A Mathematical Model for Micropolar Fluid Flow Through an Artery with the effect of Stenosis and Post Stenotic Dilatation

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 11, Issue 2 (December 2016), pp. 680 - 692

A mathematical model for micropolar fluid flow through an artery with the effect of stenosis and post stenotic dilatation

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Received: October 8, 2015; Accepted: August 6, 2016

Abstract

The effects of both stenosis and post stenotic dilatation have been studied on steady flow of micropolar fluid through an artery. Assuming the stenosis to be mild, the equations governing the flow of the proposed model are solved. Closed form expressions for the flow characteristics such as velocity, pressure drop, and volumetric flow rate, resistance to the flow and wall shear stress are derived. The effects of various parameters on resistance to the flow and wall shear stress have been analyzed through the graphs. It is found that the resistance to the flow increases with the height and length of the stenosis, but the resistance to the flow decreases with stenotic dilatation. With the increase of the coupling number the resistance to the flow increases. However, the effect of coupling number is not very significant. The resistance to the flow decreases with the micropolar fluid parameter. The wall shear stress increases with coupling number and stenosis height, but it decreases with micropolar fluid parameter and stenotic dilatation.

Keywords: Micropolar fluid; flow characteristics; stenosis and post stenotic dilatation; resistance to the flow; coupling number

MSC 2010 No.: 76A05

1. Introduction

One of the leading causes of the deaths in the world is due to heart diseases and the most commonly heard name among them is atherosclerosis or stenosis. It is the abnormal and unnatural growth in the arterial wall thickness that develops at various locations of the cardiovascular system under certain conditions. It may result in serious consequences such as cerebral strokes, myocardial infarction leading to heart failure, etc. Therefore, the study of blood flow characteristics in such blood vessels is quite important. Several researchers have conducted investigations to understand the characteristics of blood flow through a stenosed artery by treating blood as Newtonian (Young 1968; Morgan and Young 1974; Lee and Fung 1970). The Newtonian behaviour of blood may be true in larger arteries, but blood being a suspension of cells behaves like a non-Newtonian fluid at low shear rates in small arteries (Huckaba and Hahn 1968; Sapna and Shah 2011).

Micropolar fluid is a special case of non-Newtonian fluid. The micropolar fluid model was introduced by Eringen (1966), which represented fluid consisting of rigid and randomly oriented particles suspended in viscous medium where the deformation of the particles is ignored. Ariman (1974) examined the blood flow in a rigid circular tube and concluded that the micropolar fluid model is a better model because it accounts for the microrotation of blood suspensions. Abdullah and Amin (2010) studied a nonlinear micropolar fluid model for blood flow in a tapered artery with single stenosis. Mekhemier and El kot (2007) considered blood as micropolar fluid and discussed the effects of the asymmetry nature of stenosis in their steady flow analysis. Prasad et.al. (2012) studied the effect of multiple stenosis on couple stress fluid through a tube with nonuniform cross-section. kumar and Diwakar (2013) investigated the effect of post stenotic dilatation and multiple stenosis through an artery by treating blood as Bingham plastic fluid.

Motivated by these studies, an attempt is made in the present paper to analyze the flow of micropolar fluid through an artery with stenosis and post dilatation.

2. Mathematical Formulation

Consider the steady flow of an incompressible micropolar fluid of constant viscosity μ, and density ρ, in a uniform tube of length L containing multiple abnormal segments as shown in Figure 1.

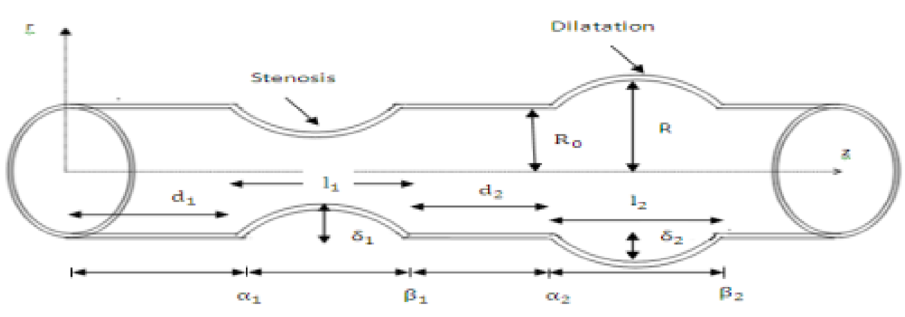

Figure 1. Geometry of arterial segment under consideration

The equations describing the geometry of the wall as shown in Figure 1 are

$$h = \frac{R(z)}{R_0} = \begin{cases} 1 - \frac{\delta_i}{2R_0}\left[1 + Cos\,\frac{2\pi}{l_i}\left(z - \alpha_i - \frac{l_i}{2}\right)\right], & \text{for } \alpha_i \le z \le \beta_i, \\ 1, & \text{otherwise,} \end{cases} \tag{1}$$

where δ_i $(i = 1,2)$ represents the maximum distance of the i^{th} abnormal segment, R represents the radius of the artery, R_0 represents the radius of the normal artery, l_i represents the length of the i^{th} abnormal segment, α_i represents the distance from the origin to the start of the i^{th} abnormal segment and is given by

$$\alpha_i = \left(\sum_{j=1}^{i}\left(d_j + l_j\right)\right) - l_i\,, \tag{2}$$

and β_i represents the distance from the origin to the end of the i^{th} abnormal segment

$$\beta_i = \left(\sum_{j=1}^{i}\left(d_j + l_j\right)\right), \tag{3}$$

where d_i represents the distance separating the start of the i^{th} abnormal segment from the end of the $(i-1)^{\text{th}}$, or from the start of the segment if $i = 1$, (where $i = 1,2$).

The governing equations for the steady flow of an incompressible micropolar fluid in the absence of body force and body couple are

$$\nabla . U = 0, \tag{4}$$

$$\rho(U.\nabla U) = -\nabla p + k\nabla \times U + (\mu + k)\nabla^2 U, \tag{5}$$

$$\rho j(U.\nabla V) = -2kV + k\nabla \times U - \gamma(\nabla \times \nabla \times V) + (\alpha + \beta + \gamma)\nabla(\nabla . V), \tag{6}$$

where p is the pressure, U is the velocity vector, V is the micro rotation vector, j is the microgyration parameter. $\mu, k, \alpha, \beta, \gamma$ are the material constants and satisfy the following inequalities (Eringen, 1966).

$$2\mu + k \ge 0, k \ge 0, 3\alpha + \beta + \gamma \ge 0, \gamma \ge |\beta|.$$

Since the flow is axisymmetric, all the variables are independent of θ. Hence, for this flow the velocity $U = (u_r, 0, u_z)$ and the microrotation vector $V = (0, v_\theta, 0)$ .Thus, the governing equations can be written as (where u_r, u_z are the velocities in *r* and *z* directions)

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0, \tag{7}$$

$$\rho\left(u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + (\mu + k)\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right) + \frac{k}{r}\frac{\partial(r v_\theta)}{\partial r}, \tag{8}$$

$$\rho\left(u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + (\mu + k)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) - k\frac{\partial v_\theta}{\partial z}\,, \tag{9}$$

$$\rho j\left(u_r\frac{\partial v_\theta}{\partial r} + u_z\frac{\partial v_\theta}{\partial z}\right) = -2kv_\theta - k\left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\right) + \gamma\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right). \tag{10}$$

$u_r = \frac{k}{r}$ and $u_z = u_z(r)$ satisfies the Equation (7) and hence 2nd term of RHS in (9) vanishes.

Introducing the following non-dimensional variables

$$\bar{z} = \frac{z}{L}, \bar{\delta} = \frac{\delta}{R_0}, \bar{r} = \frac{r}{R_0}, \bar{P} = \frac{P}{\frac{\mu u_0 L}{R_0^2}}, \bar{u}_z = \frac{u_z}{u_0}, \bar{u}_r = \frac{L u_r}{u_0 \delta}, \bar{v}_\theta = \frac{R_0 v_\theta}{u_0}, \bar{J} = \frac{j}{R_0^2}, \tag{11}$$

in Equations (7) - (10), under the assumption of mild stenosis, the convective terms in the equations can be neglected and the equations reduce as follows

$$\frac{\partial p}{\partial z} = \frac{1}{1-N}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{N}{r}\frac{\partial(r v_\theta)}{\partial r}\right), \tag{12}$$

$$\frac{\partial p}{\partial r} = 0, \tag{13}$$

$$2v_\theta = -\frac{\partial u_z}{\partial r} + \frac{2-N}{m^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right), \tag{14}$$

where $N = \frac{k}{\mu+k}$ is the coupling number $(0 \le N < 1)$ and $m^2 = \frac{R_0^2 k(2\mu+k)}{\gamma(\mu+k)}$ is the micropolar parameter.

The corresponding non-dimensional boundary conditions are

$$\frac{\partial u_z}{\partial r} = 0 \text{ at } r = 0, \tag{15}$$

$$u_z = 0 \text{ at } r = h, \tag{16}$$

$$v_\theta = 0 \text{ at } r = h, \tag{17}$$

$$u_z \text{ is finite at } r = 0, \tag{18}$$

$$v_\theta \text{ is finite at } r = 0. \tag{19}$$

3. Solution of the problem

It is noted that (12) can be written as

$$\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r} + N r v_\theta - (1-N)\frac{r^2}{2}\frac{dp}{dz} = 0. \tag{20}$$

Integrating (20), we get

$$\frac{\partial u_z}{\partial r} = -N v_\theta + (1-N)\frac{r}{2}\frac{dp}{dz} + \frac{c_1}{r}. \tag{21}$$

Substituting (21) in (14), we get

$$\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\left(m^2+\frac{1}{r^2}\right)v_\theta=\frac{m^2(1-N)}{(2-N)}\frac{r}{2}\frac{dp}{dz}+\frac{m^2}{(2-N)}\frac{c_1}{r}, \tag{22}$$

The general solution of Equation (22) is

$$v_\theta=c_2I_1(mr)+c_3K_1(mr)-\frac{(1-N)}{(2-N)}\frac{r}{2}\frac{dp}{dz}-\frac{1}{(2-N)}\frac{c_1}{r}. \tag{23}$$

where $I_1(mr)$ and $K_1(mr)$ are the modified Bessel functions of first and second kind of order one, respectively.

Substituting (23) in (21) and solving for u_z, using the boundary conditions (15) to (19), we get

$$u_z=\frac{1-N}{2-N}\frac{dp}{dz}\left\{\frac{r^2-h^2}{2}+\frac{Nh}{2mI_1(mh)}\left[I_0(mh)-I_0(mr)\right]\right\}. \tag{24}$$

The volumetric flow rate is defined by

$$Q=2\pi\int_0^h u_z r\,dr. \tag{25}$$

Integrating (25),

$$Q=\pi\left(\frac{1-N}{2-N}\right)\frac{dp}{dz}\left\{\frac{-h^4}{4}+\frac{Nh^3I_0(mh)}{2mI_1(mh)}-\frac{Nh^2}{m}\right\}, \tag{26}$$

$$\frac{dp}{dz}=\frac{Q}{\pi\left(\frac{1-N}{2-N}\right)\left\{\frac{-h^4}{4}+\frac{Nh^3I_0(mh)}{2mI_1(mh)}-\frac{Nh^2}{m}\right\}}. \tag{27}$$

When the micropolar parameter $N\to 0$, the fluid becomes Newtonian fluid.

The pressure drop Δp across the stenosis between $z=0$ to $z=1$ is obtained by integrating (27), as

$$\Delta p=\int_0^1\frac{dp}{dz}\,dz=\int_0^1\frac{Q}{\pi\left(\frac{1-N}{2-N}\right)\left\{\frac{-h^4}{4}+\frac{Nh^3I_0(mh)}{2mI_1(mh)}-\frac{Nh^2}{m}\right\}}dz. \tag{28}$$

The resistance to the flow λ is defined by

$$\lambda=\frac{\Delta p}{Q}=\int_0^1\frac{1}{\pi\left(\frac{1-N}{2-N}\right)\left\{\frac{-h^4}{4}+\frac{Nh^3I_0(mh)}{2mI_1(mh)}-\frac{Nh^2}{m}\right\}}\,dz. \tag{29}$$

The pressure drop in the absence of stenosis $(h=1)$ is denoted by Δp_N, is obtained from (27) as

$$\Delta P_N=\int_0^1\frac{Q}{\pi\left(\frac{1-N}{2-N}\right)\left\{\frac{-1}{4}+\frac{NI_0(m)}{2mI_1(m)}-\frac{N}{m}\right\}}\,dz. \tag{30}$$

The resistance to the flow in the absence of stenosis is denoted by $\lambda_{N,}$ is obtained from (24) as

$$\lambda_N=\frac{\Delta P_N}{Q}=\int_0^1\frac{1}{\pi\left(\frac{1-N}{2-N}\right)\left\{\frac{-1}{4}+\frac{NI_0(m)}{2mI_1(m)}-\frac{N}{m}\right\}}dz. \tag{31}$$

The normalized resistance to the flow denoted by

$$\bar{\lambda} = \frac{\lambda}{\lambda_N}\,. \tag{32}$$

The wall shear stress is given by

$$\tau_h = \frac{-1}{(1-N)}\left(\frac{\partial u_z}{\partial r} + N v_\theta\right)\Big|_{r=h}. \tag{33}$$

From (17) $v_\theta = 0 \; at \; r = h$

$$\tau_h = \frac{-1}{(1-N)}\left(\frac{\partial u_z}{\partial r}\right)\Big|_{r=h}. \tag{34}$$

From (15) and (21),

$$\frac{\partial u_z}{\partial r} = \left((1-N)\frac{r}{2}\frac{dp}{dz}\right)\Big|_{r=h}. \tag{35}$$

From (34) and (35),

$$\tau_h = \frac{-h}{2}\frac{dp}{dz}. \tag{36}$$

4. Results and Discussions

Using Mathematica 9.0, computer codes are developed for numerical evaluation of the analytic expressions for impedance ($\bar{\lambda}$) and wall shear stress (τ_h) given by the Equations (32) and (36). The effects of various parameters on flow resistance and wall shear stress have been calculated and shown graphically (Figures 2 - 18).

It is observed that the resistance to the flow increases with the height and length of the stenosis, but it decreases with post stenotic dilatation (Figures 2 - 6).

It is observed from Figures 7 - 9 the resistance to the flow increases with stenosis height and coupling number, but decreases with stenotic dilatation. The resistance to the flow increases with the height of the stenosis and decreases with micropolar fluid parameter (Figures 10 and 11), but it decreases with stenotic dilatation and micropolar fluid parameter (Figure 12).

It can be observed from Figures 13 - 15, the wall shear stress increases with coupling number and stenosis height, but decreases with stenotic dilatation.

It is also observed from the Figures 16 - 18, the wall shear stress increases as the height of stenosis increases and decreases with micropolar fluid parameter, but it decreases with stenotic dilatation and micropolar fluid parameter.

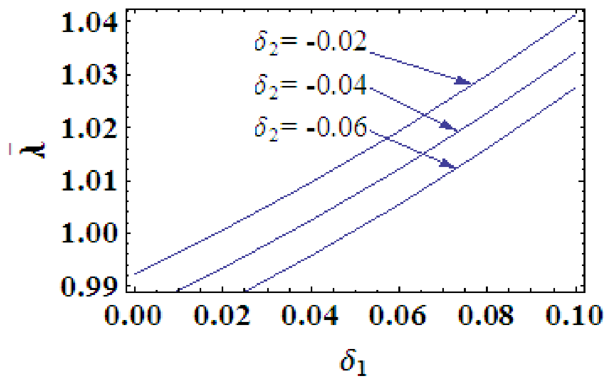

Figure 2. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different δ_2
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, m = 1)$

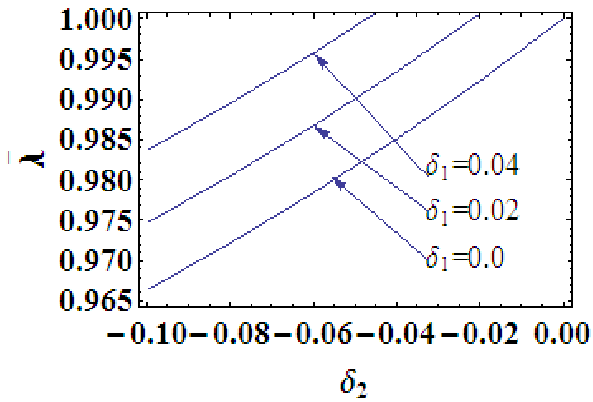

Figure 3. Variation of flow resistance $\bar{\lambda}$ with δ_2 for different δ_1
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, m = 1)$

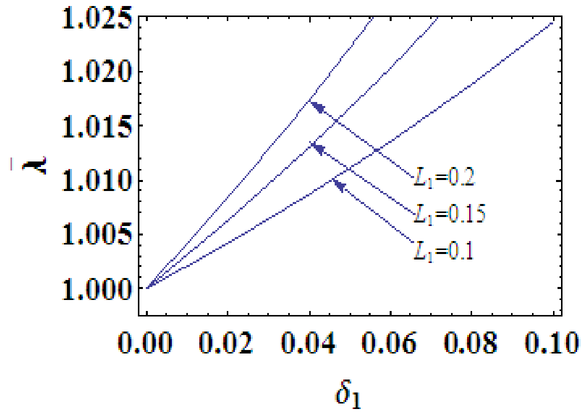

Figure 4. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different L_1
$(d_1 = 0.2, d_2 = 0.2, L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, m = 1, \delta_2 = 0.0)$

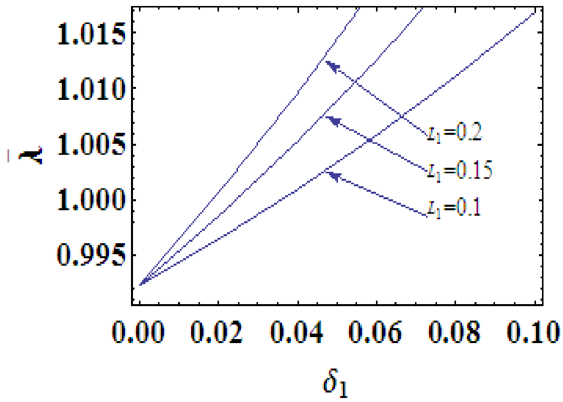

Figure 5. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different L_1
$(d_1 = 0.2, d_2 = 0.2, L_2 = 0.2, L = 1, Q = 0.1, \mathrm{N} = 0.2, m = 1, \delta_2 = -0.02)$

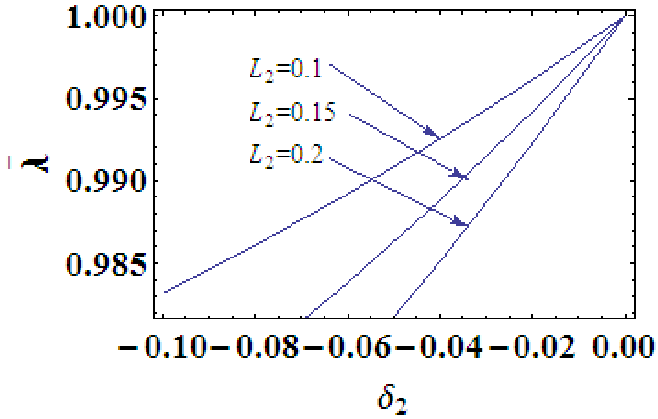

Figure 6. Variation of flow resistance $\bar{\lambda}$ with δ_2 for different L_2
$(d_1 = 0.2, d_2 = 0.2, L_1 = 0.2, L = 1, Q = 0.1, \mathrm{N} = 0.2, m = 1, \delta_1 = 0.0)$

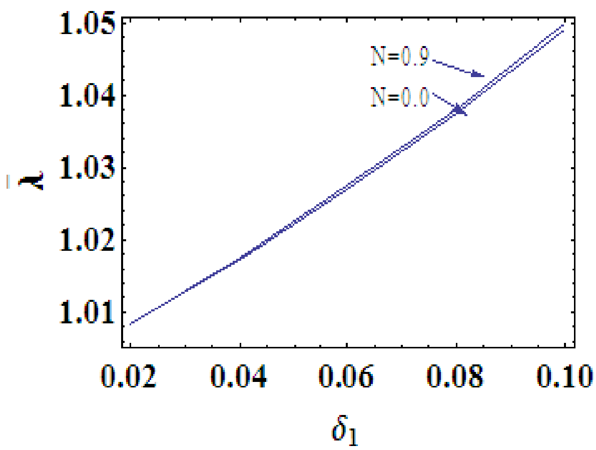

Figure 7. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different N
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_2 = 0.0)$

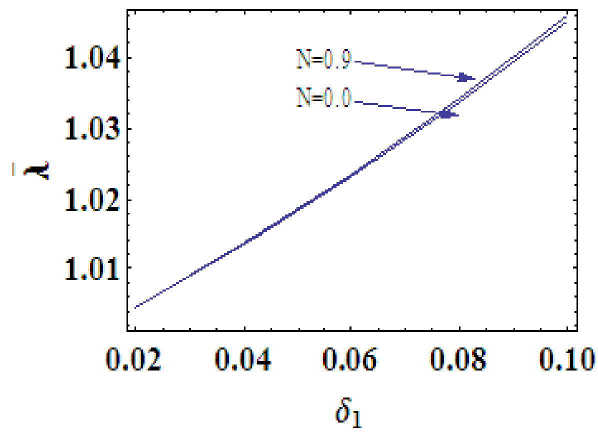

Figure 8. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different N $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_2 = -0.02)$

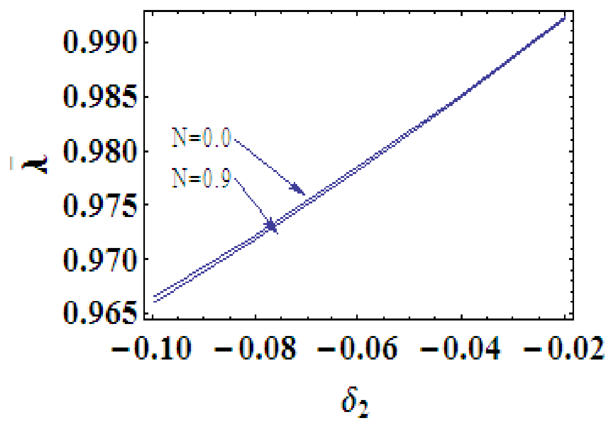

Figure 9. Variation of flow resistance $\bar{\lambda}$ with δ_2 for different N $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_1 = 0.0)$

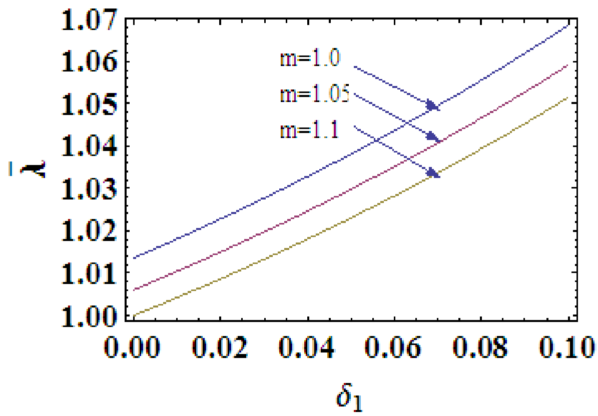

Figure 10. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different m $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_2 = 0.0)$

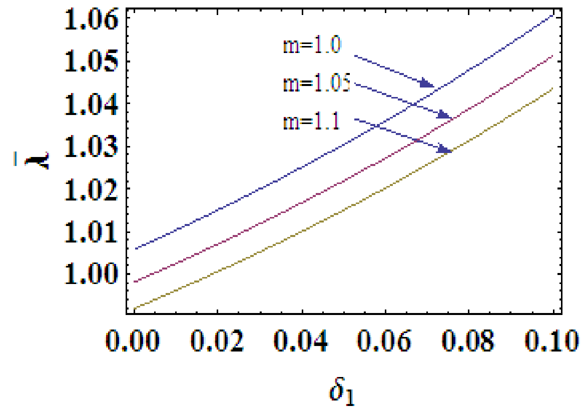

Figure 11. Variation of flow resistance $\bar{\lambda}$ with δ_1 for different m
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_2 = -0.02)$

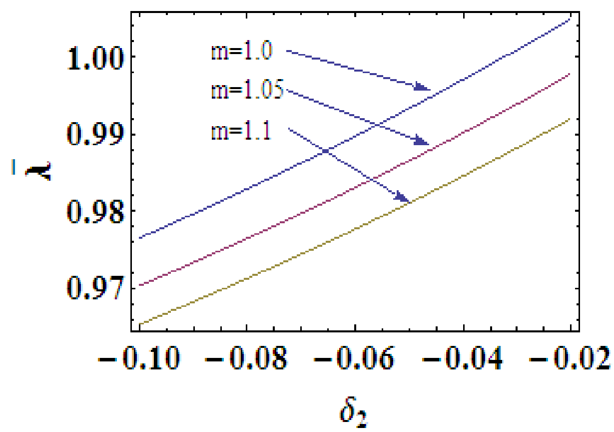

Figure 12. Variation of flow resistance $\bar{\lambda}$ with δ_2 for different m
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_1 = 0.0)$

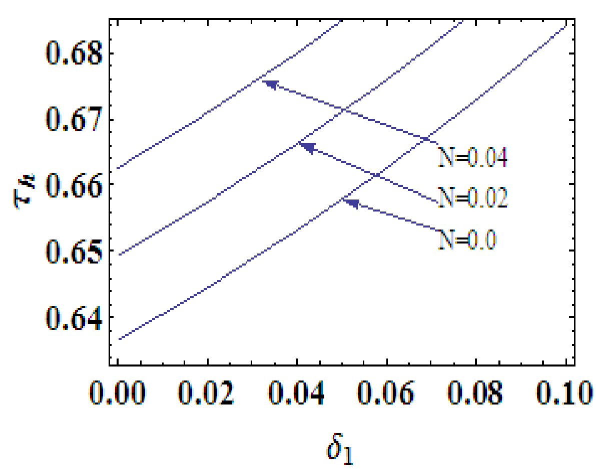

Figure 13. Variation of wall shear stress τ_h with δ_1 for different N
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_2 = 0.0)$

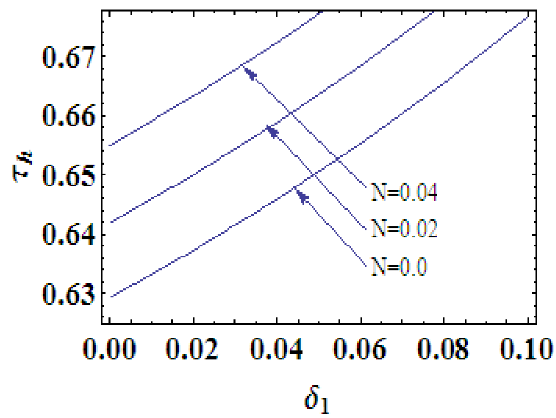

Figure 14. Variation of wall shear stress τ_h with δ_1 for different N
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_2 = -0.02)$

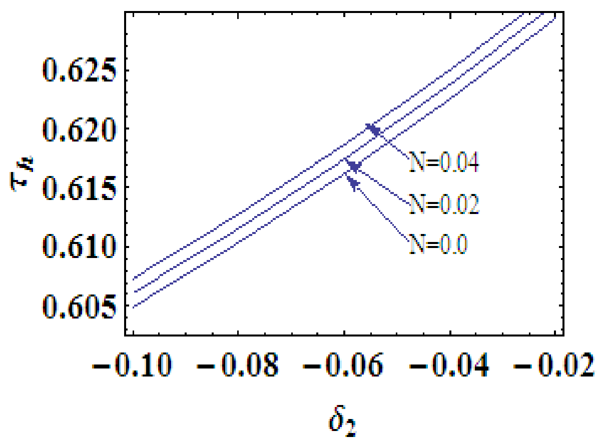

Figure 15. Variation of wall shear stress τ_h with δ_2 for different N
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, m = 1, \delta_1 = 0.0)$

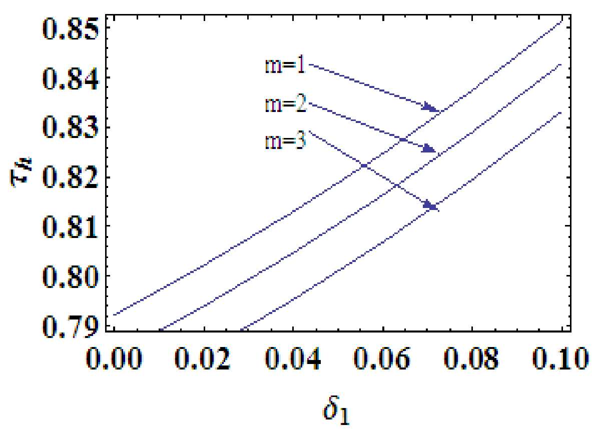

Figure 16. Variation of wall shear stress τ_h with δ_1 for different m
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_2 = 0.0)$

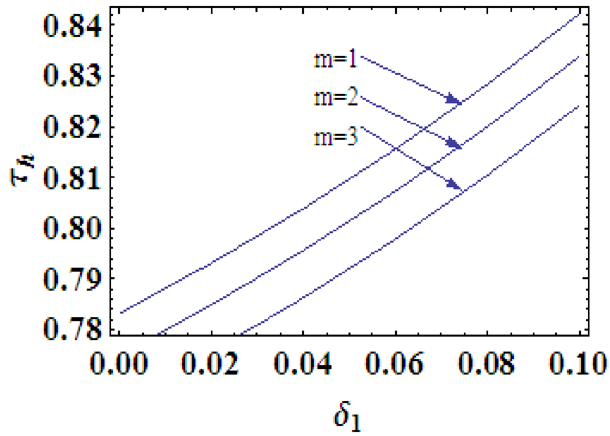

Figure 17. Variation of wall shear stress τ_{h} with δ_1 for different m
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_2 = -0.02)$

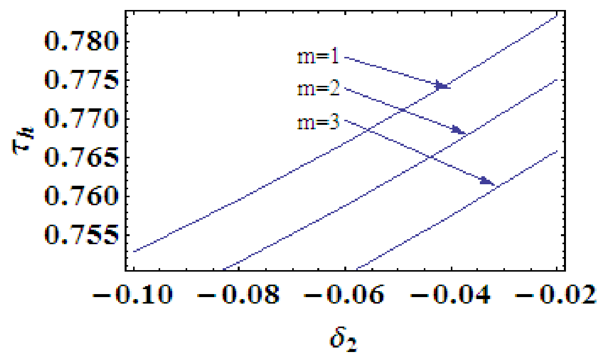

Figure 18. Variation of wall shear stress τ_{h} with δ_2 for different m
$(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, N = 0.2, \delta_1 = 0.0)$

5. Conclusion

A mathematical model for the steady flow of micropolar fluid through a stenosed artery with post stenotic dilatation has been analyzed. Results have been studied for mild stenosis and it has been shown that the resistance to the flow and the wall shear stress increase with the height of the stenosis, coupling number and decreases with the stenotic dilatation. However, the effect of coupling number is not very significant. The same parameters increase with stenosis height and decrease with micropolar fluid parameter and stenotic dilatation.

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