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Solution of a Cauchy singular fractional integro-differential equation in Bernstein polynomial basis

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Abstract

This article proposes a simple method to obtain approximate numerical solution of a singular fractional order integro-differential equation with Cauchy kernel by using Bernstein polynomials as basis. The fractional derivative is described in Caputo sense. The properties of Bernstein polynomials are used to reduce the fractional order integro-differential equation to the solution of algebraic equations. The numerical results obtained by the present method compares favorably with those obtained earlier for the first order integro-differential equation. Also the convergence of the method is established rigorously.

Keywords: Fractional integro-differential equation; singular fractional; Cauchy kernel; Bernstein polynomials; Cauchy kernel; algebraic equations

MSC 2010 No.: 45J99, 26A33

1. Introduction

Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus of integral and derivative operators in much the same way as fractional exponents which are outgrowth of exponents with integer values. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the twentieth century, mainly due to its demonstrated applications in the numerous seemingly diverse and widespread fields of science and engineering. Fractional calculus is the focus of many studies due to its frequent appearance in the theories of integral, differential and integro-differential equations, and special functions of mathematical physics as well as their generalization in one or more variables. Some of the areas of present day applications of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport, probability and statistics, control theory of dynamical systems, viscoelasticity, chemical physics, optics and so on (cf. Podlubny (1999), Kilbas et al. (2006), Das (2008), Parthiban and Balachandran (2013), Belarbi et al. (2014), Kumar et al. (2014), Singh et al. (2015), Misra et al. (2015, 2016)). Recently, different numerical methods have been proposed in the literature to solve fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs) (cf. Sweilam et al. (2007), Sweilam and Khader (2010), Al-Bar (2015)) and others.

Bernstein polynomials have been used to solve some linear as well as nonlinear differential equations approximately by Bhatti and Bracken (2007). These polynomials defined on an interval form a complete basis over the interval. The sum of these polynomials is unity, each of them being positive.

In this article, we obtain the numerical solution of the singular fractional order of Caputo type integro-differential equation

$$2\frac{d^\alpha \phi}{dx^\alpha}+\lambda\int_{-1}^{1}\frac{\phi(t)}{t-x}dt=f(x), \qquad -1<x<1, \lambda>0, 0<\alpha\le 1, \tag{1.1}$$

with Cauchy type kernel, specified end conditions $\phi(-1)=0=\phi(1)$ and a special forcing function $f(x)=-\frac{x}{2}$, using Bernstein polynomials.

For $\alpha=1$, Equation (1.1) reduces to a singular integro-differential equation which arises in some special type of mixed boundary value problems involving two dimensional Laplace equation, which was solved earlier by Frankel (1995), Chakrabarti and Hamasapriye (1999), Mandal and Bera (2007), Mandal and Bhattacharya (2008) by using various methods. Also, the forcing function $f(x)=-\frac{x}{2}$, arises in the problems of heat conduction and radiation (cf. Frankel (1995)).

Here, we have introduced a truncated expansion for $\frac{\phi(x)}{(1-x^2)^{\frac{1}{2}}}$ in terms of Bernstein polynomials and used it to reduce the fractional integro-differential equation to a system of linear equations after using collocation points. The coefficients of the truncated expansion are obtained by solving the linear system and the values of the function $\phi(x)$ at various points for different values of α $(0 < \alpha < 1)$ are found. For a value of α near 1 $(\alpha = 0.99)$, the values of $\phi(x)$ at different points are calculated and these are seen to be close to the values of $\phi(x)$ at these points for $\alpha = 1$ obtained by Frankel (1995).

The convergence of the method is established rigorously. Although the numerical computations have been carried out for $f(x) = -\frac{x}{2}$, the method can be utilized for other forms of $f(x)$.

2. Preliminaries

2.1. Basic definitions of fractional integrals and derivative operators

Definition 1.

A function $f(x) \in \mathbb{R}, x > 0$ is said to be in the $\mathbb{C}_\mu$ space, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p g(x)$, where $g(x) \in [0, \infty)$ and it is said to be in the space $\mathbb{C}_\mu^m$ iff $f^{(m)} \in \mathbb{C}_\mu,\ m \in \mathbb{N}$.

Definition 2.

The Riemann-Liouville fractional integral operator $J_a^\alpha$of order α, generalized from the repeated n-fold integration by Gamma function for the factorial expression is defined on $L_1[a, b]$ by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0,\ \ a \leq x \leq b. \tag{2.1}$$

Definition 3.

The expression for Riemann-Liouville fractional derivative operator D_a^α of order α $(n-1 < \alpha \leq n)$ (left-hand definition (LHD)) is given by (cf. Podlubny (1999))

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, \quad n \in \mathbb{N}, \alpha > 0, a \leq x \leq b. \tag{2.2}$$

Definition 4.

The expression for Caputo fractional derivative operator ${}^cD_a^\alpha$ of order α $(n-1 < \alpha \leq n)$ (right hand definition (RHD)) is given by (cf. Caputo and Mainardi (1971))

$${}^cD_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(\tau)}{(x-\tau)^{\alpha+1-n}} d\tau, \quad n \in \mathbb{N}, \alpha > 0, a \leq x \leq b. \tag{2.3}$$

Properties.

Caputo fractional derivative operator is a linear operator similar to integer order differentiation so that

$$D^{\alpha}\big(\lambda\, f(x) + \mu\, g(x)\big) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x), \tag{2.4}$$

where λ and μ are constants. Caputo derivative satisfies

$$D^{\alpha} C = 0,\ \ C \text{ being a constant},$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil\alpha\rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil\alpha\rceil, \end{cases} \tag{2.5}$$

where the ceiling function $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0,1,2,\ldots\}$. For $\alpha \in \mathbb{N}$, Caputo differential operator coincides with the usual differential operator of integer order.

2.2. Bernstein polynomials and their properties

Definition 5.

The Bernstein polynomials of degree n are defined on the interval $[a, b]$ as

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^{i}(b-x)^{n-i}}{(b-a)^{n}}, \quad i = 0,1,\ldots,n, \tag{2.6}$$

where $\binom{n}{i}$ is a binomial coefficient.

The Bernstein basis polynomials of degree n form a basis for the vector space $\prod_n$ of polynomials of degree atmost n. These polynomials defined on an interval form a complete basis over the interval. The sum of these polynomials is unity, each of them being positive.

3. General method of solution

The unknown function $\phi(x)$ satisfying (1.1) with end conditions $\phi(-1) = 0 = \phi(1)$ can be represented in the form

$$\phi(x) = \sqrt{1-x^2}\, \psi(x), \qquad -1 \leq x \leq 1, \tag{3.1}$$

where $\psi(x)$ is a well behaved function of x in the interval $[-1, 1]$.

Let us approximate $\psi(x)$ in terms of Bernstein polynomials in $[-1, 1]$ as

$$\psi(x) = \sum_{i=0}^{n} a_i B_{i,n}(x), \tag{3.2}$$

where $B_{i,n}(x)$, $(i = 0,1,\dots,n)$ are now defined on $[-1,1]$ as

$$B_{i,n}(x) = \binom{n}{i}\frac{(1+x)^i(1-x)^{n-i}}{2^n}, \quad i = 0,1,\dots,n \tag{3.3}$$

and $a_i (i = 0,1,\dots,n)$ are unknown constants. Using (3.1) and (3.2) in(1.1), we find

$$\sum_{i=0}^{n} a_i \left[2\,\frac{d^\alpha}{dx^\alpha}\left\{\sqrt{1-x^2}B_{i,n}(x)\right\} + \lambda\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}B_{i,n}(t)dt\right] = f(x),$$
$$-1 < x < 1, 0 < \alpha \leq 1. \tag{3.4}$$

Substituting Caputo right hand definition of fractional derivative operator (2.3) in (3.4), we obtain

$$\sum_{i=0}^{n} a_i \left[\frac{2}{\Gamma(p-\alpha)}\int_{-1}^{x}\frac{\left\{\sqrt{1-t^2}B_{i,n}(t)\right\}^{(p)}}{(x-t)^{\alpha+1-p}}dt + \lambda\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}B_{i,n}(t)dt\right] = f(x),$$
$$-1 < x < 1,\ \ 0 < \alpha \leq 1,\ \ p-1 < \alpha < p, \tag{3.5}$$

where $\alpha > 0$ is the order of the derivative and $p \in \mathbb{N}$ is the smallest integer greater than α.

As here $0 < \alpha \leq 1$, we can assume that $p = 1$. Hence, we get

$$\sum_{i=0}^{n} a_i \left[\frac{2}{\Gamma(1-\alpha)}\int_{-1}^{x}\frac{\left\{\sqrt{1-t^2}B'_{i,n}(t) - \frac{t}{\sqrt{1-t^2}}B_{i,n}(t)\right\}}{(x-t)^\alpha}dt + \lambda\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}B_{i,n}(t)dt\right] = f(x),$$
$$-1 < x < 1,\ \ 0 < \alpha \leq 1. \tag{3.6}$$

We know that

$$\sqrt{1-t^2} = -\sum_{m=0}^{\infty}\frac{1}{2m-1}\frac{1}{4^m}\binom{2m}{m}t^{2m},$$

$$\frac{1}{\sqrt{1-t^2}} = \sum_{m=0}^{\infty}\frac{1}{4^m}\binom{2m}{m}t^{2m},$$

$$B_{i,n}(t)=\frac{1}{2^n}\sum_{p=0}^{i}\sum_{q=0}^{n-i}(-1)^q\binom{n}{i}\binom{i}{p}\binom{n-i}{q}t^{p+q},$$

$$\frac{d}{dt}\{B_{i,n}(t)\}=n\{B_{i-1,n-1}(t)-B_{i,n-1}(t)\}$$

$$=\frac{n}{2^{n-1}}\left[\begin{array}{c}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i}(-1)^q\binom{n-1}{i-1}\binom{i-1}{p}\binom{n-i}{q}t^{p+q}\\ -\sum_{p=0}^{i}\sum_{q=0}^{n-i-1}(-1)^q\binom{n-1}{i}\binom{i}{p}\binom{n-i-1}{q}t^{p+q}\end{array}\right]. \tag{3.7}$$

Utilizing (3.7) we find that

$$-\frac{t}{\sqrt{1-t^2}}B_{i,n}(t)=\sum_{m=0}^{\infty}\sum_{p=0}^{i}\sum_{q=0}^{n-i}\boldsymbol{\mathcal{J}_{i,n}}t^{2m+p+q+1},$$

$$\sqrt{1-t^2}B'_{i,n}(t)=-\sum_{m=0}^{\infty}\sum_{p=0}^{i-1}\sum_{q=0}^{n-i}\boldsymbol{\mathcal{K}_{i,n}}t^{2m+p+q}+\sum_{m=0}^{\infty}\sum_{p=0}^{i}\sum_{q=0}^{n-i-1}\boldsymbol{\mathcal{L}_{i,n}}t^{2m+p+q},$$

and

$$\sqrt{1-t^2}B_{i,n}(t)=\sum_{m=0}^{\infty}\sum_{p=0}^{i}\sum_{q=0}^{n-i}\boldsymbol{\mathcal{M}_{i,n}}t^{2m+p+q}, \tag{3.8}$$

where

$$\boldsymbol{\mathcal{J}_{i,n}}=(-1)^{q+1}\frac{1}{2^{2m+n}}\binom{2m}{m}\binom{n}{i}\binom{i}{p}\binom{n-i}{q},$$

$$\boldsymbol{\mathcal{K}_{i,n}}=(-1)^{q}\frac{n}{2^{2m+n-1}(2m-1)}\binom{2m}{m}\binom{n-1}{i-1}\binom{i-1}{p}\binom{n-i}{q},$$

$$\boldsymbol{\mathcal{L}_{i,n}}=(-1)^{q+1}\frac{1}{2^{2m+n-1}(2m-1)}\binom{2m}{m}\binom{n-1}{i}\binom{i}{p}\binom{n-i-1}{q},$$

$$\boldsymbol{\mathcal{M}_{i,n}}=(-1)^{q+1}\frac{1}{2^{2m+n}(2m-1)}\binom{2m}{m}\binom{n}{i}\binom{i}{p}\binom{n-i}{q}. \tag{3.9}$$

Using (3.8) in (3.6) we obtain

$$\sum_{i=0}^{n}\sum_{m=0}^{\infty} a_i \left[\frac{2}{\Gamma(1-\alpha)}\left\{\sum_{p=0}^{i}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{J}_{i,n}} \int_{-1}^{x} \frac{t^{2m+p+q+1}}{(x-t)^{\alpha}}dt - \sum_{p=0}^{i-1}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{K}_{i,n}} \int_{-1}^{x} \frac{t^{2m+p+q}}{(x-t)^{\alpha}}dt + \sum_{p=0}^{i}\sum_{q=0}^{n-i-1} \boldsymbol{\mathcal{L}_{i,n}} \int_{-1}^{x} \frac{t^{2m+p+q}}{(x-t)^{\alpha}}dt\right\} + \lambda \sum_{p=0}^{i}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{M}_{i,n}} \int_{-1}^{1} \frac{t^{2m+p+q}}{t-x}dt\right] = f(x),$$
$$-1 < x < 1,\ \ 0 < \alpha \le 1. \qquad (3.10)$$

Using the results (cf. Gradshteyn and Ryzhik (1963))

$$\int_{-1}^{1} \frac{t^k}{t-x}dt = x^k\, ln\left|\frac{1-x}{1+x}\right| + \sum_{l=1}^{k}\binom{k}{l} x^{k-l}\, \frac{(1-x)^l-(-1)^l(1+x)^l}{l} \qquad (3.11)$$

and

$$\int_{-1}^{u} t^{m-1}\,(u-t)^{n-1}dt = \frac{(-1)^{m-1}(x+1)^n}{n}\, {}_2F_1(1,-m+1;1+n;x+1) \qquad (3.12)$$

in (3.10) and choosing $\lambda = 1$ we get

$$\sum_{i=0}^{n}\sum_{m=0}^{\infty} a_i \left[\frac{2}{\Gamma(1-\alpha)}\left\{\sum_{p=0}^{i}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{J}_{i,n}} \frac{(-1)^{r+1}}{1-\alpha}(x+1)^{1-\alpha}\, {}_2F_1(1,-r-1;2-\alpha;x+1) - \sum_{p=0}^{i-1}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{K}_{i,n}} \frac{(-1)^{r}}{1-\alpha}(x+1)^{1-\alpha}\, {}_2F_1(1,-r;2-\alpha;x+1) + \sum_{p=0}^{i}\sum_{q=0}^{n-i-1} \boldsymbol{\mathcal{L}_{i,n}} \frac{(-1)^{r}}{1-\alpha}(x+1)^{1-\alpha}\, {}_2F_1(1,-r;2-\alpha;x+1)\right\} + \sum_{p=0}^{i}\sum_{q=0}^{n-i} \boldsymbol{\mathcal{M}_{i,n}}\, x^r ln\left|\frac{1-x}{1+x}\right| + \sum_{p=0}^{i}\sum_{q=0}^{n-i}\sum_{l=1}^{r} \boldsymbol{\mathcal{M}_{i,n}} \binom{r}{l} x^{r-l}\, \frac{(1-x)^l-(-1)^l(1+x)^l}{l}\right] = f(x),$$
$$-1 < x < 1,\ \ 0 < \alpha \le 1, \qquad (3.13)$$

where $r = 2m + p + q$.

Taking $f(x) = -\frac{x}{2}$ and putting $x = x_j = -0.99 + \frac{2}{n}j\ (j = 0,1,\dots,n)$ in (3.13) we obtain a system of $(n+1)$ linear equations for the determination of the unknown coefficients $a_j(j = 0,1,\dots,n)$. The unknown coefficients $a_j(j = 0,1,\dots,n)$ are thus obtained for different values of $\alpha\ (0 < \alpha < 1)$. The numerical values of $\phi(x)$ for different values of x are then obtained approximately.

In the numerical calculations, we take $n = 7,11,13$ and $a_0, a_1, \dots, a_n$ are obtained by standard numerical methods. Using these coefficients the values of $\phi(x)$ at $x = (0.2)k, k = -5, -4, \dots, 0, \dots, 4, 5$ are obtained and presented in the Figures 1, 2, 3 for different values of α. Figure 1 to Figure 3 depict the solution for different values of $\alpha\ (0 < \alpha < 1)$. The values of $\phi(x)$ at different points for $\alpha = 1$ are known approximately (cf. Frankel (1995)). Our approximate values of $\phi(x)$ for $\alpha = 0.99$ are compared with the known values of $\phi(x)$ for $\alpha = 1$ given by Frankel (1995), and Figure 4 shows that the values obtained by the presented method (for $\alpha = 0.99$) compare favorably with the values obtained by Frankel (1995) (for $\alpha = 1$).

4. Error analysis

Using (3.1) in (1.1) an equation for $\psi(x)$ is obtained which can be written in the operator form as

$$\left(\boldsymbol{D}^{\alpha} + \frac{\lambda\,\pi}{2}\boldsymbol{C}\right)\psi(x) = \frac{f(x)}{2}, \quad -1 < x < 1, \tag{4.1}$$

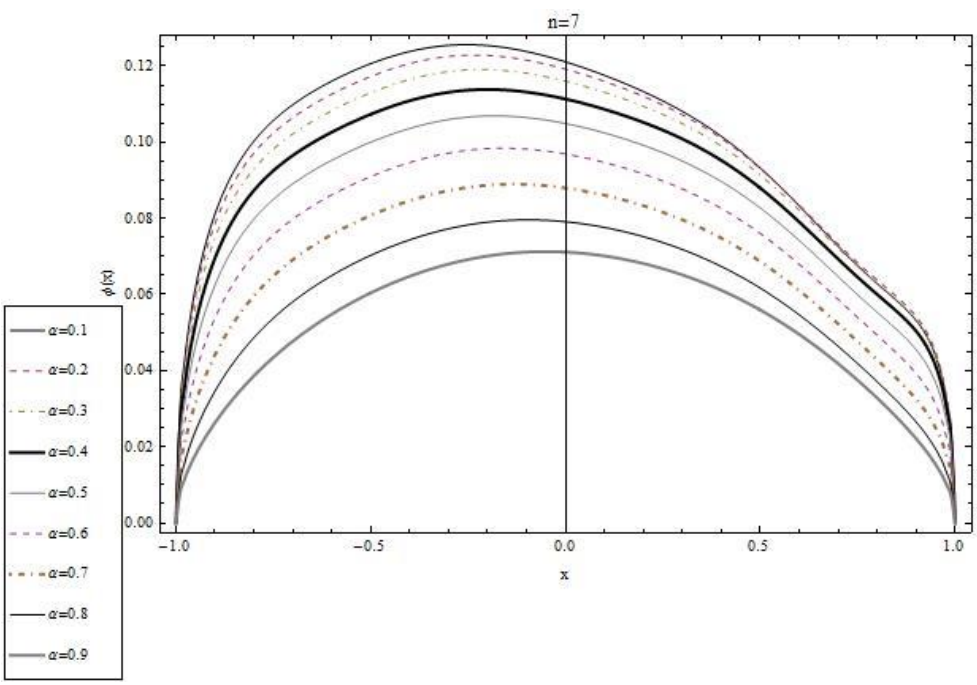

Figure 1. $\phi(x)$ for different values of α where $n = 7$

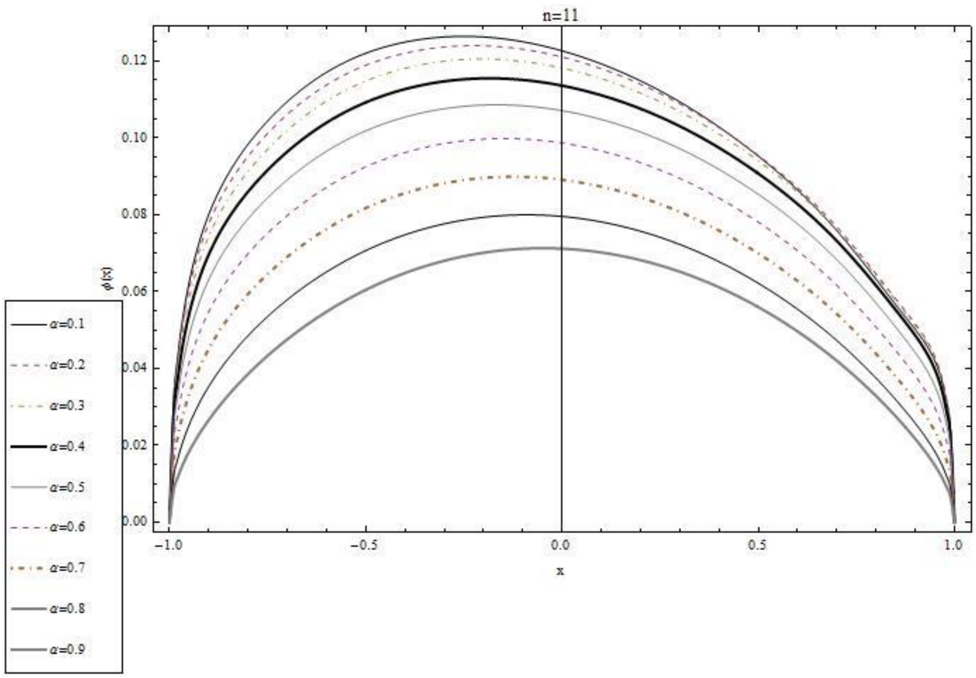

Figure 2. $\phi(x)$ for different values of α where $n = 11$

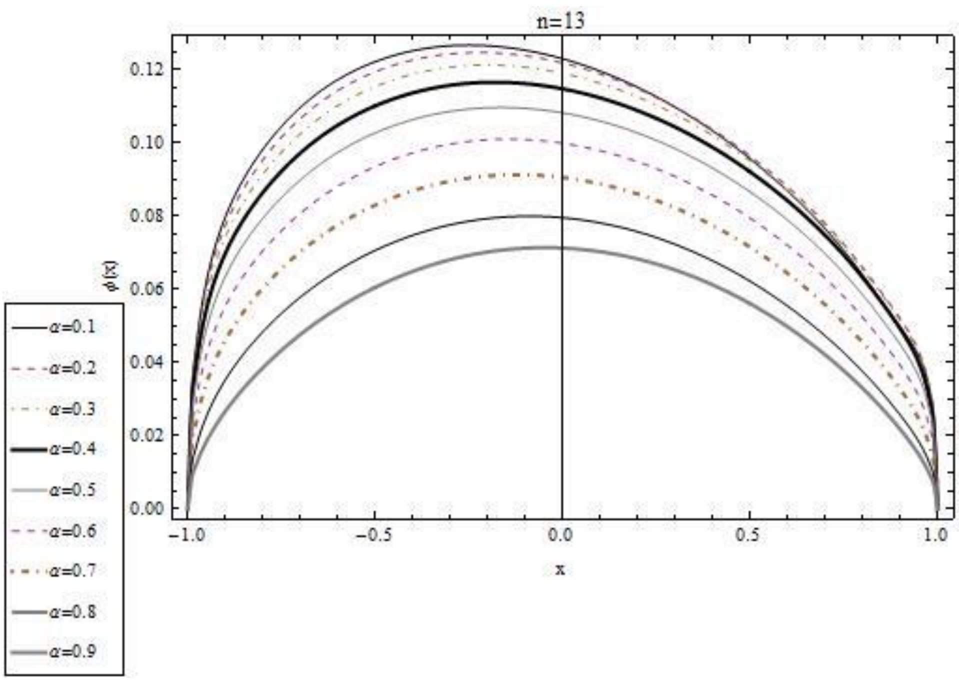

Figure 3. $\phi(x)$ for different values of α where $n = 13$

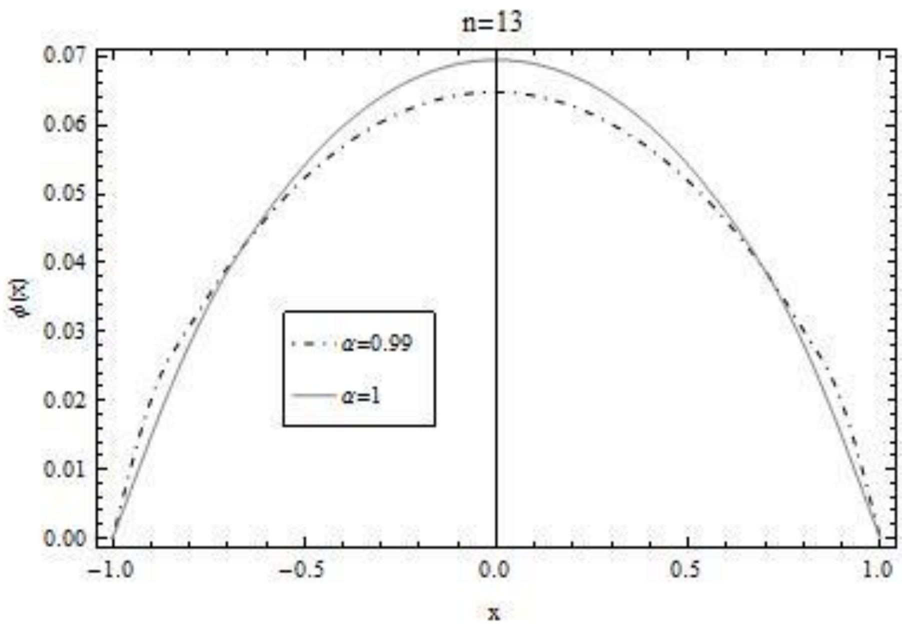

Figure 4. Comparison of presented method with Frankel's Method where $n = 13$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are operators defined as

$$\boldsymbol{C}u(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}u(t)dt, \quad -1 < x < 1 \tag{4.2}$$

and

$$\boldsymbol{D}^{\alpha}u(x) = \frac{d^{\alpha}}{dx^{\alpha}}\left[\sqrt{1-x^2}u(x)\right], \quad -1 < x < 1. \tag{4.3}$$

Letting $x = \cos\theta$, we get the Chebyshev polynomials of first the kind as $T_n(x) = \cos n\theta$ and Chebyshev polynomial of the second kind as $U_n(x) = \frac{\sin\,(n+1)\theta}{\sin\theta}$. Then,

$$\boldsymbol{C}U_n(x) = -T_{n+1}(x), n \geq 0. \tag{4.4}$$

Thus, (4.4) assures that $\boldsymbol{C}$ can be extended as a bounded linear operator from $L_1(\omega)$ to $L(\omega)$(cf. Goldberg and Chen (1997)), where $L_1(\omega)$ is the space of functions square integrable with respect to $\omega(x) = \sqrt{1-x^2}$ in $[-1,1]$ and $L(\omega)$ is the subspace of functions $f \in L(\omega)$ satisfying

$$\|f\|_1^2 = \sum_{k=0}^{\infty}(k+1)^2\,\langle f, \psi_k\rangle_{\omega}^2 < \infty, \tag{4.5}$$

where

$$\langle f, g\rangle_{\omega} = \int_{-1}^{1} f(t)g(t)\sqrt{1-t^2}dt \tag{4.6}$$

and

$$\psi_k = \sqrt{\frac{2}{\pi}}T_k. \tag{4.7}$$

Again,

$$\boldsymbol{D}^\alpha U_n(x) = \frac{1}{\Gamma(p-\alpha)}\int_{-1}^{x}\frac{\left\{\sqrt{1-t^2}U_n(t)\right\}^{(p)}}{(x-t)^{\alpha+1-p}}dt\,,\;\; p-1<x<p$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_{-1}^{x}\frac{\left\{\sqrt{1-t^2}U_n'(t)-\frac{t}{\sqrt{1-t^2}}U_n(t)\right\}}{(x-t)^{\alpha}}dt\,,\text{(since } 0<\alpha<1 \text{ gives } p=1)$$

$$= -\frac{(n+1)}{\Gamma(1-\alpha)}\int_{-1}^{x}\frac{T_{n+1}(t)}{\sqrt{1-t^2}(x-t)^{\alpha}}dt, \left(\text{using } U_n'(t)=\frac{(n+1)T_{n+1}(t)-tU_n(t)}{t^2-1}\right). \tag{4.8}$$

Let us represent Chebyshev polynomial of the first kind by

$$T_n(x) = \textstyle\sum_{j=0}^{n} b_j x^j, \tag{4.9}$$

where b_j's can easily be found. Then,

$$\boldsymbol{D}^\alpha U_n(x) = -\frac{(n+1)}{\Gamma(1-\alpha)}\sum_{j=0}^{n+1} b_j\left[\frac{\sqrt{\pi}}{2x^\alpha}\left\{\frac{(-2)^j}{j\,\Gamma\left(\frac{j}{2}\right)}\,\Gamma\left(\frac{1+j}{2}\right){}_3F_2\left(\frac{1+j}{2},\frac{1+\alpha}{2},\frac{\alpha}{2};\frac{1}{2},\frac{2+j}{2};\frac{1}{x^2}\right)\right.\right.$$
$$-\frac{(-1)^j}{x\Gamma\left(\frac{3+j}{2}\right)}\alpha\Gamma\left(\frac{2+j}{2}\right){}_3F_2\left(\frac{2+j}{2},\frac{1+\alpha}{2},1+\frac{\alpha}{2};\frac{3}{2},\frac{3+j}{2};\frac{1}{x^2}\right)$$
$$+\,2^{\alpha-j}x^{j+1}\Gamma(1+j)\Gamma(1$$
$$\left.\left.-\,\alpha)\,{}_3F_2\left(\frac{1}{2},\frac{1+j}{2},\frac{2+j}{2};\frac{2+j-\alpha}{2},\frac{3+j-\alpha}{2};x^2\right)\right\}\right], \tag{4.10}$$

which shows that the operator $\boldsymbol{D}^\alpha$ can also be extended as a bounded linear operator from $L_1(\omega)$ to $L(\omega)$. By choosing $f(x)\in L(\omega)$, we obtain that Equation (4.1) has a unique solution $\psi\in L(\omega)$ for each $f\in L(\omega)$.

Now, an approximation $d_n(x)$ of the function $\psi(x)$ in terms of Bernstein polynomials in the form

$$\psi(x) \simeq d_n(x), \tag{4.11}$$

where

$$d_n(x) = \sum_{i=0}^{n} p_i B_{i,n}(x) = \sum_{i=0}^{n} q_j U_j(x), \tag{4.12}$$

where $q_j, (j = 0,1, \dots, n)$ can be expressed in terms of p_i $(i = 0,1, \dots, n)$ and vice-versa by the transformation (cf. Snyder (1966))

$$B_{i,n}(x) = \binom{n}{i} \frac{1}{2^n} \sum_{s=0}^{n} r_{i,n}^{s} \frac{1}{2^s} \left\{ \binom{s}{m} - \binom{s}{m+1} \right\} U_{s-2m}(x), \tag{4.13}$$

where

$$r_{i,n}^{s} = \sum_{k} (-1)^{s-k} \binom{i}{k} \binom{n-i}{s-k}, \;\; k = 0,1, \dots, n, \;\; i = 0,1, \dots, n.$$

the summation over k being taken as follows:

for $i < n < n - i$,

$$(i) k = 0 \text{ to } s \qquad \text{for } s \leq i,$$

$$(ii) k = 0 \text{ to } i \qquad \text{for } i < s \leq n - i,$$

$$(iii) k = s - (n - i) \text{ to } n - i \qquad \text{for } n - i \leq n,$$

while for $i = n - i$ (n being an even integer)

$$(i) k = 0 \text{ to } s \qquad \text{for } s \leq i,$$

$$(ii) k = s - i \text{ to } i \qquad \text{for } i < s \leq n. \tag{4.14}$$

For $i > n - i$, i and $n - i$ above are to be interchanged.

Now, let us denote

$$u_n(x) = \sum_{j=0}^{n} c_j \phi_j(x) \tag{4.15}$$

with

$$c_j = \sqrt{\frac{\pi}{2}} q_j, \;\; \phi_j(x) = \sqrt{\frac{2}{\pi}} U_j(x). \tag{4.16}$$

The functions $\phi_j(x), (j = 0,1, \dots, n)$ form a set of orthonormal polynomial basis in $[-1,1]$ with respect to the weight function $\omega(x) = \sqrt{1-x^2}$. Also we have, if $f \in C^r[-1,1]$, then $u_n \to \psi$ as $n \to \infty$ in $L_1(\omega)$ and

$$||\psi - u_n||_1 < An^{-r}, \big(\text{cf. Goldberg and Chen (1997)}\big), \tag{4.17}$$

where A is a constant.

In our problem, $f(x) = -\frac{x}{2} \in C^\infty[-1,1]$. Since the method converges faster for large r, in our method it converges very rapidly.

5. Conclusion

In this paper, we proposed a simple method to approximate the unknown function in terms of truncated series involving Bernstein polynomials for solving a fractional order integro differential equation with Cauchy singular type kernel, the fractional derivative being in the Caputo sense. An approximate formula for the Caputo derivative in Bernstein polynomial basis was derived. The properties of Bernstein polynomials were utilized to reduce the fractional order integro differential equations to the solution of algebraic equations by avoiding the appearance of ill-conditioned matrices or complicated integrations. The convergence is very good as is also seen in the figures.

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