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Local Fractional Variational Iteration Method for Solving Nonlinear Partial Differential Equations within Local Fractional Operators

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Abstract

In this article, the local fractional variational iteration method is proposed to solve nonlinear partial differential equations within local fractional derivative operators. To illustrate the ability and reliability of the method, some examples are illustrated. A comparison between local fractional variational iteration method with the other numerical methods is given, revealing that the proposed method is capable of solving effectively a large number of nonlinear differential equations with high accuracy. In addition, we show that local fractional variational iteration method is able to solve a large class of nonlinear problems involving local fractional operators effectively, more easily and accurately, and thus it has been widely applicable in engineering and physics.

Keywords: Nonlinear partial differentia equation; local fractional variational iteration method; local fractional derivative operators

MSC (2010) No.: 26A33; 34A12; 34A34; 35R11

1. Introduction

The local fractional calculus has attracted a lot of interest for scientists and engineers. Several sections of local fractional derivative had been introduced, *i.e.* the local fractional derivative structured by Kolwankar and Gangal (1997), Kolwankar and Gangal (1998), and Yang

(2012), the modified Riemann- Liouville derivative given by Yang (2011), and Jumarie (2006), the fractal derivative suggested by Yang (2011), Jumarie (2011), and Parvate (2005), the fractal derivative considered by Yang (2011), Chen (2006), and Chen et al. (2010), the generalized fractal derivative proposed Chen et al. (2010), the local fractional derivative presented by Yang (2012), Adda and Cresson (2001), the local fractional derivative structured by He (2011), Fan and He (2012), and Yang (2012). As a result, the local fractional calculus theory becomes important for modeling problems for fractal mathematics and engineering on Cantor sets and it plays a key role in many applications in several fields such as theorical physics in Kolwankar (1998), and Yang (2012), the heat conduction theory in Yang (2012), He (2011), Baleanu and Yang (2013), the fracture and elasticity mechanics in Yang (2012), the fluid mechanics in Yang (2012) and Balankin and Elizarraraz (2012).

Several analytical and numerical techniques were successfully applied to deal with differential equations, fractional differential equations, and local fractional differential equations Wazwaz (2002), Schneider and Wyss (1989), Zhao and Li (2012), Momani and Odibat (2008), Laskin (2002), Zhou and Jiao (2010) Momani and Odibat (2006), Tarasov (2008), Golmankhaneh and Baleanu (2011), and Li et al. (2012), Hristov (2010). The techniques include the heat-balance integral Hristov (2010), the fractional Laplace transform Baleanu et al. (2012), the harmonic wavelet Cattani (2005, 2008), local fractional variational iteration Yang and Baleanu (2013), and Su et al. (2013), the local fractional decomposition Yang et al. (2013), Jafari and Jassim (2014), the local fractional series expansion Jafari and Jassim (2014), and the generalized local fractional Fourier transform Yang et al. (2013) methods.

In this paper, we investigate the application of local fractional variational iteration method for solving nonlinear local fractional partial differential equations. The structure of the paper is as follows. In Section 2, we give the concept of local fractional calculus. In Section 3, we give analysis of the method used. In Section 4, we consider some illustrative examples. Finally, in Section 5, we present our conclusions.

2. Basic Definitions of Local Fractional Calculus

In this section, we give some basic definitions and properties of fractional calculus theory which will be used in this paper.

Definition 1.

We say that a function $f(x)$ is local fractional continuous at $x = x_0$, Yang (2012) if

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \;, 0 < \alpha \leq 1 \;, \tag{2.1}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$. For $x\in(a,b)$, it is so called local fractional continuous on (a,b), denoted by $f(x)\in C_\alpha(a,b)$.

Definition 2.

Setting $f(x)\in C_\alpha(a,b)$, local fractional derivative of $f(x)$ at $x=x_0$ is defined as Yang (2012), Wang et al. (2014), and Yan et al. (2014)

$$D_x^\alpha f(x_0)=L_x^{(\alpha)}f(x_0)=f^{(\alpha)}(x)=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \tag{2.2}$$

where $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

Note that local fractional derivative of high order is written in the form

$$D_x^{k\alpha}f(x)=f^{(k\alpha)}(x)=\overbrace{D_x^\alpha\, D_x^\alpha \dots D_x^\alpha}^{k\,times}\ f(x), \tag{2.3}$$

and local fractional partial derivative of high order

$$\frac{\partial^{k\alpha}\ f(x,y)}{x^{k\alpha}}=\overbrace{\frac{\partial^\alpha}{\partial x^\alpha}\,\frac{\partial^\alpha}{\partial x^\alpha}\cdots\frac{\partial^\alpha}{\partial x^\alpha}}^{k\ times}\ f(x,y). \tag{2.4}$$

Definition 3.

Let us denote a partition of the interval $[a,b]$ as (t_j,t_{j+1}), $j=0,\dots,N-1$, and $t_N=b$ with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_0,\Delta t_1,\dots\}$. Local fractional integral of $f(x)$ in the interval $[a,b]$ is given by Yang (2012), Wang et al. (2014), and Yan et al. (2014)

$${}_aI_b^{(\alpha)}\ f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha. \tag{2.5}$$

Definition 4.

In fractal space, the Mittage Leffler function, sine function and cosine function are defined as Yang (2012), Wang et al. (2014), and Yan et al. (2014)

$$E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}\ ,\ 0<\alpha\le 1 \tag{2.6}$$

$$\sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}\ ,\ 0<\alpha\le 1 \tag{2.7}$$

$$\cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]}\ ,\ 0<\alpha\le 1 \tag{2.8}$$

The following results are valid:

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}\ , \tag{2.9}$$

$$\frac{d^\alpha E_\alpha(x^\alpha)}{dx^\alpha}=E_\alpha(x^\alpha)\ , \tag{2.10}$$

$$\frac{d^\alpha E_\alpha(kx^\alpha)}{dx^\alpha}=kE_\alpha(kx^\alpha)\ , \tag{2.11}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b E_\alpha(x^\alpha)(dx)^\alpha=E_\alpha(b^\alpha)-E_\alpha(a^\alpha)\ , \tag{2.12}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \sin_\alpha(x^\alpha)(dx)^\alpha=\cos_\alpha(a^\alpha)-\cos_\alpha(b^\alpha)\ . \tag{2.13}$$

3. Analysis of the Local Fractional Variational Iteration Method

We consider a general nonlinear local fractional partial differential equation:

$$L_\alpha u(x,t)+R_\alpha u(x,t)+N_\alpha u(x,t)=f(x,t)\ ,t>0,\ x\in R\ ,\ 0<\alpha\le 1, \tag{3.1}$$

where L_α denotes linear local fractional derivative operator of order 2α, R_α denotes linear local fractional derivative operator of order less than L_α, N_α denotes nonlinear local fractional operator, and $f(x,t)$ is the nondifferentiable source term.

According to the rule of local fractional variational iteration method, the correction local fractional functional for (3.1) is constructed as Yang and Baleanu (2013):

$$u_{n+1}(x)=u_n(x)+{}_0I_x^{(\alpha)}\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi)+R_\alpha\tilde{u}_n(\xi)+N_\alpha\tilde{u}_n(\xi)-f(\xi)\right]\right), \tag{3.2}$$

where $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$ is a fractal Lagrange multiplier.

Making the local fractional variation of (3.2), we have

$$\delta^\alpha u_{n+1}(x)=\delta^\alpha u_n(x)+{}_0I_x^{(\alpha)}\delta^\alpha\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi)+R_\alpha\tilde{u}_n(\xi)+N_\alpha\tilde{u}_n(\xi)-f(\xi)\right]\right). \tag{3.3}$$

The extremum condition of u_{n+1} is given by Yang (2012)

$$\delta^{\alpha} u_{n+1}(x) = 0 . \tag{3.4}$$

In view of (3.4), we have the following stationary conditions:

$$1 - \left(\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)} \right)^{(\alpha)} \Bigg|_{\xi = x} = 0, \quad \frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)} \Bigg|_{\xi = x} = 0 , \quad \left(\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)} \right)^{(2\alpha)} \Bigg|_{\xi = x} = 0. \tag{3.5}$$

So, from (3.5), we get

$$\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)} = \frac{(\xi - x)^{\alpha}}{\Gamma(1+\alpha)}. \tag{3.6}$$

The initial value $u_0(x)$ is given by

$$u_0(x) = u(0) + \frac{x^{\alpha}}{\Gamma(1+\alpha)} u^{(\alpha)}(0). \tag{3.7}$$

In view of (3.6), we have

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)} \left(\frac{(\xi - x)^{\alpha}}{\Gamma(1+\alpha)} \left[L_{\alpha} u_n(\xi) + R_{\alpha} u_n(\xi) + N_{\alpha} u_n(\xi) - f(\xi) \right] \right). \tag{3.8}$$

Finally, from (3.8), we obtain the solution of (3.1) as follows:

$$u(x,t) = \lim_{n \to \infty} u_n(x,t). \tag{3.9}$$

4. Some Illustrative Examples

In this section, we gave some illustrative examples for solving the nonlinear partial differential equations involving local fractional derivative operators by using local fractional variational iteration method.

Example 1.

Let us consider the nonlinear local fractional partial differential equation

$$\frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} u(x,y)}{\partial x^{\alpha}} \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}} - u(x,y) = - \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \tag{4.1}$$

and subject to the fractal value conditions

$$u(0,y) = \sin_{\alpha}(y^{\alpha}), \quad \frac{\partial^{\alpha} u(0,y)}{\partial x^{\alpha}} = 1. \tag{4.2}$$

From (4.2) we take the initial value, which reads

$$u_0(x,y)=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)} \tag{4.3}$$

By using (3.8) we structure a local fractional iteration procedure as

$$u_{n+1}(x,y)=u_n(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_n(\xi,y)}{\partial\xi^{2\alpha}}-\frac{\partial^\alpha u_n(\xi,y)}{\partial\xi^\alpha}\frac{\partial^{2\alpha}u_n(\xi,y)}{\partial y^{2\alpha}}-u_n(\xi,y)+\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right]\right). \tag{4.4}$$

Hence, we can derive the first approximation term

$$u_1(x,y)=u_0(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_0(\xi,y)}{\partial\xi^{2\alpha}}-\frac{\partial^\alpha u_0(\xi,y)}{\partial\xi^\alpha}\frac{\partial^{2\alpha}u_0(\xi,y)}{\partial y^{2\alpha}}-u_0(\xi,y)+\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right]\right)$$

$$=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{4.5}$$

The second approximation can be calculated in the similar way, which is

$$u_2(x,y)=u_1(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_1(\xi,y)}{\partial\xi^{2\alpha}}-\frac{\partial^\alpha u_1(\xi,y)}{\partial\xi^\alpha}\frac{\partial^{2\alpha}u_1(\xi,y)}{\partial y^{2\alpha}}-u_1(\xi,y)+\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right]\right)$$

$$=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{4.6}$$

Proceeding in this manner, we get the third approximation as

$$u_3(x,y)=u_2(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_2(\xi,y)}{\partial\xi^{2\alpha}}-\frac{\partial^\alpha u_2(\xi,y)}{\partial\xi^\alpha}\frac{\partial^{2\alpha}u_2(\xi,y)}{\partial y^{2\alpha}}-u_2(\xi,y)+\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right]\right)$$

$$=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{4.7}$$

and so on. Thus, we have the local fractional series solution

$$u_n(x,y)=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{4.8}$$

As a result, the final solution reads

$$u(x,y)=\lim_{n\to\infty}u_n(x,y)$$

$$=\sin_\alpha(y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{4.9}$$

Example 2.

Consider the nonlinear local fractional partial differential equation

$$\frac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}}-\left(\frac{\partial^\alpha u(x,u)}{\partial y^\alpha}\right)^2+u^2(x,y)=0, \tag{4.10}$$

and subject to the fractal value conditions

$$u(0,y)=0,\quad \frac{\partial^{\alpha}u(0,y)}{\partial x^{\alpha}}=E_{\alpha}(y^{\alpha}). \tag{4.11}$$

From (4.11) we take the initial value, which reads

$$u_0(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}). \tag{4.12}$$

By using (3.8) we structure a local fractional iteration procedure as

$$u_{n+1}(x,y)=u_n(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_n(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{\alpha}u_n(\xi,y)}{\partial y^{\alpha}}\right)^2+u_n^2(\xi,y)\right]\right). \tag{4.13}$$

Hence, we can derive the first approximation term

$$u_1(x,y)=u_0(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_0(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{\alpha}u_0(\xi,y)}{\partial y^{\alpha}}\right)^2+u_0^2(\xi,y)\right]\right)$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}). \tag{4.14}$$

The second approximation can be calculated in the similar way, which is

$$u_2(x,y)=u_1(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_1(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{\alpha}u_1(\xi,y)}{\partial y^{\alpha}}\right)^2+u_1^2(\xi,y)\right]\right)$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}). \tag{4.15}$$

Proceeding in this manner, we get the third approximation as

$$u_3(x,y)=u_2(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_2(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{\alpha}u_2(\xi,y)}{\partial y^{\alpha}}\right)^2+u_2^2(\xi,y)\right]\right).$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}), \tag{4.16}$$

and so on. Thus, we have the local fractional series solution

$$u_n(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}). \tag{4.17}$$

As a result, the final solution reads

$$u(x,y)=\lim_{n\to\infty}u_n(x,y)$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(y^{\alpha}). \tag{4.18}$$

Example 3.

Consider the nonlinear local fractional partial differential equation

$$\frac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}}-\left(\frac{\partial^{2\alpha}u(x,u)}{\partial y^{2\alpha}}\right)^2+u^2(x,y)=0, \tag{4.19}$$

and subject to the fractal value conditions

$$u(0,y)=0, \quad \frac{\partial^{\alpha}u(0,y)}{\partial x^{\alpha}}=\cos_\alpha(y^\alpha). \tag{4.20}$$

From (4.19) we take the initial value, which reads

$$u_0(x,y)=\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha). \tag{4.21}$$

By using (3.8) we structure a local fractional iteration procedure as

$$u_{n+1}(x,y)=u_n(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_n(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{2\alpha}u_n(\xi,y)}{\partial y^{2\alpha}}\right)^2+u_n^2(\xi,y)\right]\right). \tag{4.22}$$

Hence, we can derive the first approximation term

$$u_1(x,y)=u_0(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_0(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{2\alpha}u_0(\xi,y)}{\partial y^{2\alpha}}\right)^2+u_0^2(\xi,y)\right]\right)$$

$$=\frac{x^\alpha}{\Gamma(1+\alpha)}cox_\alpha(y^\alpha). \tag{4.23}$$

The second approximation can be calculated in the similar way, which is

$$u_2(x,y)=u_1(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_1(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{2\alpha}u_1(\xi,y)}{\partial y^{2\alpha}}\right)^2+u_1^2(\xi,y)\right]\right)$$

$$=\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha). \tag{4.24}$$

Proceeding in this manner, we get the third approximation as

$$u_3(x,y)=u_2(x,y)+{}_0I_x^{(\alpha)}\left(\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_2(\xi,y)}{\partial\xi^{2\alpha}}-\left(\frac{\partial^{2\alpha}u_2(\xi,y)}{\partial y^{2\alpha}}\right)^2+u_2^2(\xi,y)\right]\right)$$

$$=\frac{x^\alpha}{\Gamma(1+\alpha)}\cos_\alpha(y^\alpha), \tag{4.25}$$

and so on. Thus, we have the local fractional series solution

$$u_n(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)} \cos_\alpha(y^\alpha). \tag{4.26}$$

As a result, the final solution reads

$$u(x,y) = \lim_{n\to\infty} u_n(x,y)$$

$$= \frac{x^\alpha}{\Gamma(1+\alpha)} \cos_\alpha(y^\alpha). \tag{4.27}$$

5. Conclusions

In this paper, we have studied nonlinear partial differential equations involving local fractional operator with the local fractional variational iteration method (LFVIM). The exact solution of the nonlinear partial differential equations is obtained by the local fractional variational iteration method. The results showed that the variational iteration method is remarkably effective. Comparison with the local fractional decomposition method (LFDM) shows that the local fractional variational iteration method is a powerful method for nonlinear equations. The advantage of the LFVIM over the LFDM is that there is no need for the evaluations of the Adomian polynomials.

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