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The shifted Jacobi polynomial integral operational matrix for solving Riccati differential equation of fractional order

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Abstract

In this article, we have applied Jacobi polynomial to solve Riccati differential equation of fractional order. To do so, we have presented a general formula for the Jacobi operational matrix of fractional integral operator. Using the Tau method, the solution of this problem reduces to the solution of a system of algebraic equations. The numerical results for the examples presented in this paper demonstrate the efficiency of the present method.

Keywords: Fractional differential equations; Operational matrix; Jacobi polynomials; Tau method, Riccati equation

MSC 2010 No.: 34A08; 26A33; 47B36

1. Introduction

Fractional calculus has been one of the most fascinating issues that have attracted the attention of large group of scholars, particularly in the fields of mathematics and engineering. This is due to the fact that boundary value problems of fractional differential equation can be employed to explain various natural phenomena. Many scholars and authors in different fields such as physics, fluid flows, electrical networks, and viscoelasticity have attempted to introduce a model for these phenomena through using fractional differential equation [Oldham and Spanier (1974), Ross (1975), Kilbas et al. (2006), Podlubny (1999), Lakshmikantham et al. (2009)]. Interested readers can check other books and papers in the related literature to get further information about fractional calculus [Kilbas et al. (2006), Podlubny (1999)].

We know that most fractional differential equations do not lend themselves to accurate analytical solutions. Consequently, we should use approximate and numerical techniques to find solutions for fractional differential equations. Various methods have been employed in the last few decades to find such solutions.

These methods include fractional partial differential equations and fractional integro-differential equations containing fractional derivatives as Adomian decomposition method [Momani and Noor (2006), Ray et al. (2006), Wang (2006)], Variational iteration method [Inc (2008), Odibat and Momani (2006), Abbasbandy(2007)], Homotopy analysis method [Hashim (2009), Zurigat(2010)] and other methods [Kazemi (2011), Sweilam et al. (2012), Erjaee et al. (2011),].

Attempts to find accurate and efficient methods to solve fractional Riccati equations have invited a lot of active research projects. Scholars and authors have presented various analytical and numerical methods for solving this equation. Analytical method includes the ADM and VIM, Abbasbandy (2007). Another approach through which we can solve fractional Riccati equation is to use HPM, Abbasbandy (2007).

In the present research, we have employed Jacobi orthogonal polynomials to find solutions to the Riccati differential equation of fractional order

$$D^{\mu} y(x) + a(x)y(x) + b(x)\, y^2(x) = g(x), \tag{1.1}$$

$$y(0) = d, \tag{1.2}$$

in which D^{μ} signifies caputo fractional derivative operator of order $a(x)$ and $b(x)$ and $g(x)$ stand for real functions on R. The purpose of this study is to generalize Jacobi integral operational matrix to fractional calculus.

Thus, these matrices have been used along with the Tau method to reduce the solution of this problem to the solution of a system of algebraic equation.

2. Preliminaries

In this section, several definitions of fractional calculus are presented. The definitions include the Jacobi polynomials, the shifted Jacobi polynomials and some of their properties.

2.1. Fractional Calculus

Definition 1.

A real function $f(x), x > 0$ is considered to be in the space C_{ν}, $(\nu \in R)$ if there exists a real number $n(> \nu)$, so that $f(x) = x^n f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_{ν}^{k} if and only if $f^{(k)} \in C_{\upsilon}, k \in N$.

Definition 2.

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\nu, \nu \geq -1$ is given by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r)\, dr\,,$$
$$I^\alpha f(x) = I_0^\alpha f(x),\ I^0 f(x) = f(x)\,.$$

Definition 3. The Caputo's fractional derivative of f is defined as

$$D^\alpha f(x) = I^{k-\alpha} D^k f(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r)\, dr,\ x > 0,$$

where, $f \in C_{-1}^k$, $k-1 < \alpha \leq k$ and $k \in N$.

Property 1.

For $k-1 < \alpha \leq k$, $k \in N$, $f \in C_\nu^k$, $\nu \geq -1$ and $x > 0$ the following properties satisfy

$i)\, D_a^\alpha I_a^\alpha f(x) = f(x)\,,$

$ii)\, I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{j=0}^{k-1} f^{(j)}(a^+) \frac{(x-a)^j}{j!}\,.$

2.2. Jacobi polynomials

The Jacobi polynomials which are represented by $J_n^{\alpha,\beta}(z)$, are orthogonal with regard to the weight function $w(z) = (1-z)^\alpha (1-z)^\beta$ on the interval $I = (-1,1)$, i.e.,

$$\int_{-1}^1 J_n^{\alpha,\beta}(z) J_m^{\alpha,\beta}(z)\, dz = \gamma_n^{\alpha,\beta} \delta_{m,n}, \tag{2.1}$$

where

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$$

(2.2)

and $\delta_{m,n}$ is the Kronecker function.

One can easily notice that the weight function $w(z)$ belongs to $L^1(I)$ if and only if $\alpha, \beta > -1$.

The following three term-recurrence to relation results in the Jacobi polynomials

$$J_0^{\alpha,\beta} = 1, J_1^{\alpha,\beta} = \frac{1}{2}(\alpha+\beta+2)z + \frac{1}{2}(\alpha-\beta),$$

$$J_{n+1}^{\alpha,\beta}(z)=\left(a_n\, z-b_n\right)J_n^{\alpha,\beta}(z)-c_n J_{n-1}^{\alpha,\beta}(z), n\geq 1,$$

where

$$a_n=\frac{\left(2n+\alpha+\beta+1\right)\left(2n+\alpha+\beta+2\right)}{2(n+1)\left(n+\alpha+\beta+1\right)},$$

$$a_n=\frac{\left(2n+\alpha+\beta+1\right)\left(2n+\alpha+\beta+2\right)}{2(n+1)\left(n+\alpha+\beta+1\right)},$$

$$c_n=\frac{\left(2n+\alpha+\beta+1\right)\left(n+\alpha\right)\left(n+\beta\right)}{(n+1)\left(n+\alpha+\beta+1\right)\left(2n+\alpha+\beta\right)}.$$

The Jacobi polynomials $J_n^{\alpha,\beta}(z)$, of degree n are generated by

$$J_n^{\alpha,\beta}(z)=2^{-n}\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}\left(z-1\right)^i\left(z+1\right)^{n-i}. \tag{2.3}$$

2.3. The shifted Jacobi polynomials

As a result of changing variable $z=2x-1,$ we obtain new orthogonal polynomials $P_n^{\alpha,\beta}(x)$ with weight function $w_s^{(\alpha,\beta)}(x)=\left(1-x\right)^{\alpha}x^{\beta}$ on the interval [0,1] which is called shifted Jacobi polynomials. These polynomials have the following orthogonality properties

$$\int_0^1 P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)w_s^{(\alpha,\beta)}(x)\,dx=\vartheta_n^{\alpha,\beta}\delta_{mn}, \tag{2.4}$$

where $\vartheta_n^{\alpha,\beta}=\dfrac{\gamma_n^{\alpha,\beta}}{2^{\alpha+\beta+1}}$.

From (2.3), we can write $P_n^{\alpha,\beta}(x)$ as follows:

$$P_n^{\alpha,\beta}(x)=\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}\left(x-1\right)^i x^{n-i}, \tag{2.5}$$

$$P_n^{\alpha,\beta}(x)=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}\binom{i}{j}\left(-1\right)^j x^{n-i}. \tag{2.6}$$

From relations (2.5) and (2.6), we can easily notice that the following properties are satisfied.

Property 2.

$$P_n^{\alpha,\beta}(0)=\left(-1\right)^n\binom{n+\alpha}{n}.$$

Property 3.

$$\frac{d^i}{dx^i}P_n^{\alpha,\beta}(x)=\frac{\Gamma\left(n+\alpha+\beta+i+1\right)}{\Gamma\left(n+\alpha+\beta+1\right)}P_{n-i}^{\alpha+i,\beta+i}(x).$$

Property 4. The shifted Jacobi polynomial can be achieved in the following form:

$$P_n^{\alpha,\beta}(x) = \sum_{i=0}^{n} p_i^{(n)} x^i,$$

in which

$$p_i^{(n)} = (-1)^{n-i} \binom{n+\alpha+\beta+1}{i} \binom{n+\alpha}{n-i}, \quad i = 0, 1, \ldots, n.$$

Property 5. For $\mu > 0,$

$$\int_0^1 x^{\mu} P_m^{\alpha,\beta}(x)\, w_s^{(\alpha,\beta)}(x)\, dx = \sum_{l=0}^{j} p_l^{(j)} B(\mu + l + \beta + 1, \alpha + 1), \tag{2.7}$$

where is $B(t,s)$ Beta function.

2.4. The approximation of functions in the Sobolev space

Suppose $\Omega = (0,1)$, then for any $r \in N$ (N is the set of all non-negative integers), the weighted Sobolev space $H^r_{w_s^{(\alpha,\beta)}}(\Omega)$ can be defined in the usual way, which indicates its inner product, semi-norm and norm by

$$(u,v)_{w_s^{(\alpha,\beta)}}, |v|_{r,w_s^{(\alpha,\beta)}} \text{ and } \|v\|_{r,w_s^{(\alpha,\beta)}}, \text{ respectively.}$$

Particularly,

$$L^2_{w_s}(\Omega) = H^0_{w_s^{(\alpha,\beta)}}(\Omega), \text{and } \|v\|_{w_s^{(\alpha,\beta)}} = \|v\|_{r,w_s^{(\alpha,\beta)}}.$$

$$H^r_{w_s^{(\alpha,\beta)}}(\Omega) = \left\{ f \mid f \text{ can be measured}, \|v\|_{r,w_s^{(\alpha,\beta)}} < \infty \right\},$$

$$\|f\|^2_{r,w_s^{(\alpha,\beta)}} = \sum_{k=0}^{r} \left\|\partial_x^k f\right\|^2_{r,w_s^{(\alpha+k,\beta+k)}},$$

$$|f|^2_{r,w_s^{(\alpha,\beta)}} = \left\|\partial_x^r f\right\|^2_{w_s^{(\alpha+r,\beta+r)}}.$$

Now we can suppose the function $f \in H^r_{w_s^{(\alpha,\beta)}}(\Omega)$ in

$$P^{m,\alpha,\beta}(x) = \text{span}\left\{ p_0^{\alpha,\beta}(x), p_1^{\alpha,\beta}(x), \ldots, p_{m-1}^{\alpha,\beta}(x) \right\},$$

as presented in the following formula:

$$f(x) = \sum_{i=0}^{\infty} k_i\, p_i^{(\alpha,\beta)}(x). \tag{2.7}$$

in which the coefficients k_i are generated by:

$$k_i = \frac{1}{\vartheta_j^{\alpha,\beta}} \int_0^1 p_i^{\alpha,\beta}(x)\, f(x)\, w_s^{(\alpha,\beta)} dx, \quad i = 0,1,\dots . \tag{2.8}$$

In practice, only the first m- terms shifted Jacobi polynomials are taken into account. Then we have:

$$f(x) \approx \sum_{i=0}^{m-1} k_i\, p_i^{(\alpha,\beta)}(x) = k^T p, \tag{2.9}$$

with

$$K = [k_0, k_1, \dots, k_{m-1}]^T, \tag{2.10}$$

$$P = \left[p_0^{\alpha,\beta}(x), p_1^{\alpha,\beta}(x), \dots, p_{m-1}^{\alpha,\beta}(x)\right]. \tag{2.11}$$

In as much as $P^{m,\alpha,\beta}$ is a finite dimensional vector space, f has a unique best approximation from $P^{m,\alpha,\beta}$, say $f_m(x) \in P^{m,\alpha,\beta}$ that is:

$$\forall\, y \in P^{m,\alpha,\beta}, \, \|f(x) - f_m(x)\|_{w_s} \le \|f(x) - y\|_{w_s}.$$

Guo and Wang (2004), came to the conclusion that for any $f \in H^r_{w_s^{(\alpha,\beta)}}(\Omega)$, $r \in N$ and $0 \le \mu \le r$, a generic positive constant C independent of any function, m, α and β exists so that:

$$\|f(x) - f_m(x)\|_{\mu, w_s^{(\alpha,\beta)}} \le C\big((m-1)(m+\alpha+\beta-1)\big)^{\frac{\mu-r}{2}} \,|\, f(x)|_{r,w_s}.$$

3. The operational matrix of fractional integral

We can express Riemann-Liouville fractional integral operator of order μ of the vector p by:

$$I^{\mu} P \approx Q^{(\mu)} P. \tag{3.1}$$

where $Q^{(\mu)}$ is the $m \times n$ operational matrix of Riemann-Liouville fractional integral of order μ.

Theorem 3.1.

If $Q^{(\mu)}$ is the $m \times n$ operational matrix of Riemann-Liouville fractional integral of order μ**,** then the elements of this matrix are taken as:

$$Q^{(\mu)} = \left\{q_{i,j}^{(\mu)}\right\}_{i,j=0}^{m-1} = \sum_{k=0}^{i} \sum_{l=0}^{j} p_k^{(i)} p_l^{(j)} \frac{\Gamma(k+1)\, B(k+l+\mu+\beta+1, \alpha+1)}{\vartheta_j^{\alpha,\beta}\, \Gamma(k+\mu+1)}. \tag{3.2}$$

Now, we define the error vector E, as

$$E = I^{\mu} P - Q^{(\mu)} P.$$

The maximum norm of vector E is defined as follows (Guo and Wang (2004))

$$\|E\|_\infty \le \begin{cases} \dfrac{x_0^{\mu}}{m!\left|\Gamma(\mu-m+1)\right|}\left(\dfrac{L}{x_0}\right)^m \dbinom{m+\beta}{m}\sqrt{B(\alpha+1,\beta+1)}, & \beta \ge 0 \\ \dfrac{x_0^{\mu}}{m!\left|\Gamma(\mu-m+1)\right|}\left(\dfrac{L}{x_0}\right)^m \sqrt{B(\alpha+1,\beta+1)}, & \beta < 0, \end{cases} \tag{3.3}$$

where $x_0 > 0$ and $L = \max\{1-x_0, x_0\}$.

4. Main Results

Lemma 4.1.

Let

$$K = [k_0, k_1, ..., k_{m-1}]^T,\ P = [p_0^{\alpha,\beta}(x), p_1^{\alpha,\beta}(x), ..., p_{m-1}^{\alpha,\beta}(x)].$$

Now if we suppose that $Q^{(\mu)}$ is the same in Theorem (3.1). Then,

$$P\,P^T (Q^{(\mu)})^T K = HP, \tag{4.1}$$

where $H = (h_{i,j})_{i,j=0,1,...,n-1}$, with

$$h_{i,j} = \frac{2^{\alpha+\beta+1}}{\gamma_j^{\alpha,\beta}} \int_0^1 p_i^{\alpha,\beta}(x)\, p_j^{\alpha,\beta}(x) \left(\sum_{l=0}^{n-1} \left(\sum_{t=0}^{n-1} p_t^{\alpha,\beta}(x)\, q_{lt}^{(\mu)} \right) k_l \right) w_s^{(\alpha,\beta)} dx, \quad i, j = 0,1,..., n-1 .$$

Proof:

We denote $P_i^{\alpha,\beta}(x) = p_i, i = 0,1,..., n-1.$ We have

$$P\,P^T (Q^{(\mu)})^T K = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} \begin{bmatrix} p_0 & p_0 & \cdots & p_0 \end{bmatrix} \begin{bmatrix} q_{00}^{(\mu)} & q_{10}^{(\mu)} & \cdots & q_{n-10}^{(\mu)} \\ q_{01}^{(\mu)} & q_{11}^{(\mu)} & \cdots & q_{n-11}^{(\mu)} \\ \vdots & & \vdots & \vdots \\ q_{0n-1}^{(\mu)} & q_{1n-1}^{(\mu)} & \cdots & q_{n-1n-1}^{(\mu)} \end{bmatrix} \begin{bmatrix} k_0 \\ k_0 \\ \vdots \\ k_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} k_0 \sum_{i=0}^{n-1} p_0 p_i q_{0i}^{(\mu)} + k_1 \sum_{i=0}^{n-1} p_0 p_i q_{1i}^{(\mu)} + \ldots + k_{n-1} \sum_{i=0}^{n-1} p_0 p_i q_{n-1i}^{(\mu)} \\ k_0 \sum_{i=0}^{n-1} p_1 p_i q_{0i}^{(\mu)} + k_1 \sum_{i=0}^{n-1} p_1 p_i q_{1i}^{(\mu)} + \ldots + k_{n-1} \sum_{i=0}^{n-1} p_1 p_i q_{n-1i}^{(\mu)} \\ \vdots \\ k_0 \sum_{i=0}^{n-1} p_{n-1} p_i q_{0i}^{(\mu)} + k_1 \sum_{i=0}^{n-1} p_{n-1} p_i q_{1i}^{(\mu)} + \ldots + k_{n-1} \sum_{i=0}^{n-1} p_{n-1} p_i q_{n-1i}^{(\mu)} \end{bmatrix}.$$

If we consider $f = (f_0, f_1, \ldots, f_{n-1})$ with

$$f_j = k_0 \sum_{i=0}^{n-1} p_k p_i q_{0i}^{(\mu)} + k_1 \sum_{i=0}^{n-1} p_k p_i q_{1i}^{(\mu)} + \ldots + k_j \sum_{i=0}^{n-1} p_k p_i q_{n-1i}^{(\mu)}, \quad j = 0,1,\ldots,n-1,$$

and applying (2.8)-(2.9), the following is obtained:

$$P\, P^T (Q^{(\mu)})^T K = \begin{bmatrix} \sum_{j=0}^{n-1} h_{0j} p_j \\ \sum_{j=0}^{n-1} h_{1j} p_j \\ \vdots \\ \sum_{j=0}^{n-1} h_{n-1j} p_j \end{bmatrix} = HP,$$

where

$$h_{i,j} = \frac{2^{\alpha+\beta+1}}{\gamma_j^{\alpha,\beta}} \int_0^1 p_i^{\alpha,\beta}(x)\, p_j^{\alpha,\beta}(x) \left(\sum_{l=0}^{n-1} \left(\sum_{t=0}^{n-1} p_t^{\alpha,\beta}(x)\, q_{lt}^{(\mu)} \right) k_l \right) w_s^{(\alpha,\beta)} dx, \quad i, j = 0,1,\ldots,n-1 .$$

The proof is complete.

Now, we consider the Riccati equation with fractional orders of the form

$$D^{\mu} y(x) + a\, y(x) + b\, y^2(x) = g(x), \; 0 < \mu \le 1, \tag{4.2}$$

$$y(0) = d, \tag{4.3}$$

where a, b, d are real constant coefficients and D^{μ} stand for the Caputo fractional derivative of order μ .

Using Definition (3), we can rewrite Equation (4.2):

$$I^{1-\mu} Dy(x) + a\, y(x) + b\, y^2(x) = g(x), \; 0 < \mu \le 1. \tag{4.4}$$

To solve problems (4.2)-(4.3) we approximate $D\, y(x)$ and $g(x)$ by the shifted Jacobi polynomials as:

$$D\, y(x) \approx \sum_{i=0}^{m-1} k_i p_i^{\alpha,\beta}(x) = K^T P, \tag{4.5}$$

$$g(x) \approx \sum_{i=0}^{m-1} g_i p_i^{\alpha,\beta}(x) = G^T P. \tag{4.6}$$

From (4.5), we get

$$\int_0^x D\, y(x)\, dx \approx I\left(K^T P\right) = K^T \left(IP\right) \tag{4.7}$$

Consequently,

$$y(x) = K^T Q^{(1)} P + y(0), \tag{4.8}$$

and

$$y^2(x) = K^T Q^{(1)} P P^T (Q^{(1)})^T K + 2K^T Q^{(1)} P\, y(0) + \left(y(0)\right)^2, \tag{4.9}$$

Moreover, we have

$$D^{\mu} y(x) = I^{1-\mu} D\, y(x) = I^{1-\mu}\left(K^T P\right) = K^T Q^{(1-\mu)} P. \tag{4.10}$$

Using (4.6) and (4.8)-(4.10), problems (4.2)-(4.3) can be rewritten as:

$$K^T Q^{(1-\mu)} P + a\left(K^T Q^{(1)} P + y(0)\right) + b\left(K^T Q^{(1)} P P^T (Q^{(1)})^T K\right) + 2K^T Q^{(1)} P y(0) + \left(y(0)\right)^2 = G^T P.$$

Applying lemma (4.1), this relation reduces to the following relation

$$\begin{aligned} K^T Q^{(1-\mu)} P + a\left(K^T Q^{(1)} P + \overline{C_1}\right) + b\left(K^T Q^{(1)} H P\right) \\ + 2K^T Q^{(1)} \overline{C_2}\, P + \overline{C_3}\, P = G^T P, \end{aligned} \tag{4.11}$$

where $y(0) = \overline{C_1} P$, $P y(0) = \overline{C_2} P$ and $\left(y(0)\right)^2 = \overline{C_3} P$ can be calculated in the same way as (4.1).

By applying the typical Tau method see Canuto et al. (1988), a system of algebraic equation

$$K^T L = F, \tag{4.12}$$

$$L = Q^{(1-\mu)} + a\, Q^{(1)} + b\, Q^{(1)} H + Q^{(1)} \overline{C_2}\, P, \tag{4.13}$$

$$F = G^T - \overline{C_1} - \overline{C_3}, \tag{4.14}$$

is obtained.

5. Numerical results

In this section, we applied the method presented in this paper and solved some examples. The examples reported in this section were selected from a large collection of problem to which this method could be applied.

Example 1.

We consider the following fractional Riccati differential equation

$$D^{\mu} y(x) - 2\, y(x) + y^2(x) = g(x), \quad 0 < x < 1,\ 0 < \mu \le 1 \tag{5.1}$$

with initial condition

$$y(0) = 0. \tag{5.2}$$

The exact solution of this problem for $\mu = 1$ was found to be of the form

$$y(x) = 1 + \sqrt{2}\, \tanh\left(\sqrt{2} x + \frac{1}{2} \log\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)\right).$$

By using the method that was elaborated in previous section, we have the approximations (4.5)-(4.10).

Using (4.11) and Tau method, the problems (5.1)-(5.2) are transformed to the following relation

$$K^T\left(Q^{(1-\mu)}-2Q^{(1)}+Q^{(1)}H\right)=G^T. \quad (5.3)$$

Letting $m=2$, $\alpha=\beta=1$ and $\mu=\frac{1}{2}$, we obtain

$$Q^{\left(\frac{1}{2}\right)}=\begin{bmatrix}0.773748 & 0.21493\\ 0.401202 & 0.351704\end{bmatrix},\ Q^{(1)}=\begin{bmatrix}0.5 & 0.25\\ -0.4 & 0.0\end{bmatrix},\ G=\begin{bmatrix}1\\0\end{bmatrix},$$

$$H=\begin{bmatrix}0.773748k_0-0.687780k_1 & 0.967185k_0-0.859725k_1\\ 0.773748k_0-0.687780k_1 & 0.96718k_0-0.859725k_1\end{bmatrix}.$$

Now, from (5.3) we conclude that $K=[6.91457 \quad 8.40791]^T$. So

$$y(x)=1.0000x+6.1391\times 10^{-7}.$$

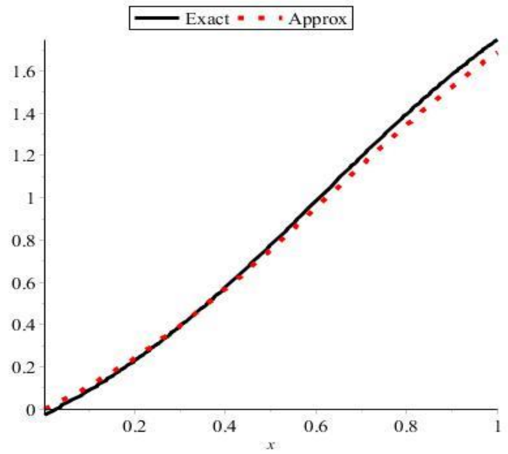

Figure1. The approximate solution in the case $\mu=1, m=5, \alpha=3$ and $\beta=2$ of Example1.

Example 2.

We consider the following fractional Riccati differential equation

$$D^{\mu}y(x)+y(x)-y^2(x)=0, \quad 0<x<1,\ 0<\mu\leq 1 \quad (5.4)$$

with initial condition

$$y(0)=\frac{1}{2}. \quad (5.6)$$

The exact solution of this problem is

$$y(x)=\frac{e^{-x}}{1+e^{-x}}.$$

By using the method that was elaborated in previous section, we have the approximations (4.5)-(4.10).

Using (4.11) and Tau method, the problems (5.4)-(5.5) are transformed to the following relation

$$K^T\left(Q^{(1-\mu)}+Q^{(1)}H\right)P+\frac{1}{4}=0. \tag{5.7}$$

Letting $m=3$, $\alpha=\beta=1$ and $\mu=0.8$, we obtain

$$Q^{(0.2)}=\begin{bmatrix} 0.935031 & 0.0703778 & -0.00754058 \\ -0.392111 & 0.727248 & 0.0446918 \\ 0.642244 & -0.437149 & 0.643899 \end{bmatrix},$$

$$Q^{(1)}=\begin{bmatrix} 0.5 & 0.16667 & 0.0 \\ -0.642857 & 0.0 & 0.0535714 \\ 1.33333 & -0.437149 & 0.0 \end{bmatrix},$$

$$H=\begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}.$$

It is easy to see that

$$-\frac{1}{4}=\begin{bmatrix} -\frac{1}{4} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 6x-3 \\ 28x^2-28x+6 \end{bmatrix}=\begin{bmatrix} -\frac{1}{4} & 0 & 0 \end{bmatrix}P.$$

Now, from (5.6) we conclude that $K=\begin{bmatrix}-63.4958 & -63.7014 & -25.8788\end{bmatrix}^T$. So,

$$y(x)=0.2351x^2-0.035180\,x-0.37578.$$

$$h_1=\begin{bmatrix} 0.608934k_0-0.881562k_1+0.725841k_2 \\ 0.214920k_0-0.140508k_1-1.41862k_2 \\ -0.0110388k_0-0.199772k_1+0.0245617k_2 \end{bmatrix},$$

$$h_2=\begin{bmatrix} 0.608934k_0-0.881562k_1+0.725841k_2 \\ 0.167160k_0+0.109284k_1-1.103372k_2 \\ -0.00858573k_0+1.55379k_1-0.190802k_2 \end{bmatrix},$$

$$h_3=\begin{bmatrix} 0.228350k_0-0.330586k_1+0.272190k_2 \\ 0.0805950k_0-0.526905k_1-0.531984k_2 \\ 0.00413955k_0+0.0749146k_1-0.00919940k_2 \end{bmatrix},$$

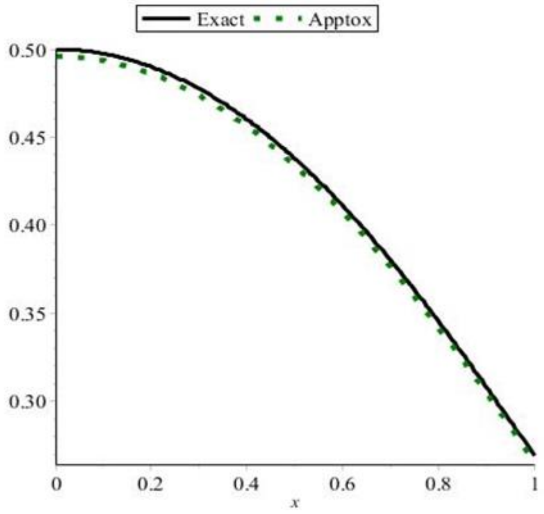

Figure 2. The approximate solution in the case $\mu = 1, m = 5, \alpha = 3$ and $\beta = 2$ of Example2.

Example 3.

As a final example, we consider the following fractional Riccati differential equation

$$D^{\mu} y(x) - y^2(x) = x^2, \quad 0 < x < 1, 0 < \mu \le 1 \tag{5.7}$$

with initial condition

$$y(0) = 1. \tag{5.8}$$

The exact solution of this problem is

$$y(x) = \frac{x\left(J_{\frac{-3}{4}}\left(\frac{x^2}{2}\right)\Gamma\left(\frac{1}{4}\right) + J_{\frac{3}{4}}\left(\frac{x^2}{2}\right)\Gamma\left(\frac{3}{4}\right)\right)}{J_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\Gamma\left(\frac{1}{4}\right) - 2J_{\frac{-1}{4}}\left(\frac{x^2}{2}\right)\Gamma\left(\frac{3}{4}\right)},$$

where $J_{\nu}(t)$ is the Bessel function of the first kind.

We suppose that $m = 3$, $\alpha = \beta = 1$ and $\mu = \frac{1}{2}$. It is easy to see that

$$x^2 + 1 = \begin{bmatrix} \frac{13}{15} & \frac{1}{30} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} 1 \\ 4x - 2 \\ 15x^2 - 2x + 3 \end{bmatrix} = \begin{bmatrix} \frac{13}{15} & \frac{1}{30} & \frac{1}{15} \end{bmatrix} P.$$

In the same way as in the previous examples, by using (4.11), the problems (5.7)-(5.8) are transformed to the following relation:

$$K^T \left(Q^{(\frac{1}{2})} - Q^{(1)} H - 2Q^{(1)} \right) = \begin{bmatrix} \frac{13}{15} & \frac{1}{30} & \frac{1}{15} \end{bmatrix} P.$$

Now, using Tau method, we reduce the problem to solve the following system of algebraic equation

$$K^T\left(Q^{(\frac{1}{2})}-Q^{(1)}H-2Q^{(1)}\right)=\begin{bmatrix}\frac{13}{15} & \frac{1}{30} & \frac{1}{15}\end{bmatrix}. \quad (5.9)$$

Now, from (5.9) we conclude

$$K=\begin{bmatrix}1.65702 & -2.45478 & 1.81873\end{bmatrix}^T.$$

So

$$y(x)=-2.554778x^2+3.33234x+2.79004.$$

In Table 1, the approximate solutions for test problems 1, 2 and 3 obtained by different values of m,μ,α and β using the presented method.

Table 1. The approximate solutions for examples 1, 2 and 3

EX	m	μ	α	β	y
$EX1$	2	0.5	1	1	$1.000x+6.1391\times10^{-7}$
	3	0.8	2	2	$0.96249x^2+0.80401x+6.1391\times10^{-7}$
	4	0.9	1	3	$0.21268x^3+0.65203x^2+0.99962x+7.2732\times10^{-7}$
	5	1.0	3	2	$-0.36169x^4+0.73652x^3+0.96249x^2+0.80401x+7.1306\times10^{-7}$
$EX2$	2	0.5	1	1	$-0.029326x+0.38318$
	3	0.8	2	2	$0.23478x^2-0.035191x+0.38941$
	4	0.9	1	3	$0.0031179x^3+0.052465x^2-0.032258x+0.41260$
	5	1.0	3	2	$-.00027185x^4+.00010234x^3-.28132x^2-.0000032236x+.52133$
$EX3$	2	0.5	1	1	$0.0001230466x+0.9931299$
	3	0.8	2	2	$0.6529539x^2+-0.5156453x+0.9761941$
	4	0.9	1	3	$6.608623x^3+0.6529539x^2-0.005250264x+0.9931489$
	5	1.0	3	2	$-14.36735x^4+14.45792x^3+.6528798x^2+1283269\times10^{-5}x+.9931370$

5. Conclusion

In this paper, we have proposed a numerical method for solving Riccati differential equation of fractional order. The shifted Jacobi polynomial integral operational matrix was developed to solve this equation. The numerical results showed this method is powerful, new and interesting. All of the numerical computations in this study have been done on a PC applying some programs written in MAPLE.

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