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Shashikant B. Mulay
*University of Tennessee*

John J. Quinn
*University of Tennessee*

Mark A. Shattuck
*University of Tennessee*

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# A Generalized Polynomial Identity Arising from Quantum Mechanics

**[1]Shashikant B. Mulay, [2]John J. Quinn, & [1]Mark A. Shattuck**

[1]Department of Mathematics
University of Tennessee
Knoxville, TN 37996
mulay@math.utk.edu; shattuck@math.utk.edu
&
[2]Department of Physics
University of Tennessee
Knoxville, TN 37996
jjquinn@utk.edu

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**Abstract**

We establish a general identity that expresses a Pfaffian of a certain matrix as a quotient of homogeneous polynomials. This identity arises in the study of weakly interacting many-body systems and its proof provides another way of realizing the equivalence of two proposed types of trial wave functions used to describe such systems. In the proof of our identity, we make use of only elementary linear algebra and combinatorics and thereby avoid use of more advanced conformal field theory in establishing the aforementioned equivalence.

**Keywords:** Polynomial Identity; Correlation Function; Trial Wave Function

**MSC 2010 No.:** 15A15, 05A05, 81V70

## 1. Introduction

Let $n \geq 1$ be a positive integer and $z_1, z_2, \ldots, z_{2n}$ be indeterminates. Let $z_{ij} = z_i - z_j$ for

$1 \leq i, j \leq 2n$. Recall that, given a $2n \times 2n$ skew-symmetric matrix $A = \{a_{i,j}\}$, the *Pfaffian* of $A$ (see, e.g., Hazewinkel (2012)) is defined as

$$Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in \mathcal{S}_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)},$$

where $\mathcal{S}_{2n}$ is the symmetric group on $2n$ objects and $\text{sgn}(\sigma)$ is the sign of a permutation $\sigma$. Let $Pf(z_{ij}^{-1})$ denote the Pfaffian of the square matrix of size $2n$ whose $(i,j)$-th entry is $z_{ij}^{-1} = \frac{1}{z_i - z_j}$ for $i \neq j$, with zeros along the main diagonal. Our main result may be expressed as follows:

$$\sum_{A \in \mathcal{T}_n} \left( \prod_{i,j \in A} (z_i - z_j)^2 \prod_{k < \ell \in A^c} (z_k - z_\ell)^2 \right) = 2^{n-1} Pf(z_{ij}^{-1}) \prod_{1 \leq i < j \leq 2n} (z_i - z_j), \qquad n \geq 1, \tag{1}$$

where $\mathcal{T}_n$ denotes the collection of all subsets of the set $[2n] = \{1, 2, \ldots, 2n\}$ of size $n$ that contain 1. For example, when $n = 2$, we have

$$Pf(z_{ij}^{-1}) = (z_{12} z_{34})^{-1} - (z_{13} z_{24})^{-1} + (z_{14} z_{23})^{-1},$$

and multiplying this by $2 \prod_{1 \leq i < j \leq 4} (z_i - z_j)$ implies that the right-hand side of (1) is given by

$$2(z_{12} z_{13} z_{24} z_{34} - z_{12} z_{14} z_{23} z_{34} + z_{13} z_{14} z_{23} z_{24}).$$

On the other hand, the left-hand side of (1) when $n = 2$ equals $z_{12}^2 z_{34}^2 + z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2$, and expansion in terms of the $z_i$ shows that the two sides are in fact equal.

We now discuss briefly a physical context in which identity (1) arises. In quantum mechanics, trial wave functions (see, e.g., Laughlin (1983), Moore and Read (1991), and Quinn (2013) and (2014)) describing a system of interacting electrons can always be written in terms of the product of an antisymmetric Fermion factor $F\{z_{ij}\} = \prod_{i<j} z_{ij}$ caused by the Pauli principle, and a symmetric correlation function $G\{z_{ij}\}$ caused by interactions. The Moore-Read trial wave function (see Greiter et al. (1992), Moore and Read (1991)) describing the half filled spin polarized quantum liquid state in the first excited Landau level has a correlation factor, which we will denote by $G_{MR}$, that can be expressed as a product of $F\{z_{ij}\}$ and an antisymmetric Pfaffian defined by

$$Pf(z_{ij}^{-1}) = \widehat{A}_N \left\{ \prod_{m=1}^{N/2} (z_{2m-1,2m})^{-1} \right\}.$$

Here $z_j = x_j + iy_j$ is the complex coordinate of the $j$-th electron on the $x$-$y$ plane and $\widehat{A}_N$ is an operator that antisymmetrizes the expression in curly brackets over all $N = 2n$ particles. A simpler but seemingly different quadratic correlation function $G_Q\{z_{ij}\}$ described in Quinn (2014) (see also Cappelli et al. (2001), Mulay et al. (2016)) can be obtained by partitioning the set of $N$ particles into subsets $g_A$ and $g_B$, each containing $N/2$ particles, and defining

$$G_Q = \widehat{S}_N \left\{ \prod_{i<j \in g_A} z_{ij}^2 \prod_{k<\ell \in g_B} z_{k\ell}^2 \right\}.$$

Here $\widehat{S}_N$ symmetrizes the product in curly brackets over all $N$ particles. Correlation diagrams can be constructed for a term of either function by representing each factor $z_{ij}$ by a line connecting particles $i$ and $j$. The diagrams for $G_{MR}$ and $G_Q$ are different, though when these correlation functions are expressed as homogeneous polynomials in the independent coordinates (i.e., in terms of the variables $z_i$), the two functions are the same, up to a multiplicative constant, as has been witnessed for $N = 4$ and $N = 6$ particle systems. Identity (1) is then seen to imply the equivalence of these trial wave functions for all even $N$. A similar related identity that also implies this equivalence was shown in Cappelli et al. (2001), using various conformal field theories. Here, using only elementary linear algebra and some combinatorial observations, we prove identity (1) above in the subsequent section.

## 2. Proof of the main result

Let $n$ be a positive integer and $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two disjoint ordered sets of indeterminates. Let $L(X,Y) = L := [a_{ij}]$ be an $n \times n$ matrix with

$$a_{ij} = (x_i - y_j)^{-1} \quad \text{for } 1 \le i \le n \text{ and } 1 \le j \le n.$$

Define the polynomials $G(X)$ and $G(Y)$ by

$$G(X) := \prod_{1 \le i < j \le n} (x_i - x_j) \quad \text{and} \quad G(Y) := \prod_{1 \le i < j \le n} (y_i - y_j),$$

and let $F(X,Y) := G(X)G(Y)$.

We will denote the determinant of a square matrix $A$ by $\det(A)$. Define $z_i$ for $1 \le i \le 2n$ by letting $z_{2i-1} := x_i$ and $z_{2i} := y_i$ for $1 \le i \le n$.

**Lemma 2.1.**

We have

$$\left( \prod_{1 \le i < j \le 2n} (z_i - z_j) \right) \cdot \det(L) = F(X,Y)^2.$$

***Proof:***

It is well-known (see, e.g., Krattenthaler (1999)) that

$$\det_{1 \le i,j \le n} \left( \frac{1}{x_i + y_j} \right) = \frac{\prod_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j)}{\prod_{1 \le i,j \le n} (x_i + y_j)}. \tag{2}$$

Replacing $y_j$ by $-y_j$ for all $j$ in (2) gives

$$\begin{aligned}
\det(L) &= \frac{(-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j)}{\prod_{1 \le i,j \le n} (x_i - y_j)} \\
&= \frac{F(X,Y)^2}{(-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} (x_i - x_j)(y_i - y_j) \prod_{1 \le i,j \le n} (x_i - y_j)} \\
&= \frac{F(X,Y)^2}{(-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} (z_{2i-1} - z_{2j-1})(z_{2i} - z_{2j}) \prod_{1 \le i,j \le n} (x_i - y_j)}.
\end{aligned}$$

The desired result then follows, upon noting

$$\begin{aligned}\prod_{1\le i,j\le n}(x_i-y_j) &= \prod_{i=1}^{n}\prod_{j=i}^{n}(z_{2i-1}-z_{2j})\prod_{i=1}^{n}\prod_{j=1}^{i-1}(z_{2i-1}-z_{2j})\\ &= (-1)^{\binom{n}{2}}\prod_{i=1}^{n}\prod_{j=i}^{n}(z_{2i-1}-z_{2j})\prod_{j=1}^{n-1}\prod_{i=j+1}^{n}(z_{2j}-z_{2i-1}).\end{aligned}$$

□

We will need to consider the following various orderings of the sets $X$ and $Y$.

**Definition.**

Let $Z$ be an unordered set of $2n$ indeterminates where $n \geq 1$. Given a partition $X, Y$ of the set $Z$ into two sets $X$ and $Y$ of cardinality $n$ each, fix a listing $\sigma$ of the elements of $X$ to obtain the ordered set $X_\sigma := \{x_1, \ldots, x_n\}$ and fix a listing $\tau$ of the elements of $Y$ to obtain the ordered set $Y_\tau := \{y_1, \ldots, y_n\}$. Define the corresponding ordering $(\sigma, \tau)$ of $Z$ by declaring $Z_{(\sigma,\tau)} = \{z_1, \ldots, z_{2n}\}$, where

$$z_{2i-1} := x_i \quad \text{and} \quad z_{2i} := y_i \quad \text{for} \quad 1 \leq i \leq n,$$

and then define

$$\delta(X_\sigma, Y_\tau) := \prod_{1\le i<j\le 2n}(z_i - z_j).$$

We have the following result.

**Theorem 2.2.**

Let $Z$ be as in the preceding definition.

1. The product $\delta(X_\sigma, Y_\tau)\cdot\det(L(X_\sigma, Y_\tau))$ is independent of $(\sigma,\tau)$ and hence may be written simply as $\delta(X,Y)\cdot\det(L(X,Y))$.
2. Also, we have $\delta(X,Y)\cdot\det(L(X,Y)) = \delta(Y,X)\cdot\det(L(Y,X))$.
3. Letting $\mathcal{P}_2$ be the set of unordered partitions $\{X,Y\}$ of $Z$ into two parts of cardinality $n$ each, we have

$$\sum_{\{X,Y\}\in\mathcal{P}_2}\delta(X,Y)\cdot\det(L(X,Y)) = \sum_{\{X,Y\}\in\mathcal{P}_2}G(X)^2G(Y)^2. \tag{3}$$

***Proof:***

The first statement follows from Lemma 2.1 since $F(X,Y)^2 = G(X)^2G(Y)^2$, with $G(X)^2$ and $G(Y)^2$ invariant under any ordering of $X$ and $Y$, respectively. The second statement also follows from Lemma 2.1 upon observing $F(X,Y) = F(Y,X)$. Summing the equalities in Lemma 2.1 corresponding to all possible members of $\mathcal{P}_2$ gives the last statement. □

We now consider further the summands $\delta(X,Y)\cdot\det(L(X,Y))$ appearing in (3). Let us assume that the elements of $X$ and $Y$ are arranged in increasing order of their indices and that $X$ contains $z_1$. That is, $X = \{z_{a_1}, z_{a_2}, \ldots, z_{a_n}\}$ and $Y = \{z_{a'_1}, z_{a'_2}, \ldots, z_{a'_n}\}$, for some $1 = a_1 < a_2 < \cdots <$

$a_n$ and $a'_1 < a'_2 < \cdots < a'_n$. Let $A$ be the subset of $[2n]$ comprising the $a_i$. Consider the permutation $a_1 a'_1 a_2 a'_2 \cdots a_n a'_n$ of $[2n]$, which we will denote by $A_\rho$. By the *number of inversions* of a word $w = w_1 w_2 \cdots w_m$ in a totally ordered alphabet, we will mean the number of ordered pairs $(i, j)$ such that $1 \le i < j \le m$ and $w_i > w_j$, which we will denote by $\mathrm{inv}(w)$. From the definitions, we have

$$\frac{\delta(X,Y)}{\prod_{1\le i<j\le 2n}(z_i - z_j)} = (-1)^{\mathrm{inv}(A_\rho)} = \mathrm{sgn}(A_\rho), \tag{4}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of a permutation $\sigma$.

Let $\mathcal{T}_n$ denote the set of all $A \subseteq [2n]$ containing the element $1$ such that $|A| = n$. By (4) and the definition of the determinant in terms of the symmetric group (see, e.g., Stanley (2012)), we may write

$$\sum_{\{X,Y\}\in\mathcal{P}_2} \delta(X,Y)\cdot\det(L(X,Y)) = \prod_{1\le i<j\le 2n}(z_i - z_j)\sum_{A\in\mathcal{T}_n}\sum_{\sigma\in\mathcal{S}_n}\frac{\mathrm{sgn}(A_\rho)\mathrm{sgn}(\sigma)}{\prod_{i=1}^n (z_{a_i} - z_{a'_{\sigma(i)}})}, \tag{5}$$

where $\mathcal{S}_n$ denotes the symmetric group on $n$ objects and the $a_i$ and $a'_i$ are as in the preceding paragraph.

We now group together $(A, \sigma)$ terms in the sum on the right-hand side of (5) such that the product

$$\prod_{i=1}^n (z_{a_i} - z_{a'_{\sigma(i)}})$$

is the same (up to the sign). Suppose $B \in \mathcal{T}_n$, with $B = \{1 = b_1 < b_2 < \cdots < b_n\}$ and $B^c = \{b'_1 < b'_2 < \cdots < b'_n\}$. Consider the term corresponding to $(B, \tau)$ in the sum on the right-hand side of (5), where $\tau$ is some member of $\mathcal{S}_n$. Note that in order for

$$\prod_{i=1}^n (z_{a_i} - z_{a'_{\sigma(i)}}) = \pm\prod_{i=1}^n (z_{b_i} - z_{b'_{\tau(i)}}) \tag{6}$$

to hold, it must be the case that

$$\{a_1, a'_{\sigma(1)}\}, \{a_2, a'_{\sigma(2)}\}, \ldots, \{a_n, a'_{\sigma(n)}\} = \{b_1, b'_{\tau(1)}\}, \{b_2, b'_{\tau(2)}\}, \ldots, \{b_n, b'_{\tau(n)}\}$$

as *partitions*, by unique factorization. Since $a_1 = b_1 = 1$, it follows that given $(A, \sigma)$, there are exactly $2^{n-1}$ terms $(B, \tau)$ in the sum such that (6) holds. Note that of these $2^{n-1}$ terms $(B, \tau)$, exactly one satisfies $b_i < b'_{\tau(i)}$ for all $1 \le i \le n$.

We will show that each term in the sum corresponding to some $(B, \tau)$ for which (6) holds, where $(A, \sigma)$ is fixed, has the same sign. We will need the following lemma. By the *standardization* of a word $w = w_1 w_2 \cdots w_m$ having $m$ distinct letters, which is denoted $\mathrm{stan}(w)$, we mean the member of $\mathcal{S}_m$ obtained by replacing the $i$-th smallest letter of $w$ with $i$ for all $i$.

**Lemma 2.3.**

Suppose $\alpha = \alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$ is a permutation of $[2n]$ and that $\gamma = \gamma_1\gamma_2\cdots\gamma_n$ is the increasing rearrangement of $\beta = \beta_1\beta_2\cdots\beta_n$. Let $\delta = \mathrm{stan}(\beta)$. Then,

$$\mathrm{sgn}(\alpha) = \mathrm{sgn}(\alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_n\gamma_n)\mathrm{sgn}(\delta). \tag{7}$$

***Proof:***

Let $\epsilon_1\epsilon_2\cdots\epsilon_n$ denote the permutation $\delta^{-1}$ taken on the support set $\{2, 4, \ldots, 2n\}$. Then, we have

$$(1\epsilon_1 3\epsilon_3\cdots(2n-1)\epsilon_n)(\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n) = (\alpha_1\gamma_1\alpha_2\gamma_2\cdots\alpha_n\gamma_n), \tag{8}$$

where it is understood that the left permutation is applied first and that permutations are expressed in one-line notation (despite the enclosing parentheses). For example, if $\alpha = 351642$, then $\beta = 562$, $\gamma = 256$, $\delta = 231$, $\delta^{-1} = 312$, $\epsilon = 624$, and $(163254)(351642) = (321546)$.

Then, (7) follows from (8) upon using the facts that sgn is multiplicative, preserved under inverse, preserved under changing the support set, and preserved under "splicing" a permutation with an identity permutation as in the first permutation in (8) (for the last assertion, consider the decomposition into cycles). □

**Definition.**

Suppose $A \in \mathcal{T}_n$ and $\sigma \in \mathcal{S}_n$, with $A = \{a_1 < a_2 < \cdots < a_n\}$ and $A^c = \{a'_1 < a'_2 < \cdots < a'_n\}$. Let $A(\sigma)$ denote the permutation of $[2n]$ given by

$$A(\sigma) = a_1 a'_{\sigma(1)} a_2 a'_{\sigma(2)} \cdots a_n a'_{\sigma(n)}.$$

For example, if $n = 6$, $A = \{1, 4, 5, 7, 8, 11\}$ and $\sigma = 624315$, then $A^c = \{2, 3, 6, 9, 10, 12\}$ and

$$A(\sigma) = 1(12)43597682(11)(10).$$

Lemma 2.3 implies that equation (5) may be rewritten as

$$\sum_{\{X,Y\}\in\mathcal{P}_2} \delta(X,Y)\cdot\det(L(X,Y)) = \prod_{1\le i<j\le 2n}(z_i - z_j)\sum_{A\in\mathcal{T}_n}\sum_{\sigma\in\mathcal{S}_n}\frac{\mathrm{sgn}(A(\sigma))}{\prod_{i=1}^{n}(z_{a_i} - z_{a'_{\sigma(i)}})}. \tag{9}$$

Given $\omega = a_1b_1a_2b_2\cdots a_nb_n \in \mathcal{S}_{2n}$ of the form $A(\sigma)$, let $\omega^* \in \mathcal{S}_{2n}$ be obtained as follows. First fix some subset $S \subseteq \{2, 3, \ldots, n\}$ of size $k$, and let $s_1 < s_2 < \cdots < s_k$ be the elements of $S$. Form the set $T = \{a_1, a_2, \ldots, a_{s_1-1}, b_{s_1}, a_{s_1+1}, \ldots, a_n\}$. Suppose that $b_{s_1}$ is the $\ell$-th smallest member of $T$. Note that $a_1 = 1$ and $s_1 \ge 2$ implies $\ell \ge 2$. Now write the sequence

$$J = a_1b_1\cdots a_{s_1-1}b_{s_1-1}a_{s_1+1}b_{s_1+1}\cdots a_nb_n$$

of length $2n-2$ consisting of all $a_jb_j$ pairs with $j \ne s_1$. Insert $b_{s_1}a_{s_1}$ into this sequence directly following the $a_{\ell-1}b_{\ell-1}$ pair. Let $\omega_1$ denote the resulting permutation of $[2n]$.

In general, let $\omega_i$ be obtained recursively from $\omega_{i-1}$ for $1 \le i \le k$, where $\omega_0 = \omega$, by switching the order of the pair $a_{s_i}b_{s_i}$ within $\omega_{i-1}$ and moving the pair $b_{s_i}a_{s_i}$ so that the odd-indexed positions form an increasing subsequence, with $b_{s_i}$ occupying an odd position. Let $\omega^* = \omega_k$.

For example, if $n=6$, $k=3$, $S=\{2,3,5\}$ and $\omega = 19254(10)6378(12)(11)$, then we switch the order of the pairs $25$, $4(10)$ and $78$ and move them in three steps. This gives

$$\begin{aligned}\omega_1 &= 194(10)526378(12)(11),\\ \omega_2 &= 19526378(10)4(12)(11),\\ \omega_3 &= 19526387(10)4(12)(11),\end{aligned}$$

so that $\omega^* = \omega_3$.

**Lemma 2.4.**

We have $\mathrm{sgn}(\omega^*) = (-1)^k \mathrm{sgn}(\omega)$.

***Proof:***

If $k=0$, then the result is obvious, so assume $k \geq 1$. We will show that $\mathrm{sgn}(\omega_i) = -\mathrm{sgn}(\omega_{i-1})$ for $1 \leq i \leq k$, whence the result follows. To do so, suppose that the pair $xy$ is switched in the transition from $\omega_{i-1}$ to $\omega_i$ for some $i$, and that $yx$ is translated exactly $t$ positions (from where $xy$ was positioned in $\omega_{i-1}$). Then the change in the sign due to the translation of $yx$ is $(-1)^{2t}$, i.e., the sign is unaffected.

To show this, suppose that the translation is to the right, without loss of generality. Then the change in sign resulting from the translation that is attributable to the letter $x$ is $(-1)^t$. To see this, let $d_1$ be the number of elements greater than $x$ that $x$ moves past when $yx$ is repositioned and let $d_2$ be the number of elements less than $x$ that $x$ moves past. Then the change in sign caused by $x$ is $(-1)^{d_1-d_2} = (-1)^{t-2d_2} = (-1)^t$. By the same reasoning, the change in sign caused by $y$ is also $(-1)^t$. Finally, an inversion is either created or lost by switching the order of $x$ and $y$. Thus, we have $\mathrm{sgn}(\omega_i) = (-1)^{2t+1}\mathrm{sgn}(\omega_{i-1}) = -\mathrm{sgn}(\omega_{i-1})$, as desired. □

By $Pf(z_{ij}^{-1})$, it is meant the Pfaffian of the square matrix of size $2n$ whose $(i,j)$-th entry is $z_{ij}^{-1} = (z_i - z_j)^{-1}$ if $i \neq j$, with zeros along the main diagonal. (Note that this matrix is skew symmetric.) By an alternative formulation of the Pfaffian (see, for example, http://en.wikipedia.org/wiki/Pfaffian, 2001), we then have

$$Pf(z_{ij}^{-1}) = \sum_{\sigma \in \mathcal{M}_{2n}} \frac{\mathrm{sgn}(\sigma)}{\prod_{i=1}^{n}(z_{\sigma(2i-1)} - z_{\sigma(2i)})},$$

where $\mathcal{M}_{2n}$ denotes the set of all $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$ such that $\sigma_{2i-1} < \sigma_{2i}$ for $1 \leq i \leq n$ and $\sigma_{2i-1} < \sigma_{2i+1}$ for $1 \leq i \leq n-1$ (termed *perfect matchings*). We can now complete the proof of our main result.

**Theorem 2.5.**

For $n \geq 1$,

$$\sum_{A \in \mathcal{T}_n} \left( \prod_{i,j \in A} (z_i - z_j)^2 \prod_{k<\ell \in A^c} (z_k - z_\ell)^2 \right) = 2^{n-1} Pf(z_{ij}^{-1}) \prod_{1 \leq i < j \leq 2n} (z_i - z_j), \tag{10}$$

where $\mathcal{T}_n$ denotes the collection of all subsets of $[2n]$ of size $n$ that contain $1$.

***Proof:***

Let us say that a permutation $x$ is *related* to $y$ if $y$ can be obtained from $x$ by the procedure described above, which we will denote by $x \sim y$. One can verify that $\sim$ defines an equivalence relation. Furthermore, observe that $A(\sigma) \sim B(\tau)$ if and only if equation (6) is satisfied. By Lemma 2.4, we have in this case that $\text{sgn}(A(\sigma)) = (-1)^k \text{sgn}(B(\tau))$, where $k$ is the number of steps in the transition from $A(\sigma)$ to $B(\tau)$. Note that then exactly $k$ of the factors in the product $\prod_{i=1}^{n}(z_{a_i} - z_{a'_{\sigma(i)}})$ appearing on the right-hand side of (9) corresponding to the $(A, \sigma)$ term are switched when one considers the comparable product for the $(B, \tau)$ term. This factor of $(-1)^k$ thus cancels the earlier factor of $(-1)^k$ introduced when one considers the signs. Therefore, given any $(A, \sigma)$, all $(B, \tau)$ terms in (9) such that $B(\tau) \sim A(\sigma)$ (and only those terms) are equal to the $(A, \sigma)$ term.

Note that the equivalence classes of $\sim$ are of size $2^{n-1}$, and within each class, there is exactly one perfect matching. Given $\delta \in \mathcal{M}_{2n}$, let $S_\delta$ denote the set of all permutations of the form $A(\sigma)$, where $A \in \mathcal{T}_n$ and $\sigma \in \mathcal{S}_n$, such that $A(\sigma) \sim \delta$. Grouping together terms corresponding to members of the same equivalence class of $\sim$ yields

$$\begin{aligned}\sum_{A \in \mathcal{T}_n} \sum_{\sigma \in \mathcal{S}_n} \frac{\text{sgn}(A(\sigma))}{\prod_{i=1}^{n}(z_{a_i} - z_{a'_{\sigma(i)}})} &= \sum_{\delta \in \mathcal{M}_{2n}} \sum_{(A,\sigma) \in S_\delta} \frac{\text{sgn}(A(\sigma))}{\prod_{i=1}^{n}(z_{a_i} - z_{a'_{\sigma(i)}})} \\ &= \sum_{\delta \in \mathcal{M}_{2n}} \frac{2^{n-1}\text{sgn}(\delta)}{\prod_{i=1}^{n}(z_{\delta(2i-1)} - z_{\delta(2i)})}.\end{aligned}$$

Combining (3) and (9) with this last equality gives (10). □

## 3. Conclusion

We have established (1) above from which the equivalence of the correlation functions $G_{MR}$ and $G_Q$ follows. Our proof is self-contained and combines elementary linear algebra with some combinatorial observations concerning permutations. In the process, we have shown that both sides of (1) can be expressed in terms of a certain weighted sum of determinants. From a physical standpoint, we note that one advantage of using the correlation function $G_Q$ associated with the left-hand side of (1) is that it can easily be visualized for any even value of $N$. First partition $N$ particles into two subsets $g_A$ and $g_B$ each containing $N/2$ particles. In constructing the correlation diagram, you insert Laughlin correlations (two cf lines) between each pair of particles in the same subset. In the trial wave function, a single cf line must also be inserted between every pair of electrons due to the Fermion factor $F\{z_{ij}\}$. The total number of cf lines in the trial wave function is then equal to $\frac{N(2N-3)}{2}$, which equals $N\ell$ since $2\ell = 2N - 3$ for the Moore-Read state. For a state of angular momentum zero (see Fano et al., 1986), it is indeed necessary that the number of correlation lines be equal $N\ell$.

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