)}\sum_{k=0}^{i}\frac{(-1)^k\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ k!\ (i-k)!\ 2^k},$$

$$(2)\ \frac{d^i}{dx^i}\ P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(n+\alpha+\beta+i+1)}{2^i\ \Gamma(n+\alpha+\beta+1)}\ P_{n-i}^{(\alpha+i,\ \beta+i)}(x).$$

Lemma 1.

The Jacobi polynomial $P_i^{(\alpha,\beta)}(x)$ can be obtained in the form of:

$$P_i^{(\alpha,\beta)}(x)=\sum_{k=0}^{i}\gamma_k^{(i)}\ x^k,$$

where $\gamma_k^{(i)}$ are

$$\gamma_k^{(i)}=(-1)^{i-k}\ \begin{pmatrix} i+k+\alpha+\beta \\ k \end{pmatrix}\frac{\Gamma(i+\beta+1)}{2^k\ \Gamma(i+k+\alpha+\beta+1)}$$
$$\times\sum_{l=0}^{i-k}\frac{(-1)^l\ \Gamma(i+k+l+\alpha+\beta+1)}{2^l\ \Gamma(l+k+\beta+1)\ l!\ (i-k-l)!}.$$

Proof:

The coefficients $\gamma_k^{(i)}$ can be obtained as:

$$\gamma_k^{(i)}=\frac{1}{k!}\frac{d^k}{dx^k}\ P_i^{(\alpha,\beta)}(x)\bigg|_{x=0}.$$

Now, by using properties (1) and (2) the lemma can be proved. □

Lemma 2.

If $P_j^{(\alpha,\beta)}(x)$ and $P_k^{(\alpha,\beta)}(x)$ are $j-$th and $k-$th shifted Jacobi polynomials, respectively, then the product of $P_j^{(\alpha,\beta)}(x)$ and $P_k^{(\alpha,\beta)}(x)$ can be written as

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{r=0}^{j+k} \lambda_r^{(j,k)}\ x^r. \tag{9}$$

Proof:

Defining the $Q_{j+k}^{(\alpha,\beta)}(x) = P_j^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(x)$, as a polynomial of degree $j+k$, leads to:

$$\begin{aligned} Q_{j+k}^{(\alpha,\beta)}(x) &= \left(\sum_{m=0}^{j} \gamma_m^{(j)}\ x^m\right) \left(\sum_{n=0}^{k} \gamma_n^{(k)}\ x^n\right) \\ &= \sum_{r=0}^{j+k} \lambda_r^{(j,k)}\ x^r. \end{aligned}$$

The relation between coefficients $\lambda_n^{(j,k)}$ with coefficients $\gamma_m^{(j)}$ and $\gamma_m^{(k)}$ will be as follows:

If $j \geq k$:

$r = 0, 1, ..., j+k,$
$if\ \ r > j\ \ then$
$\lambda_r^{(j,k)} = \sum_{l=r-j}^{k} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$else$
$r_1 = min\{r, k\},$
$\lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$end.$

If $j < k$:

$r = 0, 1, ..., j+k,$
$if\ \ r \leq j\ \ then$
$r_1 = min\{r, j\},$
$\lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$else$
$r_2 = min\{r, k\},$
$\lambda_r^{(j,k)} = \sum_{l=r-j}^{r_2} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$end.$

where the coefficients $\gamma_m^{(n)}$ are defined by Lemma 1. Thus, the coefficients $\lambda_r^{(j,k)}$ is determined. □

Lemma 3.

$Q_{j+k}^{(\alpha,\beta)}(x)$ can be expanded as follows:

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{l=0}^{j+k} \Lambda_l^{(j,k)} (x+1)^l, \tag{10}$$

where $\Lambda_i^{(j,k)}$ are obtained from the following relations,

$$\begin{aligned}
&\Lambda_{j+k}^{(j,k)} = \lambda_{j+k}^{(j,k)},\\
&\Lambda_l^{(j,k)} = \lambda_l^{(j,k)} - \sum_{m=l+1}^{j+k} \binom{m}{l} \Lambda_m^{(j,k)}, \qquad l = j+k-1, j+k-2, ..., 1, 0,
\end{aligned} \tag{11}$$

where $\lambda_l^{(j,k)}$ have been introduced by Lemma 2.

Proof:

The $(x+1)^l$ can be written as:

$$(x+1)^l = \sum_{m=0}^{l} \binom{l}{m} x^m.$$

The goal is to obtain the $\Lambda_l^{(j,k)}$ in terms of the $\lambda_l^{(j,k)}$. To this end, we rewrite Equation (10) as follows:

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{l=0}^{j+k} \sum_{m=0}^{l} \Lambda_l^{(j,k)} \binom{l}{m} x^m. \tag{12}$$

The coefficient of x^r in Equation (12) will be as:

$$\sum_{m=r}^{j+k} \binom{m}{r} \Lambda_m^{(j,k)}. \tag{13}$$

By using Equation (13), we rewrite Equation (12) as follows:

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{l=0}^{j+k} \left\{ \sum_{m=l}^{j+k} \binom{m}{l} \Lambda_m^{(j,k)} \right\} x^l.$$

According to Lemma 2, we have

$$\lambda_l^{(j,k)} = \sum_{m=l}^{j+k} \binom{m}{l} \Lambda_m^{(j,k)}, \qquad l = 0, 1, ..., j+k. \tag{14}$$

Solving the upper triangular system (14) as backwards leads to determine the unknown coefficients, $\Lambda_l^{(j,k)}$. Consequently, the desired result is achieved and we can use the Equation (10) instead the Equation (9). $\square$

Lemma 4.

If $l \geq 0$, then we have

$$\int_{-1}^{1} (x+1)^l \; P_i^{(\alpha,\beta)}(x) \; w^{(\alpha,\beta)}(x) \; dx = \sum_{k=0}^{i} \left\{ \frac{(-1)^{i-k} \; 2^{l+\alpha+\beta+1} \; \Gamma(i+\beta+1)}{\Gamma(k+\beta+1) \; \Gamma(i+\alpha+\beta+1)} \right. \\ \left. \times \frac{\Gamma(i+k+\alpha+\beta+1) \; \Gamma(l+k+\beta+1) \; \Gamma(\alpha+1)}{k! \; (i-k)! \; \Gamma(l+k+\alpha+\beta+2)} \right\}.$$

Proof:

Using the Equation (5) and definition of weight function, we have

$$\begin{aligned}\int_{-1}^{1}(x+1)^l\ P_i^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx &= \sum_{k=0}^{i}\frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{2^k\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ k!\ (i-k)!}\\ &\times\int_{-1}^{1}(x+1)^{l+k+\beta}\ (1-x)^{\alpha}\ dx\\ &=\sum_{k=0}^{i}\frac{(-1)^{i-k}\ 2^{l+\alpha+\beta+1}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ k!\ (i-k)!}\\ &\times\int_{0}^{1}t^{l+k+\beta}\ (1-t)^{\alpha}\ dt\\ &=\sum_{k=0}^{i}\frac{(-1)^{i-k}\ 2^{l+\alpha+\beta+1}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ k!\ (i-k)!}\\ &\times B(l+k+\beta+1,\alpha+1),\end{aligned}$$

where $B(s,t)$ is the Beta function and defined as:

$$B(s,t)=\int_{0}^{1}v^{s-1}(1-v)^{t-1}dv=\frac{\Gamma(s)\ \Gamma(t)}{\Gamma(s+t)}.\ \square$$

Lemma 5.

If $P_i^{(\alpha,\beta)}(x)$, $P_j^{(\alpha,\beta)}(x)$, and $P_k^{(\alpha,\beta)}(x)$ are $i-$th, $j-$th, and $k-$th shifted Jacobi polynomials, respectively, then the following relation is established,

$$\begin{aligned}q_{ijk}&=\int_{-1}^{1}P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx\\ &=\sum_{l=0}^{j+k}\left\{\sum_{m=0}^{i}\frac{(-1)^{i-m}\Lambda_l^{(j,k)}\ 2^{l+\alpha+\beta+1}\ \Gamma(l+m+\beta+1)}{\Gamma(m+\beta+1)\ \Gamma(i+\alpha+\beta+1)}\right.\\ &\left.\times\frac{\Gamma(i+\beta+1)\ \Gamma(i+m+\alpha+\beta+1)\ \Gamma(\alpha+1)}{\Gamma(l+m+\alpha+\beta+2)\ (i-m)!\ m!}\right\},\end{aligned}$$

where $\Lambda_l^{(j,k)}$ were introduced in Lemma 3.

Proof:

Assume $P_j^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(x)=Q_{j+k}^{(\alpha,\beta)}(x)$. Using Equation (5), Lemmas 3 and 4 leads to:

$$
\begin{aligned}
\int_{-1}^{1} P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx &= \int_{-1}^{1} P_i^{(\alpha,\beta)}(x)\ Q_{j+k}^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx \\
&= \sum_{l=0}^{j+k} \Lambda_l^{(j,k)} \int_{-1}^{1} (x+1)^l\ P_i^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx \\
&= \sum_{l=0}^{j+k} \Bigg\{ \sum_{m=0}^{i} \frac{(-1)^{i-m} \Lambda_l^{(j,k)}\ 2^{l+\alpha+\beta+1}\ \Gamma(l+m+\beta+1)}{\Gamma(m+\beta+1)\ \Gamma(i+\alpha+\beta+1)} \\
&\times \frac{\Gamma(i+\beta+1)\ \Gamma(i+m+\alpha+\beta+1)\ \Gamma(\alpha+1)}{\Gamma(l+m+\alpha+\beta+2)\ (i-m)!\ m!} \Bigg\}.
\end{aligned}
$$

Thus, desirable result is obtained. □

In performing arithmetic and other operations on the Jacobi bases, we frequently encounter the integration of the vector $\Phi(x)$ defined by Equation (8) which is called the operational matrix of the integration. Hence, the matrix relations must be obtained. In this section, this operational matrix will be derived. To this end, some useful lemmas and theorems are stated.

Theorem 6.

Let $\Phi(x)$ be the Jacobi vector in Equation (8). Then,

$$
\int_0^x \Phi(t)\ dt \simeq M\ \Phi(x),
$$

where M is the $(N+1) \times (N+1)$ operational matrix of the integration and is defined by:

$$
M = \begin{bmatrix} \Omega(0,0) & \Omega(0,1) & \dots & \Omega(0,N) \\ \Omega(1,0) & \Omega(1,1) & \dots & \Omega(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(N,0) & \Omega(N,1) & \dots & \Omega(N,N) \end{bmatrix},
$$

where

$$
\Omega(i,0) = \omega_{i0k} + \theta_i, \quad i = 0,1,...,N, \quad k = 0,1,...,i, \tag{15}
$$

$$
\Omega(i,j) = \sum_{k=0}^{i} \omega_{ijk}, \quad i = 0,1,...,N, \quad j = 1,2,...,N, \tag{16}
$$

and θ_i and ω_{ijk} are given by

$$
\begin{aligned}
\theta_i &= \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (k+1)!\ (i-k)!\ 2^k}, \\
\omega_{ijk} &= \frac{(-1)^{i-k}\ 2^{\alpha+\beta+2}\ \Gamma(j+\beta+1)\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(\alpha+1)}{h_j\ \Gamma(k+\beta+1)\ \Gamma(j+\alpha+\beta+1)\ \Gamma(i+\alpha+\beta+1)(k+1)!\ (i-k)!} \\
&\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ \Gamma(j+l+\alpha+\beta+1)\ \Gamma(l+k+\beta+2)}{\Gamma(l+\beta+1)\ \Gamma(l+k+\alpha+\beta+3)\ l!\ (j-l)!}, \\
& i,j = 0,1,...,N, \ \ k = 0,1,...,i.
\end{aligned}
$$

Proof:

Integration of the analytical form of $P_i^{(\alpha,\beta)}(x)$, i.e. Equation (5), from 0 to x leads to:

$$\int_0^x P_i^{(\alpha,\beta)}(t)\ dt = \sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ (x+1)^{k+1}}{2^k\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (k+1)!\ (i-k)!} + \sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{2^k\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (k+1)!\ (i-k)!}.$$

Now, one can approximate $(x+1)^{k+1}$ in terms of shifted Jacobi polynomials as:

$$(x+1)^{k+1} = \sum_{j=0}^{N} a_{k,j}\ P_j^{(\alpha,\beta)}(x),$$

where

$$a_{k,j} = \frac{1}{h_j}\ \int_{-1}^{1} (x+1)^{k+1}\ P_j^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx.$$

But according to Lemma 4 one has,

$$\int_{-1}^{1} (x+1)^{k+1}\ P_j^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx = \sum_{l=0}^{j} \left\{ \frac{(-1)^{(j-l)}\ 2^{k+\alpha+\beta+2}\ \Gamma(j+l+\alpha+\beta+1)}{\Gamma(l+\beta+1)\ \Gamma(j+\alpha+\beta+1)} \times \frac{\Gamma(j+\beta+1)\ \Gamma(k+l+\beta+2)\ \Gamma(\alpha+1)}{\Gamma(k+l+\alpha+\beta+3)\ l!\ (j-l)!} \right\}.$$

Therefore, Equation (15) will be as follows:

$$\begin{aligned}\int_0^x P_i^{(\alpha,\beta)}(t)\ dt &= \sum_{k=0}^{i} \frac{(-1)^{i-k}\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)}{2^k\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (k+1)!\ (i-k)!} \\ &+ \sum_{j=0}^{N} \left\{ \sum_{k=0}^{i} \frac{(-1)^{i-k}\ 2^{\alpha+\beta+2}\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(j+\beta+1)\ \Gamma(i+\beta+1)\ \Gamma(\alpha+1)}{h_j\ \Gamma(k+\beta+1)\ \Gamma(j+\alpha+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (i-k)!\ (k+1)!} \right. \\ &\times \left. \sum_{l=0}^{j} \frac{(-1)^{j-l}\ \Gamma(j+l+\alpha+\beta+1)\ \Gamma(l+k+\beta+2)}{\Gamma(l+\beta+1)\ \Gamma(l+k+\alpha+\beta+3)\ (j-l)!\ l!} \right\} P_j^{(\alpha,\beta)}(x) \\ &= \sum_{j=0}^{N}\ \Omega(i,j)\ P_j^{(\alpha,\beta)}(x),\end{aligned}$$

where $\Omega(i,j)$ have been given in Equations (15) and (16). Accordingly, rewriting the last relation as a vector form gives

$$\int_0^x P_i^{(\alpha,\beta)}(t)\ dt = [\Omega(i,0), \Omega(i,1), ..., \Omega(i,N)]\ \Phi(x), \quad i = 0, 1, .., N.$$

This leads to the desired result. □

The following property of the product of two vectors $\Phi(x)$ and $\Phi^T(x)$ will also be used.

$$\Phi(x)\ \Phi^T(x)\ C \simeq \tilde{C}\ \Phi(x), \tag{17}$$

where C and $\tilde{C}$ are a $(N+1)\times 1$ vector and a $(N+1)\times(N+1)$ operational matrix of product, respectively.

Theorem 7.

The entries of the matrix $\tilde{C}$, in Equation (17), are computed as:

$$\tilde{C}_{jk} = \frac{1}{h_k}\sum_{i=0}^{N} C_i \; q_{ijk}, \qquad j,k = 0,1,...,N,$$

where q_{ijk} are computed by Lemma 5 and C_i are the entries of the vector C in Equation (17).

Proof:

The left side of equality (17) is as follows:

$$\Phi(x)\;\Phi^T(x)\;C = \begin{bmatrix} \sum_{i=0}^{N} C_i\; P_0^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x) \\ \sum_{i=0}^{N} C_i\; P_1^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x) \\ \vdots \\ \sum_{i=0}^{N} C_i\; P_N^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x) \end{bmatrix}.$$

We put,

$$P_j^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{N} a_k\; P_k^{(\alpha,\beta)}(x), \qquad i,j = 0,1,...,N. \tag{18}$$

Multiplying both sides of Equation (18) by $P_m^{(\alpha,\beta)}(x),\; m,n = 0,1,...,N$, and integration of the result from -1 to 1 yields:

$$\begin{aligned} &\int_{-1}^{1} P_j^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x)\; P_m^{(\alpha,\beta)}(x)\; w^{(\alpha,\beta)}(x)\; dx \\ &\qquad = \sum_{k=0}^{N} a_k \int_{-1}^{1} P_k^{(\alpha,\beta)}(x)\; P_m^{(\alpha,\beta)}(x)\; w^{(\alpha,\beta)}(x)\; dx \\ &\qquad = a_m\; h_m, \qquad m = 0,1,...,N. \end{aligned}$$

Therefore,

$$a_m = \frac{1}{h_m}\int_{-1}^{1} P_j^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x)\; P_m^{(\alpha,\beta)}(x)\; w^{(\alpha,\beta)}(x)dx.$$

Now, suppose

$$\int_{-1}^{1} P_j^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x)\; P_m^{(\alpha,\beta)}(x)\; w^{(\alpha,\beta)}(x)\; dx = q_{ijm}.$$

We get

$$a_m = \frac{q_{ijm}}{h_m}, \qquad i,j,m = 0,1,...,N.$$

By substituting a_m into Equation (18) we have,

$$P_j^{(\alpha,\beta)}(x)\; P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{N} \frac{q_{ijk}}{h_k}\; P_k^{(\alpha,\beta)}(x), \qquad i,j = 0,1,...,N.$$

Therefore, each component of the vector $\Phi(x)\ \Phi^T(x)\ C$ in Equation (18) is

$$\sum_{i=0}^{N} C_i\ P_j^{(\alpha,\beta)}(x)\ P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{N}\left\{\frac{1}{h_k}\sum_{i=0}^{N} C_i\ q_{ijk}\right\}\ P_k^{(\alpha,\beta)}(x).$$

Thus, the desirable result is obtained. □

The next theorem presents the general formula to approximate the nonlinear terms, as $\upsilon^r(x)\ u^s(x)$, which may appear in nonlinear equations.

Theorem 8.

If c and υ are the $(N+1)\times 1$ vectors, $\tilde{c}$ and $\tilde{\upsilon}$ are the $(N+1)\times(N+1)$ operational matrices of the product such that,

$$u(x) \simeq \Phi^T(x)\ c = c^T\ \Phi(x),\quad \upsilon(x) \simeq \Phi^T(x)\ \upsilon = \upsilon^T\ \Phi(x),\quad \Phi(x)\ \Phi^T(x)\ c \simeq \tilde{c}\ \Phi(x),$$

and $\Phi(x)\ \Phi^T(x)\ \upsilon \simeq \tilde{\upsilon}\ \Phi(x)$, then the following proposition hold:

$$\upsilon^r(x)\ u^s(x) \simeq \upsilon^T\ (\tilde{\upsilon})^{r-1}\ \tilde{B}_{s-1}\ \Phi(x),\quad B_{s-1} = (\tilde{c}^T)^{s-1}\ c,\quad r,s = 1,2,\dots\ .$$

Proof:

We have,

$$u^2(x) \simeq (\Phi^T(x)\ c)^2 = c^T\ \Phi(x)\ \Phi^T(x)\ c \simeq c^T\ \tilde{c}\ \Phi(x),$$

So, by use of induction, $u^s(x)$ can be approximated as:

$$u^s(x) \simeq c^T\ (\tilde{c})^{s-1}\ \Phi(x),\qquad s = 1,2,\dots\ .$$

To similar way, $\upsilon^r(x)$ is approximated as follows:

$$\upsilon^r(x) \simeq \upsilon^T\ (\tilde{\upsilon})^{r-1}\ \Phi(x),\qquad r = 1,2,\dots\ .$$

Using the expressed relations and induction is easily resulted,

$$\upsilon^r(x)\ u^s(x) \simeq \upsilon^T\ (\tilde{\upsilon})^{r-1}\ \tilde{B}_{s-1}\ \Phi(x),\quad B_{s-1} = (\tilde{c}^T)^{s-1}\ c,\quad r,s = 1,2,\dots\ .\ \square$$

3. Description of proposed scheme

Let us consider a general nonlinear wave equation in the form

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \tag{19}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its derivatives in which higher order derivatives and nonlinear terms are involved. To obtain approximate solutions, the following steps are followed:

Step 1.

We combine the real variables x and t by a compound variable η,

$$u(x,t) = u(\eta), \qquad \eta = K(x \pm Vt), \tag{20}$$

where K and V are arbitrary constants. The wave transformation (20) converts Equation (19) to a nonlinear ordinary differential equation (ODE) for $u = u(\eta)$ as:

$$Q(u, u', u'', u''', ...) = 0, \tag{21}$$

where Q is a polynomial of u and its derivatives with respect to η.

Step 2.

Suppose $u^{(n)}(\eta)$ is the highest order derivative in Equation (21) which can be expressed as follows:

$$u^{(n)}(\eta) \simeq \sum_{i=0}^{N} C_i \; P_i^{(\alpha,\beta)}(\eta) = \Phi^T(\eta) \; C, \tag{22}$$

where $P_i^{(\alpha,\beta)}(\eta)$ are the Jacobi polynomials of degree i and C_i are the unknown coefficients which should be determined. Then the other terms in Equation (21) can be approximated by integrating of Equation (22) from 0 to η as follows:

$$\begin{aligned}
&u^{(n-1)}(\eta) \simeq \Phi^T(\eta) \; M^T \; C + u^{(n-1)}(0) \simeq \Phi^T(\eta) \; M^T \; C + \Phi^T(\eta) \; U_1, \\
&u^{(n-2)}(\eta) \simeq \Phi^T(\eta) \; (M^T)^2 \; C + \Phi^T(\eta) \; M^T \; U_1 + \Phi^T(\eta) \; U_2, \\
&\qquad \vdots \\
&u(\eta) \simeq \Phi^T(\eta) \; (M^T)^n \; C + \Phi^T(\eta) \; \left\{ \sum_{k=1}^{n} (M^T)^{n-k} \; U_k \right\},
\end{aligned} \tag{23}$$

where M is the integral operational matrix, $u^{(r)}(0) \simeq \Phi^T(\eta) \; U_{n-r}, \;\; r = 0, 1, ..., n-1$, and $u^{(r)}(0)$ are obtained from the initial conditions of the principle problem. Substituting the approximations (23) into Equation (21) leads to the nonlinear algebraic equation. By using the roots of $P_{N+1}^{(\alpha,\beta)}(\eta)$, the algebraic equation resulted is collocated in $N+1$ nodes. Hereby, the algebraic equation reduces the problem to solve a system of nonlinear algebraic equations and unknown coefficients are obtained.

Step 3.

Because the approximate solution has a polynomial form it can not indicate the soliton nature of the exact solution; we define a rational exponential approximation (REA) as:

$$u_{REA}(\eta) = \frac{\sum_{j=-e}^{f} a_j \; e^{j\eta}}{\sum_{k=-p}^{q} b_k \; e^{k\eta}}, \qquad b_q = 1, \quad \eta = K(x \pm Vt), \tag{24}$$

where the numerator and denominator have no common factors and a_j and b_k are unknown coefficients which are determined by using the coefficients in the Taylor expansion of the approximate solution obtained of the Step 2. For this reason, suppose $\sum_{i=0}^{e+f+p+q+1} d_i^{(1)}\eta^i$ and $\sum_{i=0}^{e+f+p+q+1} d_i^{(2)}\eta^i$ are the Taylor expansion of (24) and solution series in Step 2. Equating powers of η in two series leads to $e+f+p+q+1$ algebraic equations and completes the determination of the coefficients a_j and b_k.

4. Applications of method

Example 1.

The nonlinear dispersive equation formulated by Korteweg and de Vries (KdV) in simplest form is given by Wazwaz (2009),

$$u_t - 6uu_x + u_{xxx} = 0. \tag{25}$$

This equation models a variety of nonlinear phenomena including ion acoustic waves in plasma and shallow water waves. The first three conservation laws for the KdV equation are as follows (Wazwaz (2009)),

$$I_1 = \int_a^b u(x,t)\ dx, \quad I_2 = \int_a^b u^2(x,t)\ dx, \quad I_3 = \int_a^b \left(u^3(x,t) - \frac{1}{2}u_x^2(x,t)\right)\ dx, \tag{26}$$

To seek the soliton solution of Equation (25), we make the transformation $\eta = K(x - Vt)$. Then Equation (25) is reduced to the following ordinary differential equation,

$$-KVu' - 6Kuu' + K^3u''' = 0, \tag{27}$$

where K and V are arbitrary constants and superscripts indicate the ordinary derivatives with respect to η. The exact solution obtained by Ebaid (2007) in the form,

$$u(\eta) = -\frac{2K^2}{e^{-\eta} + 2 + e^{\eta}}, \quad \eta = K(x - Vt). \tag{28}$$

We need three conditions for Equation (27) as:

$$u(0) = \frac{-K^2}{2}, \quad u'(0) = 0, \quad u''(0) = \frac{K^2}{4}, \tag{29}$$

which are obtained from Equation (28). The nonlinear Equation (27) with the initial conditions (29) is approximated using the Jacobi Collocation method introduced in Section 3. For this end, let us consider the following approximation,

$$u'''(\eta) \simeq \Phi^T(\eta)\ C. \tag{30}$$

Integration of Equation (30) with respect to η from 0 to η, one gets the following approximation for $u''(\eta)$,

$$u''(\eta) \simeq \Phi^T(\eta)\ M^T\ C + u''(0) \simeq \Phi^T(\eta)\ M^T\ C + \Phi^T(\eta)\ U_1, \tag{31}$$

where $u''(0)$ is approximated by $\Phi^T(\eta)\ U_1$ which U_1 is a $(N+1)\times 1$ known vector. Again, integrating Equation (31) with respect to η from 0 to η, one gets the following approximation for $u'(\eta)$,

$$u'(\eta) \simeq \Phi^T(\eta)\ (M^T)^2\ C + \Phi^T(\eta)\ M^T\ U_1 + \Phi^T(\eta)\ U_2, \tag{32}$$

where $\Phi^T(\eta)\ U_2 \simeq u'(0)$. Finally, integrating Equation (32) with respect to η leads to get the following approximation for $u(\eta)$,

$$u(\eta) \simeq \Phi^T(\eta)\ (M^T)^3\ C + \Phi^T(\eta)\ (M^T)^2\ U_1 + \Phi^T(\eta)\ M^T\ U_2 + \Phi^T(\eta)\ U_3, \tag{33}$$

where $\Phi^T(\eta)\ U_3 \simeq u(0)$. Also, the nonlinear term in Equation (27) is approximated as follows:

$$u(\eta)\ u'(\eta) \simeq V_1^T \Phi(\eta)\ \Phi^T(\eta) V_2 \simeq V_1^T \tilde{V}_2\ \Phi(\eta) = \Phi^T(\eta)\ \tilde{V}_2^T V_1, \tag{34}$$

where $V_1 = (M^T)^3\ C + (M^T)^2\ U_1 + M^T\ U_2 + U_3$, $V_2 = (M^T)^2\ C + M^T\ U_1 + U_2$, and $\tilde{V}_2$ is operational matrix of product and its entries are determined in terms of the components of the vector V_2. Substituting the approximations (30)–(34) into Equation (27), leads to the following nonlinear algebraic equation.

$$\Phi^T(\eta)\Big\{ -KV\ \Big((M^T)^2 C + M^T U_1 + U_2\Big) - 6K\ \tilde{V}_2^T\ V_1 + K^3\ C\Big\} \approx 0. \tag{35}$$

Setting $K=1$, $V=K^2$, $N=20$, and using the roots of $P_{21}^{(\alpha,\beta)}(\eta)$, Equation (35) is collocated in 21 collocation nodes for different values of α and β parameters. Hereby, Equation (35) reduces the problem to solve a system of nonlinear algebraic equations and unknown coefficients are obtained for some values of α and β parameters. Substituting the resultant vector C into Equation (33) leads to an approximate solution to the exact solution (28). Now, in order to improve the approximate solution (28) and achieve the soliton solutions, four cases are considered for the rational approximation (24):

Case I

$e = f = 1$, $p = q = 1$,

In this case, Equation (24) will be as follows:

$$u_{ERA}(\eta) = \frac{a_{-1}e^{-\eta} + a_0 + a_1 e^{\eta}}{b_{-1}e^{-\eta} + b_0 + e^{\eta}}. \tag{36}$$

Equating the Taylor expansions of (36) and (33) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, and b_0. Solving the system of algebraic equations with the aide of Maple, the coefficients are obtained. Table 1 shows the maximum absolute errors of the approximate solutions for different values of α and β parameters. Table 2 displays the numerical values of conservation quantities (26) in various times for $\alpha = \beta = 1/2$, $\alpha = 2$, $\beta = 2$, $a = -20$, and $b = 100$. The analytical values are: $I_1 = -2$, $I_2 = 0.6666666666$, and $I_3 = -0.3333333333$. As it can be seen from Table 2, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. Also, the absolute errors functions obtained by the proposed method are displayed in Figure 1 for $\alpha = 1, \beta = 2$, and $t = 1, 15, 30$.

Table 1. Maximum absolute errors of Example 1 for different values of α and β, $e = f = 1$ and $p = q = 1$

(α, β)	$Error_{Abs}$	(α, β)	$Error_{Abs}$
$(0, 0)$	1.5×10^{-10}	$(1, 2)$	2.5×10^{-12}
$(\frac{1}{2}, \frac{1}{2})$	4.0×10^{-12}	$(1, 1)$	3.0×10^{-12}
$(-\frac{1}{2}, \frac{1}{2})$	6.0×10^{-12}	$(2, 2)$	2.0×10^{-12}
$(\frac{1}{2}, -\frac{1}{2})$	6.0×10^{-12}	$(8, 7)$	2.0×10^{-13}
$(-\frac{1}{2}, 1)$	5.0×10^{-12}	$(10, 10)$	9.0×10^{-14}

Table 2. Numerical values of I_1, I_2, and I_3 of Example 1 for $\alpha = \beta = \frac{1}{2}$, $\alpha = \beta = 2$, and $e = f = p = q = 1$

		$(\alpha, \beta) = (2, 2)$	
t	I_1	I_2	I_3
0	-1.9999999956366	0.666666667464	-0.33333323040
5	-1.9999999997311	0.666666672909	-0.33333331863
10	-1.9999999997587	0.666666662266	-0.33333318910
15	-1.9999999997589	0.666666672924	-0.33333331072
20	-1.9999999997589	0.666666672061	-0.33333326198
25	-1.9999999997589	0.666666665480	-0.33333347100
		$(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$	
t	I_1	I_2	I_3
0	-1.999999995370	0.666666665155	-0.333333335579
5	-1.999999999464	0.6666666669404	-0.333333285760
10	-1.999999999492	0.666666665837	-0.333333329469
15	-1.999999999492	0.666666667028	-0.333333293896
20	-1.999999999492	0.666666667870	-0.333333313797
25	-1.999999999492	0.666666661850	-0.333333377706

Case II

$e = f = 2$, $p = q = 2$,

In this case, Equation (24) will be as follows:

$$u_{ERA}(\eta) = \frac{a_{-2}\ e^{-2\eta} + a_{-1}\ e^{-\eta} + a_0 + a_1\ e^{\eta} + a_2\ e^{2\eta}}{b_{-2}\ e^{-2\eta} + b_{-1}\ e^{-\eta} + b_0 + b_1\ e^{\eta} + e^{2\eta}}. \tag{37}$$

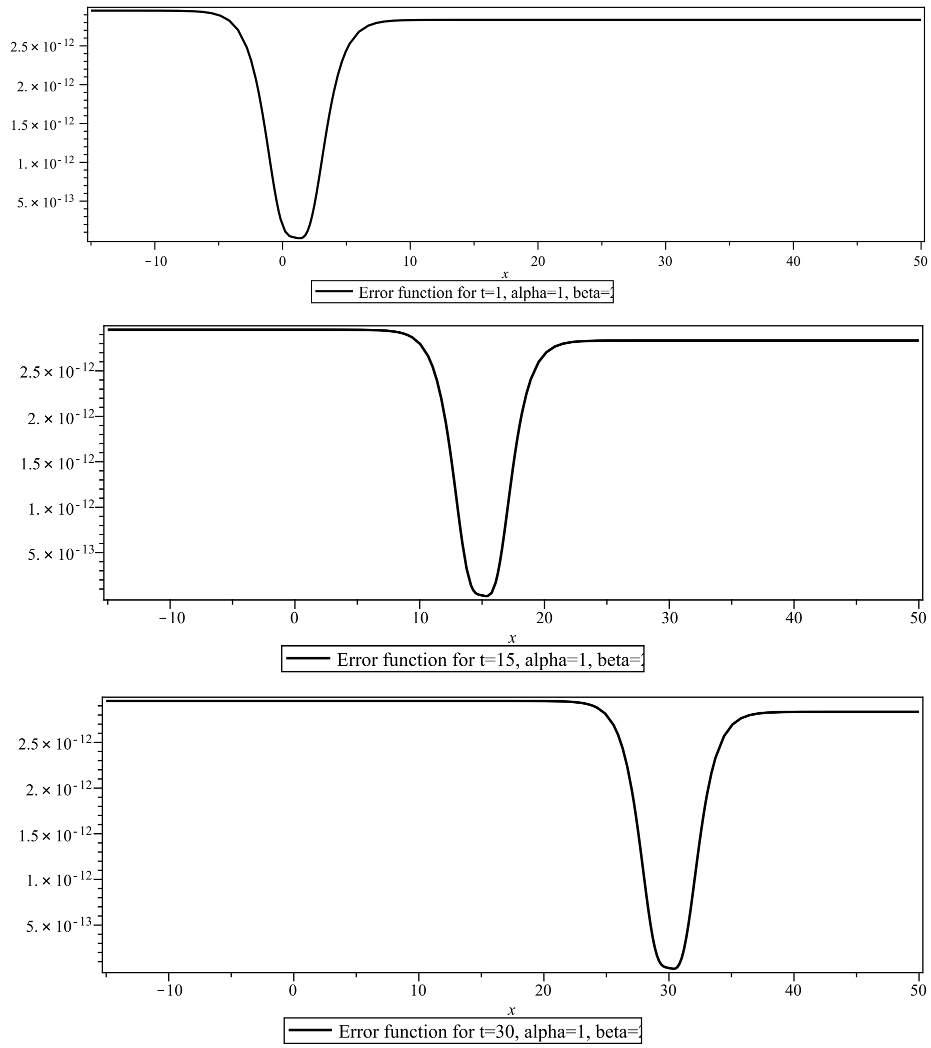


Figure 1. Behavior of error function of Example 1 for $\alpha = 1, \beta = 2, t = 1, 15, 30$

Equating the Taylor expansions of (37) and (33) yields a set of algebraic equations for a_{-2}, a_{-1}, a_0, a_1, a_2, b_{-2}, b_{-1}, b_0, and b_1. Solving the system of algebraic equations the coefficients are obtained. Table 3 shows the maximum absolute errors of the approximate solutions for different values of α and β parameters. Table 4 displays the numerical values of conservation quantities (26) in various times for $\alpha = \beta = 1/2$, $\alpha = 2, \beta = 2$, $a = -20$, and $b = 100$. As can be seen from Table 4, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. Also, the approximate solution and absolute error function obtained by the proposed method are displayed in Figure 2 for $\alpha = 8$ and $\beta = 7$.

Case III

$e = f = 1$, $p = q = 2$,

In this case, Equation (24) will be as follows:

$$u_{ERA}(\eta) = \frac{a_{-1}\ e^{-\eta} + a_0 + a_1\ e^{\eta}}{b_{-2}\ e^{-2\eta} + b_{-1}\ e^{-\eta} + b_0 + b_1\ e^{\eta} + e^{2\eta}}. \tag{38}$$

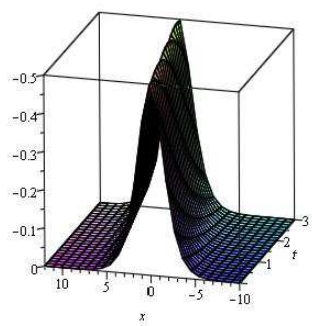
Equating the Taylor expansions of (38) and (33) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-2}, b_{-1}, b_0, and b_1. Solving the system of algebraic equations with the aide of Maple, the coefficients are obtained. Table 5 shows the maximum absolute errors of the approximate solutions for different values of α and β. Table 6 displays the numerical values of conservation quantities (26) in various times for $\alpha = \beta = 1/2$, $\alpha = 2, \beta = 2$, $a = -20$, and $b = 100$. As can be seen from Table 6, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. Table 7 displays different values of the exact and approximate solutions in various points for $\alpha = -1/2, \beta = 1$, $\alpha = \beta = 10$. As can be seen from Table 7 the results of the solutions obtained by the proposed scheme are almost the same as the results of the exact solutions.

Table 3. Maximum absolute errors of Example 1 for different values α and β, $e = f = 2$, and $p = q = 2$

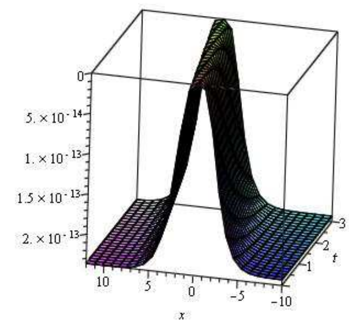
(α, β)	$Error_{Abs}$	(α, β)	$Error_{Abs}$
$(0, 0)$	1.8×10^{-10}	$(1, 2)$	2.0×10^{-12}
$(\frac{1}{2}, \frac{1}{2})$	4.0×10^{-12}	$(1, 1)$	3.0×10^{-12}
$(-\frac{1}{2}, \frac{1}{2})$	6.0×10^{-12}	$(2, 2)$	2.0×10^{-12}
$(\frac{1}{2}, -\frac{1}{2})$	6.0×10^{-12}	$(8, 7)$	2.0×10^{-13}
$(-\frac{1}{2}, 1)$	5.0×10^{-12}	$(10, 10)$	9.0×10^{-14}

Table 4. Numerical values of I_1, I_2, and I_3 of Example 1 for $\alpha = \beta = \frac{1}{2}$, $\alpha = \beta = 2$ and $e = f = p = q = 2$

		$(\alpha, \beta) = (2, 2)$	
t	I_1	I_2	I_3
0	-1.999999995805	0.66666666666822547	-0.33333333333320121
5	-2.000000000201	0.66666666666822550	-0.33333333333320121
10	-2.000000000230	0.66666666666822548	-0.33333333333320122
15	-2.000000000230	0.66666666666822550	-0.33333333333320121
20	-2.000000000230	0.66666666666822548	-0.33333333333320122
25	-2.000000000230	0.66666666666822550	-0.33333333333320121
		$(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$	
t	I_1	I_2	I_3
0	-1.99999999605	0.66666666666988152	-0.33333333333305633
5	-2.00000000044	0.66666666666988157	-0.33333333333305631
10	-2.00000000047	0.66666666666988153	-0.33333333333305634
15	-2.00000000048	0.66666666666988157	-0.33333333333305631
20	-2.00000000048	0.66666666666988153	-0.33333333333305634
25	-2.00000000048	0.66666666666988157	-0.33333333333305631



(a)



(b)

Figure 2. (a) Plot of approximate solution, (b) Plot of error function of Example 1 for $\alpha = 8$, $\beta = 7$, $e = f = p = q = 2$

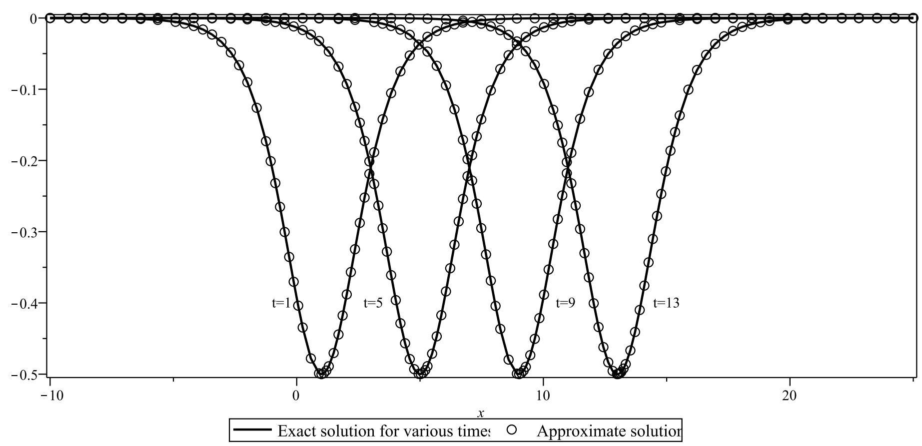


Figure 3. Plot of exact and approximate solutions of Example 1 for $\alpha = \beta = 1/2$ and various times: $t = 1, 5, 9, 13$

Case IV

$e = 0,\ f = p = q = 2,$

In this case, Equation (24) will be as follows:

$$u_{ERA}(\eta) = \frac{a_0 + a_1\ e^{\eta} + a_2\ e^{2\eta}}{b_{-2}\ e^{-2\eta} + b_{-1}\ e^{-\eta} + b_0 + b_1\ e^{\eta} + e^{2\eta}}. \tag{39}$$

Equating the Taylor expansions of (39) and (33) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-2}, b_{-1}, b_0, and b_1. Solving the system of algebraic equations, the coefficients are obtained. Table 8 shows the maximum absolute errors of the approximate solutions for different values of α and β parameters. Table 9 displays the numerical values of conservation quantities (26) in various times for $\alpha = \beta = 1$, $\alpha = -1/2,\ \beta = 1/2$, $a = -20$, and $b = 100$. As can be seen from Table 9, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. In Figure 3 the exact and approximate solutions are compared for the case $\alpha = \beta = 1/2$ and various times: $t = 1, 5, 9, 13$.

Table 5. Maximum absolute errors of Example 1 for different values β, $e = f = 1$ and $p = q = 2$

(α,β)	Error$_{Abs}$	(α,β)	Error$_{Abs}$
$(0,0)$	3.0×10^{-13}	$(1,2)$	6.0×10^{-15}
$(\frac{1}{2},\frac{1}{2})$	1.0×10^{-14}	$(1,1)$	8.0×10^{-15}
$(-\frac{1}{2},\frac{1}{2})$	1.6×10^{-14}	$(2,2)$	4.0×10^{-15}
$(\frac{1}{2},-\frac{1}{2})$	1.6×10^{-14}	$(8,7)$	3.0×10^{-16}
$(-\frac{1}{2},1)$	1.4×10^{-14}	$(10,10)$	1.2×10^{-16}

Table 6. Numerical values of I_1, I_2, and I_3 of Example 1 for $\alpha = \beta = \frac{1}{2}$, $\alpha = \beta = 2$, and $e = f = 1,\ p = q = 2$

		$(\alpha,\beta) = (2,2)$	
t	I_1	I_2	I_3
0	-1.99999999999997449	0.66666666666665677	-0.33333333333330047
5	-1.99999999999997450	0.66666666666665678	-0.33333333333330044
10	-1.99999999999997448	0.66666666666665678	-0.33333333333330047
15	-1.99999999999997315	0.66666666666665678	-0.33333333333330044
20	-1.99999999999977361	0.66666666666665678	-0.33333333333330047
25	-1.99999999997016208	0.66666666666665678	-0.33333333333330044
		$(\alpha,\beta) = (\frac{1}{2},\frac{1}{2})$	
t	I_1	I_2	I_3
0	-1.99999999999994007	0.6666666666666435	-0.33333333333325685
5	-1.99999999999994009	0.6666666666666435	-0.33333333333325681
10	-1.99999999999994006	0.6666666666666435	-0.33333333333325685
15	-1.99999999999993874	0.6666666666666435	-0.33333333333325681
20	-1.99999999999973919	0.6666666666666435	-0.33333333333325685
25	-1.99999999997012767	0.6666666666666435	-0.33333333333325681

Table 7. Comparison of exact and approximate solutions of Example 1 for $N = 20$, $\alpha = -\frac{1}{2}, \beta = 1$, and $\alpha = \beta = 10$

(x_i,t_i)	u_{Exact}	$u_{REA}(x,t)$ $(\alpha,\beta) = (-\frac{1}{2},1)$	$Error_{Abs}$	$u_{REA}(x,t)$ $(\alpha,\beta) = (10,10)$	$Error_{Abs}$
$(-3,0.2)$	-0.07526353790914	-0.07526353790913	7.6405×10^{-15}	-0.07526353790914	6.9076×10^{-17}
$(-2.5,0.4)$	-0.09886713787329	-0.09886713787328	9.4728×10^{-15}	-0.09886713787329	8.5609×10^{-17}
$(-2,0.6)$	-0.12871659835155	-0.12871659835155	1.1405×10^{-14}	-0.12871659835155	1.0302×10^{-16}
$(-1.5,1)$	-0.14020743309021	-0.14020743309020	1.2032×10^{-14}	-0.14020743309022	1.0868×10^{-16}
$(-1,1.2)$	-0.17960065808014	-0.17960065808012	1.3700×10^{-14}	-0.17960065808014	1.2365×10^{-16}
$(3,2)$	-0.39322386648296	-0.39322386648296	8.1597×10^{-15}	-0.39322386648296	8.6290×10^{-17}
$(5,10)$	-0.01329611334158	-0.01329611334158	1.5486×10^{-15}	-0.01329611334158	1.4014×10^{-17}
$(10,6)$	-0.03532541242658	-0.03532541242657	3.6503×10^{-15}	-0.03532541242658	3.5526×10^{-17}

Table 8. Maximum absolute errors of Example 1 for different values α and β, $e = 0$, and $f = p = q = 2$

(α, β)	Error_{Abs}	(α, β)	Error_{Abs}
$(0, 0)$	5.0×10^{-10}	$(1, 2)$	3.0×10^{-12}
$(\frac{1}{2}, \frac{1}{2})$	9.0×10^{-12}	$(1, 1)$	1.8×10^{-11}
$(-\frac{1}{2}, \frac{1}{2})$	6.0×10^{-12}	$(2, 2)$	2.0×10^{-11}
$(\frac{1}{2}, -\frac{1}{2})$	6.0×10^{-12}	$(8, 7)$	2.0×10^{-13}
$(-\frac{1}{2}, 1)$	6.0×10^{-12}	$(10, 10)$	3.0×10^{-13}

Table 9. Numerical values of I_1, I_2 and I_3 of Example 1 for $\alpha = \beta = 1$, $\alpha = -\beta = -1/2$ and $e = 0,\ f = p = q = 2$

		$(\alpha, \beta) = (1, 1)$	
t	I_1	I_2	I_3
0	-1.99999999975	0.6666666666597	-0.333418
5	-1.99999999977	0.6666666666618	-0.333376
10	-1.99999999978	0.6666666666597	-0.333418
15	-1.99999999980	0.6666666666618	-0.333376
20	-1.99999999982	0.6666666666597	-0.333418
25	-1.99999999981	0.6666666666618	-0.333376
		$(\alpha, \beta) = (-\frac{1}{2}, \frac{1}{2})$	
t	I_1	I_2	I_3
0	-1.99999999954	0.66666666666144339	-0.333333333333714859
5	-1.99999999957	0.66666666666144339	-0.333333333333714855
10	-1.99999999960	0.66666666666144339	-0.333333333333714859
15	-1.99999999963	0.66666666666144339	-0.333333333333714855
20	-1.99999999967	0.66666666666144339	-0.333333333333714859
25	-1.99999999967	0.66666666666144339	-0.333333333333714855

Example 2.

A well known model of nonlinear dispersive waves was proposed by Boussinesq in the form of Wazwaz (2009):

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx}. \tag{40}$$

The Boussinesq equation describes motion of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice. The first four conservation laws for the Boussinesq equation are as follows (Wazwaz (2009)),

$$I_1 = \int_a^b u_t\, dx = 0, \quad I_2 = \int_a^b x u_t\, dx, \quad I_3 = \int_a^b (-u + t u_t)\, dx, \quad I_4 = \int_a^b (-xu + xtu_t)\, dx = 0, \tag{41}$$

To seek the soliton solution of Equation (40), we introduced the wave variable $\eta = Kx + Vt$. Then, Equation (40) is reduced to the following ordinary differential equation,

$$(V^2 - K^2)u'' - 6K^2(u')^2 - 6K^2 uu'' - K^4 u^{(4)} = 0, \tag{42}$$

where K and V are arbitrary constants and superscripts indicate the ordinary derivatives with respect to η. The exact solution obtained by Ebaid (2007) in the form,

$$u(\eta) = \frac{2K^2\ e^{2\eta}}{e^{\eta} + 2\ e^{2\eta} + e^{3\eta}}, \quad \eta = Kx + Vt. \tag{43}$$

We need four conditions for Equation (42) as:

$$u(0) = \frac{K^2}{2}, \quad u'(0) = 0, \quad u''(0) = \frac{-K^2}{4}, \quad u'''(0) = 0, \tag{44}$$

which are obtained from Equation (43). The nonlinear Equation (42) with the initial conditions (44) is approximated using the Jacobi Collocation method introduced in Section 3. For this end, let us consider the following approximations,

$$\begin{aligned}
&u^{(4)}(\eta) \simeq \Phi^T(\eta)\ C,\\
&u'''(\eta) \simeq \Phi^T(\eta)\ M^T\ C + \Phi^T(\eta)\ U_1,\\
&u''(\eta) \simeq \Phi^T(\eta)\ (M^T)^2\ C + \Phi^T(\eta)\ M^T\ U_1 + \Phi^T(\eta)\ U_2,\\
&u'(\eta) \simeq \Phi^T(\eta)\ (M^T)^3\ C + \Phi^T(\eta)\ (M^T)^2\ U_1 + \Phi^T(\eta)\ M^T\ U_2 + \Phi^T(\eta)\ U_3,\\
&u(\eta) \simeq \Phi^T(\eta)\ (M^T)^4\ C + \Phi^T(\eta)\ (M^T)^3\ U_1 + \Phi^T(\eta)\ (M^T)^2\ U_2 + \Phi^T(\eta)\ M^T\ U_3 + \Phi^T(\eta)\ U_4,
\end{aligned} \tag{45}$$

where $u^{(k)}(0)$ are approximated by $\Phi^T(\eta)\ U_{4-k},\ k = 0, 1, 2, 3, 4,$ which U_k are a $(N + 1) \times 1$ known vectors. Also, the nonlinear terms in Equation (42) are approximated as follows:

$$\begin{aligned}
&(u')^2(\eta) \simeq \Phi^T(\eta)\ \tilde{V}_1^T V_1,\\
&u(\eta)\ u''(\eta) \simeq \Phi^T(\eta)\ \tilde{V}_3\ V_2,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
V_1 &= (M^T)^3\ C + (M^T)^2\ U_1 + M^T\ U_2 + U_3,\\
V_2 &= (M^T)^4\ C + (M^T)^3\ U_1 + (M^T)^2\ U_2 + M^T\ U_3 + U_4,\\
V_3 &= (M^T)^2\ C + M^T\ U_1 + U_2,
\end{aligned}$$

and $\tilde{V}_1$ and $\tilde{V}_3$ are operational matrices of product and their entries are determined in terms of the components of the vectors V_1 and V_3, respectively. Substituting the approximations (45)–(46) into Equation (42), leads to the following nonlinear algebraic equation.

$$\Phi^T(\eta)\left\{(V^2 - K^2)\ \left((M^T)^2\ C + M^T\ U_1 + U_2\right) - 6K^2\ \tilde{V}_1^T\ V_1 - 6K^2\ \tilde{V}_3^T\ V_2 - K^4\ C\right\} \approx 0. \tag{47}$$

Setting $K = 1$, $V = -K\sqrt{1 + K^2}$, $N = 20$, and using the roots of $P_{21}^{(\alpha,\beta)}(\eta)$, Equation (47) is collocated in 21 collocation nodes for different values of parameters α and β. Hereby, the Equation (47) reduces the main problem to solve a system of nonlinear algebraic equations and unknown coefficients are obtained for some values of α and β parameters. Substituting the vector C resulted into approximations (45) leads to an approximate solution for the exact solution (43). Now, in order to improve the approximate solution (45) and achieve the soliton solutions, two cases are considered for the rational approximation (24):

Case I

$e = f = 1$, $p = q = 1$,

In this case, equating the Taylor expansions of (36) and (45) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, and b_0. Solving the system of algebraic equations with the aide of Maple, the coefficients are obtained. Table 10 shows the maximum absolute errors of the approximate solutions for different values of α and β parameters. Table 11 displays the numerical values of conservation quantities (41) in various times for $\alpha = \beta = 0$, $a = -50$, and $b = 50$. As it can be seen from Table 11, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. Also, the plots of approximate solution and absolute error function are displayed in Figure 4 for $\alpha = \beta = 0$.

Table 10. Maximum absolute errors of Example 2 for different values α and β, $e = f = 1$ and $p = q = 1$

(α, β)	Error_{Abs}	(α, β)	Error_{Abs}
$(0, 0)$	1.5×10^{-10}	$(1, 1)$	3.0×10^{-12}
$(\frac{1}{2}, \frac{1}{2})$	4.0×10^{-12}	$(3, 3)$	1.2×10^{-12}
$(-\frac{1}{2}, \frac{1}{2})$	6.0×10^{-12}	$(5, 7)$	4.0×10^{-13}
$(\frac{1}{2}, -\frac{1}{2})$	6.0×10^{-12}	$(12, 10)$	8.0×10^{-14}

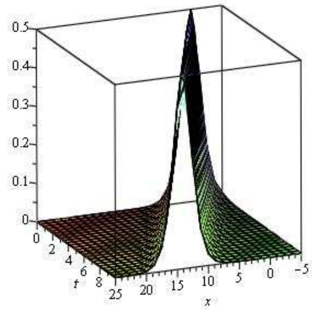
Table 11. Numerical values of I_1, I_2, I_3, and I_4 of Example 2 for $\alpha = \beta = 0$, $e = f = 1$, $p = q = 1$

t	I_1	I_2	I_3	I_4
0	-0.0000	2.828427125942	-1.999999985	7.5659×10^{-13}
5	-1.5616×10^{-19}	2.828427125942	-1.999999985	1.0010×10^{-12}
10	-4.4699×10^{-17}	2.828427125942	-1.999999985	1.1782×10^{-12}
15	-1.2795×10^{-14}	2.828427125941	-1.999999985	7.6976×10^{-12}
20	-3.6625×10^{-12}	2.828427125755	-1.999999985	2.8534×10^{-9}
25	-1.0484×10^{-9}	2.828427072474	-2.000000005	1.0315×10^{-6}

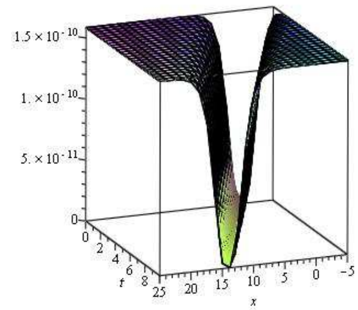
Case II

$e = f = 1$, $p = q = 2$,

In this case, equating the Taylor expansions of (38) and (45) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-2}, b_{-1}, b_0, and b_1. Solving the system of algebraic equations, the coefficients are obtained. Table 12 shows the maximum absolute errors of the approximate solutions for different values of α and β parameters. Table 13 displays the numerical values of conservation quantities (41) in various times for $\alpha = 5$, $\beta = 7$, $a = -50$, and $b = 50$. As can be seen from Table 13, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable. In Figure 5 the exact and approximate solutions are compared for the case $\alpha = \beta = 0$ and various times: $t = 1, 5, 9, 13$.



(a)



(b)

Figure 4. (a) Plot of approximate solution, (b) Plot of error function of Example 2 for $\alpha = \beta = 0$, $e = f = p = q = 1$

Table 12. Maximum absolute errors of Example 2 for different values α and β, $e = f = 1$ and $p = q = 2$

(α, β)	$Error_{Abs}$	(α, β)	$Error_{Abs}$
$(0, 0)$	3.0×10^{-13}	$(1, 1)$	8.0×10^{-15}
$(\frac{1}{2}, \frac{1}{2})$	1.0×10^{-14}	$(3, 3)$	2.5×10^{-15}
$(-\frac{1}{2}, \frac{1}{2})$	1.6×10^{-14}	$(5, 7)$	8.0×10^{-16}
$(\frac{1}{2}, -\frac{1}{2})$	1.6×10^{-14}	$(12, 10)$	1.0×10^{-16}

Table 13. Numerical values of I_1, I_2, I_3 and I_4 of Example 2 for $\alpha = 5,\ \beta = 7$ and $e = f = 1,\ p = q = 2$

t	I_1	I_2	I_3	I_4
0	2.2673×10^{-18}	2.82842712475	-1.9999999999999958	-4.0081×10^{-16}
5	1.8568×10^{-17}	2.82842712475	-1.9999999999999959	-6.4615×10^{-16}
10	8.1641×10^{-16}	2.82842712475	-2.0000000000000033	-3.8436×10^{-13}
15	9.5569×10^{-13}	2.82842712470	-2.0000000000136556	-6.9563×10^{-10}
20	1.1253×10^{-9}	2.82842706744	-2.0000000217095266	1.1056×10^{-6}
25	1.3248×10^{-9}	2.82835965426	-2.0000321853046230	1.6390×10^{-3}

Example 3.

The Kaup–Kuperschmidt equation fifth–order equation is given by (Wazwaz (2009)),

$$u_t + 20\ uu_{xxx} + 50\ u_x u_{xx} - 80\ u^2 u_x - u_{xxxxx} = 0. \tag{48}$$

The first two conservation laws for the Kaup–Kuperschmidt equation are as follows, Wazwaz (2009),

$$I_1 = \int_a^b u\ dx, \qquad I_2 = \int_a^b \left(\frac{1}{3}u^3 - \frac{1}{8}u_x^2\right)\ dx. \tag{49}$$

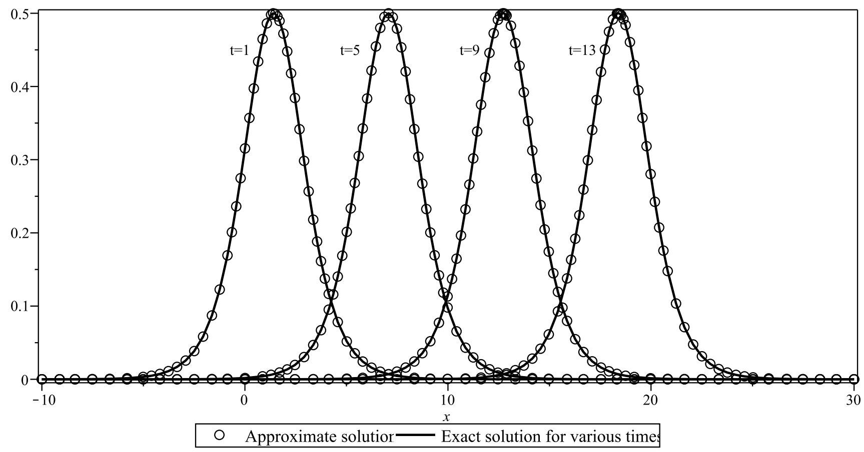


Figure 5. Plot of exact and approximate solutions of Example 2 for $\alpha = \beta = 0$ and various times: $t = 1, 5, 9, 13$

To seek the soliton solution of Equation (48), we introduced the wave variable $\eta = x - Vt$. Then Equation (48) reduce to the following ordinary differential equation,

$$-Vu' + 20\ uu''' + 50\ u'u'' - 80\ u'u^2 - u^{(5)} = 0, \tag{50}$$

where V is arbitrary constant and superscripts indicate the ordinary derivatives with respect to η. The exact solution is obtained by Wazwaz (2009) in the form,

$$u(\eta) = \frac{1}{16}\frac{1 - 10\ e^{\eta} + e^{2\eta}}{1 + 2\ e^{\eta} + e^{2\eta}}, \qquad \eta = x - Vt. \tag{51}$$

We need five conditions for Equation (50) as:

$$u(0) = -\frac{1}{8}, \quad u'(0) = 0, \quad u''(0) = \frac{3}{32}, \quad u'''(0) = 0, \quad u''''(0) = -\frac{3}{16}, \tag{52}$$

which are obtained from Equation (51). The nonlinear Equation (50) with the initial conditions Equation (52) is approximated by using the Jacobi Collocation method introduced in Section 3. For this end, let us consider the following approximations,

$$\begin{aligned}
&u^{(5)}(\eta) \simeq \Phi^T(\eta)\ C,\\
&u''''(\eta) \simeq \Phi^T(\eta)\ M^T\ C + \Phi^T(\eta)\ U_1,\\
&u'''(\eta) \simeq \Phi^T(\eta)\ (M^T)^2\ C + \Phi^T(\eta)\ M^T\ U_1 + \Phi^T(\eta)\ U_2,\\
&u''(\eta) \simeq \Phi^T(\eta)\ (M^T)^3\ C + \Phi^T(\eta)\ (M^T)^2\ U_1 + \Phi^T(\eta)\ M^T\ U_2 + \Phi^T(\eta)\ U_3,\\
&u'(\eta) \simeq \Phi^T(\eta)\ (M^T)^4\ C + \Phi^T(\eta)\ (M^T)^3\ U_1 + \Phi^T(\eta)\ (M^T)^2\ U_2 + \Phi^T(\eta)\ M^T\ U_3\\
&\qquad + \Phi^T(\eta)\ U_4,\\
&u(\eta) \simeq \Phi^T(\eta)\ (M^T)^5\ C + \Phi^T(\eta)\ (M^T)^4\ U_1 + \Phi^T(\eta)\ (M^T)^3\ U_2 + \Phi^T(\eta)\ (M^T)^2\ U_3\\
&\qquad + \Phi^T(\eta)\ M^T\ U_4 + \Phi^T(\eta)\ U_5,
\end{aligned} \tag{53}$$

where $u^{(k)}(0)$ are approximated by $\Phi^T(\eta)\ U_{5-k},\ k = 0, 1, ..., 5$, which U_k are a $(N+1)\times 1$ known vectors. Also, the nonlinear terms in Equation (42) are approximated as follows:

$$u(\eta)u'''(\eta) \simeq \Phi^T(\eta)\ \tilde{V}_2^T V_1, \quad u'(\eta)u''(\eta) \simeq \Phi^T(\eta)\ \tilde{V}_4^T V_3, \quad u(\eta)u^2(\eta) \simeq \Phi^T(\eta)\ \tilde{W}^T V_3, \tag{54}$$

where

$$V_1 = (M^T)^5\ C + (M^T)^4\ U_1 + (M^T)^3\ U_2 + (M^T)^2\ U_3 + M^T\ U_4 + U_5,$$
$$V_2 = (M^T)^2\ C + M^T\ U_1 + U_2,$$
$$V_3 = (M^T)^4\ C + (M^T)^3\ U_1 + (M^T)^2\ U_2 + M^T\ U_3 + U_4,$$
$$V_4 = (M^T)^3\ C + (M^T)^2\ U_1 + M^T\ U_2 + U_3, \qquad W = \tilde{V}_1^T\ V_1,$$

and $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_4$, and $\tilde{W}$ are operational matrices of product and their entries are determined in terms of the components of the vectors V_1, V_2, V_4 and W, respectively. Substituting the approximations (53)–(54) into Equation (50), this leads to the following nonlinear algebraic equation.

$$\Phi^T(\eta)\left\{-V\left((M^T)^4\ C+(M^T)^3\ U_1+(M^T)^2\ U_2+M^T\ U_3+U_4\right)+20\ \tilde{V}_2^T\ V_1+50\ \tilde{V}_4^T V_3-80\ C\right\} \approx 0. \tag{55}$$

Setting $V = -1/16$, $N = 20$ and using the roots of $P_{21}^{(\alpha,\beta)}(\eta)$, Equation (55) is collocated in 21 collocation nodes for different values of parameters α and β. Thus, Equation (55) reduces the problem to solve a system of nonlinear algebraic equations and unknown coefficients are obtained for some values of α and β parameters. Substituting the vector C resulted into Equation (53) leads to an approximate solution for the exact solution (51). Now in order to improve the approximate solution (53) and achieve the soliton solutions and for simplicity we set $e = f = p = q = 1$. So, equating the Taylor expansions of (36) and (53) yields a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, and b_0. Solving the system of algebraic equations with the aide of Maple, the coefficients are obtained. Table 14 shows the maximum absolute errors of the approximate solutions for different values of α and β. Table 15 displays the numerical values of conservation quantities (49) in various times for $\alpha = \beta = 0$, $\alpha = \beta = 1/2$, $\alpha = \beta = 10$, $a = -50$, and $b = 50$. As can be seen from Table 15, the numerical values are almost equal the analytical values of conservation quantities. This shows the proposed scheme is stable.

Table 14. Maximum absolute errors of Example 3 for different values α and β, $e = f = p = q = 1$

(α,β)	Error_{Abs}	(α,β)	Error_{Abs}
$(0,0)$	1.8×10^{-10}	$(1,1)$	1.2×10^{-12}
$(\frac{1}{2},\frac{1}{2})$	1.6×10^{-12}	$(4,4)$	3.0×10^{-13}
$(-\frac{1}{2},\frac{1}{2})$	2.0×10^{-12}	$(6,8)$	1.2×10^{-13}
$(\frac{1}{2},-\frac{1}{2})$	2.0×10^{-12}	$(10,7)$	7.0×10^{-14}
$(1,\frac{1}{2})$	1.4×10^{-12}	$(10,10)$	3.0×10^{-14}

Table 15. Numerical values of I_1 and I_2 of Example 3 for $\alpha = \beta = 0$, $\alpha = \beta = 1/2$, $\alpha = \beta = 10$, and $e = f = p = q = 1$

	$(\alpha,\beta)=(0,0)$		$(\alpha,\beta)=(\frac{1}{2},\frac{1}{2})$		$(\alpha,\beta)=(10,10)$	
t	I_1	I_2	I_1	I_2	I_1	I_2
0	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.00403623922
5	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.00403621428
10	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.00403621065
15	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.004036239218
20	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.004036235813
25	5.50000970	0.0040364929	5.50000774	0.0040364861	5.500000257	0.004036239218

5. Conclusion

In this paper, a computational method based on the Collocation method was presented for solving some of nonlinear partial differential equations in terms of one variable Jacobi polynomials by converting them to a nonlinear system of algebraic equations and then improving the approximate solutions obtained by introducing the rational exponential approximation. This study shows that the proposed method is efficient and practically well suited for use in finding approximate solutions for the KdV, Boussinesq, and Kaup–Kuperschmidt equations. The results indicate the proposed approach can be regarded as simple and applicable approach to numerically solve these type of nonlinear equations. It is predicted that the Jacobi collocation method will be a powerful tools for investigating approximate solutions and even analytic to wide classes of nonlinear wave equations. For numerical purposes the computer programs have been written in Maple 13.

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