Applications and Applied Mathematics: An International Journal (AAM)

Volume 10 | Issue 2 | Article 11

12-2015

The fuzzy over-relaxed proximal point iterative scheme for generalized variational inclusion with fuzzy mappings

Rais Ahmad
Aligarh Muslim University

Mijanur Rahaman
Aligarh Muslim University

Haider A. Rizvi
Aligarh Muslim University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 10, Issue 2 (December 2015), pp. 805 – 816

Applications and Applied Mathematics:
An International Journal
(AAM)

The fuzzy over-relaxed proximal point iterative scheme for generalized variational inclusion with fuzzy mappings

Rais Ahmad, Mijanur Rahaman and Haider Abbas Rizvi

Department of Mathematics, Aligarh Muslim University
Aligarh-202002, Uttar Pradesh, India
raisain_123@rediffmail.com; mrahman96@yahoo.com, haider.alig.abbas@gmail.com

Received: January 26, 2015; Accepted: June 26, 2015

Abstract

This paper deals with the introduction of a fuzzy over-relaxed proximal point iterative scheme based on $H(\cdot,\cdot)$-cocoercivity framework for solving a generalized variational inclusion problem with fuzzy mappings. The resolvent operator technique is used to approximate the solution of generalized variational inclusion problem with fuzzy mappings and convergence of the iterative sequences generated by the iterative scheme is discussed. Our results can be treated as refinement of many previously-known results.

Keywords: Variational inclusion; Fuzzy mapping; Convergence; Over-relaxed

MSC 2010 No.: 47J22; 47J25; 47S40

1. Introduction

The fuzzy set theory which was introduced by Zadeh (1965) has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of the fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering sciences. Of course, fuzzy sets are powerful mathematical tools for modeling and controlling uncertain systems in industry and nature. They are helpful for approximating reasoning in decision making in the absence of complete and precise information.

Variational inequality theory provides us a unified frame work for dealing with a wide class of problems arising in elasticity, structural analysis, physical and engineering sciences, etc. (see Ahmad et al. (2000), Aubin et al. (1984), Baiocchi et al. (1984), Giannessi et al. (1995), Harker et al. (1990), Hassouni et al. (1994), and references therein). Variational inclusions are an important generalization of classical variational inequalities and thus have wide applications to many fields including mechanics, optimization and control, engineering sciences, and nonlinear programming (see Ahmad et al. (2005), Hassouni et al. (1994), Huang (2001)). Chang et al. (1989) first introduced the concept of variational inequalities for fuzzy mappings in abstract spaces. Since then, several classes of variational inequalities and complementarity problems with fuzzy mappings were studied by several authors (see Chang et al. (1999), Lan et al. (2008), Lee et al. (1999), Park et al. (1998)). Variational inequalities and related problems with fuzzy mappings have been useful in the study of equilibria and optimal control problem (see Chang et al. (2000)).

Recently, Verma (2009) introduced and studied the over-relaxed A-proximal point algorithm based on A-maximal monotonicity to approximate the solvability of a general variational inclusion problem. In 2011, Pan et al. introduced the over-relaxed proximal point algorithm based on (A, η)-accretive mapping for approximation solvability of a general class of variational inclusion problems. On the other hand, Li (2012) also studied over-relaxed proximal point algorithms for approximating solvability of some nonlinear operator equation. Quite reasonable work is done in this direction to solve some classes of variational inclusion problems. For more details of the related work, we refer to Deepmala (2014), Deepmala et al. (2013), Husain et al. (2013a), Husain et al. (2013b), Husain et al. (2013c), Mishra (2007), Pathak et al. (2013) and references therein.

Motivated and inspired by recent research works mentioned above, in this manuscript we introduce the fuzzy over-relaxed proximal point iterative scheme based on $H(\cdot,\cdot)$-cocoercive operators (Ahmad et al. (2011)), since cocoercive operators are generalized forms of monotone mappings and accretive mappings. By using this new scheme, we approximate the solvability of a generalized variational inclusion problem with fuzzy mappings and discuss the convergence of fuzzy over-relaxed proximal point iterative scheme generated by the suggested iterative algorithms in Hilbert space.

2. Preliminaries

Let X be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(X)$ be a collection of all fuzzy sets over X. A mapping $F : X \longrightarrow \mathcal{F}(X)$ is called a fuzzy mapping. For each $x \in X$, $F(x)$ (denote it by F_x in the sequel) is a fuzzy set on X and $F_x(y)$ is the membership function of y in F_x.

A fuzzy mapping $F : X \longrightarrow \mathcal{F}(X)$ is said to be closed if, for each $x \in X$, the function $y \to F_x(y)$ is upper semicontinuous, i.e., for any given net $\{y_\alpha\} \in X$ satisfying $y_\alpha \to y_0 \in X$, $\limsup_\alpha F_x(y_\alpha) \leq F_x(y_0)$. For $B \in \mathcal{F}(X)$ and $\lambda \in [0,1]$, the set $(B)_\lambda = \{x \in X : B(x) \geq \lambda\}$ is called a λ-cut set of B.

A closed fuzzy mapping $F : X \longrightarrow \mathcal{F}(X)$ is said to satisfy the $condition(*)$: if there exists a mapping $a : X \longrightarrow [0,1]$ such that for each $x \in X$, then $(F_x)_{a(x)} = \{y \in X : F_x(y) \geq a(x)\}$ is a nonempty bounded subset of X. Clearly, F is a closed fuzzy mapping satisfying $condition(*)$, then for each $x \in X$, the set $(F_x)_{a(x)} \in CB(X)$, where $CB(X)$ denotes the family of all nonempty bounded closed subsets of X. In fact, let $\{y_\alpha\}_{\alpha\in\Lambda} \in (F_x)_{a(x)}$ be a net and $y_\alpha \to y_0 \in X$. Then, $F_x(y_\alpha) \geq a(x)$, for each $\alpha \in \Lambda$. Since F is closed, we have

$$F_x(y_0) \geq \limsup_{\alpha\in\Lambda} F_x(y_\alpha) \geq a(x),$$

which implies that $y_0 \in (F_x)_{a(x)}$ and hence, $(F_x)_{a(x)} \in CB(X)$.

The following definitions and results are needed to prove the main result of this paper.

Definition. A mapping $g : X \longrightarrow X$ is said to be

(i) Lipschitz continuous if there exists a constant $\lambda_g > 0$ such that

$$\|g(x) - g(y)\| \leq \lambda_g \|x - y\|, \quad \forall x, y \in X;$$

(ii) monotone if

$$\langle g(x) - g(y), x - y\rangle \geq 0, \quad \forall x, y \in X;$$

(iii) α-expansive if there exists a constant $\alpha > 0$ such that

$$\|g(x) - g(y)\| \geq \alpha\|x - y\|, \quad \forall x, y \in X;$$

If $\alpha = 1$, then it is expansive.

Definition. A multi-valued mapping $T : X \longrightarrow CB(X)$ is said to be $\mathcal{D}$-Lipschitz continuous if there exists a constant $\delta_T > 0$ such that

$$\mathcal{D}\big(T(x), T(y)\big) \leq \delta_T\|x - y\|, \quad \forall x, y \in X,$$

where $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric defined on $CB(X)$.

Definition. A mapping $T : X \longrightarrow X$ is said to be cocoercive if there exists a constant $\varrho > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \varrho\|T(x) - T(y)\|^2, \quad \forall x, y \in X.$$

Definition. A multi-valued mapping $M : X \longrightarrow 2^X$ is said to be cocoercive if there exists a constant $\xi > 0$ such that

$$\langle u - v, x - y\rangle \geq \xi\|u - v\|^2, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y).$$

Definition. Let $H : X \times X \longrightarrow X$ and $A, B : X \longrightarrow X$ be mappings. Then

(i) $H(A,\cdot)$ is said to be cocoercive with respect to A if for a fixed $u \in X$, there exists a constant $\mu > 0$ such that

$$\langle H(A(x), u) - H(A(y), u), x - y\rangle \geq \mu\|A(x) - A(y)\|^2, \quad \forall x, y \in X;$$

(ii) $H(\cdot, B)$ is said to be relaxed cocoercive with respect to B if for a fixed $u \in X$ there exists a constant $\gamma > 0$ such that

$$\langle H(u, B(x)) - H(u, B(y)), x - y \rangle \geq (-\gamma)\|B(x) - B(y)\|^2, \quad \forall x, y \in X;$$

(iii) $H(A, \cdot)$ is said to be ρ-Lipschitz with respect to A if there exists a constant $\rho > 0$ such that

$$\|H(A(x), \cdot) - H(A(y), \cdot)\| \leq \rho\|x - y\|, \quad \forall x, y \in X;$$

(iv) $H(\cdot, B)$ is said to be ζ-Lipschitz with respect to B if there exists a constant $\zeta > 0$ such that

$$\|H(\cdot, B(x)) - H(\cdot, B(y))\| \leq \zeta\|x - y\|, \quad \forall x, y \in X.$$

Definition. A sequence $\{x_i\}$ is said to converge linearly to x^* if there exists a constant $0 < c < 1$ such that

$$\|x_{i+1} - x^*\| \leq c\|x_i - x^*\|,$$

for all $i > m$ for some natural number $m > 0$.

Definition. (Ahmad et al. (2011)) Let $A, B : X \longrightarrow X$ and $H : X \times X \longrightarrow X$ be three single-valued mappings and $M : X \longrightarrow 2^X$ be a multi-valued mapping. The mapping M is said to be $H(\cdot, \cdot)$-cocoercive with respect to A and B if H is μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, M is cocoercive, and $[H(A, B) + \lambda M](X) = X$, for every $\lambda > 0$.

Theorem 1. (Ahmad et al. (2011))

Let $H(\cdot, \cdot)$ be μ-cocoercive with respect to A, γ-relaxed cocoercive with respect to B, A be α-expansive, B be β-Lipschitz continuous, $\mu > \gamma$, and $\alpha \geq \beta$. Let M be an $H(\cdot, \cdot)$-cocoercive operator with respect to A and B. Then the resolvent operator $R^H_{\lambda,M} : X \longrightarrow X$, defined by

$$R^H_{\lambda,M}(u) = [H(A, B) + \lambda M]^{-1}(u), \quad \forall u \in X,$$

is single-valued and $\dfrac{1}{\mu\alpha^2 - \gamma\beta^2}$-Lipschitz continuous.

Let $T, F : X \longrightarrow \mathcal{F}(X)$ be closed fuzzy mappings satisfying $condition(*)$. Then, there exists mappings $a, b : X \longrightarrow [0, 1]$ such that for each $x \in X$, we have $(T_x)_{a(x)}, (F_x)_{b(x)} \in CB(X)$. Therefore, we can define the multi-valued mappings $\widetilde{T}, \widetilde{F} : X \longrightarrow CB(X)$ by

$$\widetilde{T}(x) = (T_x)_{a(x)}, \quad \widetilde{F}(x) = (F_x)_{b(x)}, \quad \forall x \in X.$$

In the sequel, $\widetilde{T}$ and $\widetilde{F}$ are called multi-valued mappings induced by the fuzzy mappings T and F, respectively.

Let $S, H : X \times X \longrightarrow X$, let $A, B, g : X \longrightarrow X$ be single-valued mappings, and $T, F : X \longrightarrow \mathcal{F}(X)$ be fuzzy mappings. Let the multi-valued mapping $M : X \longrightarrow 2^X$ be $H(\cdot, \cdot)$-cocoercive with respect to A and B such that $g(x) \cap dom(M) \neq \emptyset$. We consider the following generalized variational inclusion problem with fuzzy mappings:

Find $x \in X$, $u \in (T_x)_{a(x)}$ and $v \in (F_x)_{b(x)}$ such that

$$0 \in S(u, v) + M(g(x)). \tag{1}$$

If $T, F : X \longrightarrow CB(X)$ are the classical multi-valued mappings, we can define the fuzzy mappings T, F by

$$x \mapsto \chi_{T(x)}, \quad x \mapsto \chi_{F(x)},$$

where $\chi_{T(x)}, \chi_{F(x)}$ are the characteristic functions of $T(x)$ and $F(x)$, respectively. Taking $a(x) = b(x) = 1$, $g = I$, for all $x \in X$, then Problem (1) is equivalent to the following problem:

Find $x \in X$, $u \in T(x)$ and $v \in F(x)$ such that

$$0 \in S(u, v) + M(x). \tag{2}$$

Problem (2) was considered by Verma (2008) and many other authors in different settings.

It is clear that for suitable choices of operators involved in the formulation of Problem (1), one can obtain many variational inclusions studied in recent past.

In support of Problem (1), we provide the following example.

Example 1.

Let $X = [0, 1]$, and suppose that

(i) the closed fuzzy mappings $T, F : X \longrightarrow \mathcal{F}(X)$ are defined by

$$T_x(u) = \begin{cases} \frac{x+u}{2} & , \text{ if } x \in [0, \frac{1}{2}), u \in [0, 1]; \\ (1-x)u & , \text{ if } x \in [\frac{1}{2}, 1], u \in [0, 1], \end{cases}$$

and

$$F_x(v) = \begin{cases} 2xv & , \text{ if } x \in [0, \frac{1}{2}), v \in [0, 1]; \\ (1-x) + \frac{v}{2} & , \text{ if } x \in [\frac{1}{2}, 1], v \in [0, 1]. \end{cases}$$

(ii) the mappings $a, b : X \longrightarrow [0, 1]$ are defined by

$$a(x) = \begin{cases} \frac{x}{2} & , \text{ if } x \in [0, \frac{1}{2}); \\ 0 & , \text{ if } x \in [\frac{1}{2}, 1], \end{cases}$$

and

$$b(x) = \begin{cases} 0 & , \text{ if } x \in [0, \frac{1}{2}); \\ (1-x)x & , \text{ if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $T_x(u) \geq a(x)$ and $F_x(v) \geq b(x)$, for all $x, u, v \in X$. Also,

(iii) the mappings $S : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are defined by

$$S(u, v) = \begin{cases} u - v & , \text{ if } u > v; \\ v - u & , \text{ if } u < v; \\ 0 & , \text{ if } u = v, \end{cases}$$

and

$$g(x) = \frac{x}{2-x}, \quad \forall x \in [0, 1].$$

(iv) the mappings $A, B : X \longrightarrow X$ are defined by

$$A(x) = \frac{x}{2}, \quad B(x) = -x, \quad \forall x \in [0,1].$$

Suppose that $H(A, B) : X \times X \longrightarrow X$ is defined by

$$H(A(x), B(y)) = A(x) + B(y), \quad \forall x, y \in [0,1].$$

Now,

$$\begin{aligned} \langle H(A(x), u) - H(A(y), u), x - y \rangle &= \langle A(x) - A(y), x - y \rangle \\ &= \left\langle \frac{x-y}{2}, x - y \right\rangle \\ &= \frac{(x-y)^2}{2}. \end{aligned}$$

Also,

$$\begin{aligned} \|A(x) - A(y)\|^2 &= \langle A(x) - A(y), A(x) - A(y) \rangle \\ &= \left\langle \frac{x-y}{2}, \frac{x-y}{2} \right\rangle \\ &= \frac{(x-y)^2}{4}, \end{aligned}$$

which implies that

$$\langle H(A(x), u) - H(A(y), u), x - y \rangle = 2\|A(x) - A(y)\|^2,$$

i.e., $H(A, B)$ is 2-cocoercive with respect to A.
Further,

$$\begin{aligned} \langle H(u, B(x)) - H(u, B(y)), x - y \rangle &= \langle B(x) - B(y), x - y \rangle \\ &= \langle -(x - y), x - y \rangle \\ &= -(x-y)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \|B(x) - B(y)\|^2 &= \langle B(x) - B(y), B(x) - B(y) \rangle \\ &= \langle -(x - y), -(x - y) \rangle \\ &= (x-y)^2, \end{aligned}$$

which implies that

$$\langle H(u, B(x)) - H(u, B(y)), x - y \rangle = (-1)\|B(x) - B(y)\|^2,$$

i.e., $H(A, B)$ is 1-relaxed cocoercive with respect to B.

Let $M : X \longrightarrow 2^X$ be defined by $M(g(x)) = \{g(x)\}$, for all $x \in [0,1]$ such that

$g(x) \cap dom(M) \neq \emptyset$. Then, M is cocoercive and for any $\lambda > 0$, one can easily check that $[H(A,B)+\lambda M](X) = X$. Therefore, M is $H(\cdot,\cdot)$-cocoercive with respect to A and B. In view of assumptions $(i)-(iv)$, it is easy to perceive that all the conditions of Problem (1) are satisfied.

3. The Fuzzy Over-Relaxed Proximal Point Scheme and Existence Results

First of all, we show that the generalized variational inclusion problem (1) is equivalent to the following fixed point equation.

Lemma 1.

The triplet (x, u, v), where $x \in X$, $u \in (T_x)_{a(x)}$ and $v \in (F_x)_{b(x)}$, is the solution of generalized variational inclusion problem (1) if and only if it satisfies the equation

$$g(x) = R^H_{\lambda,M}[H(A(g(x)), B(g(x))) - \lambda S(u, v)], \tag{3}$$

where $R^H_{\lambda,M}(x) = [H(A,B) + \lambda M]^{-1}(x)$ and $\lambda > 0$ is a constant.

Proof:

The proof is a direct consequence of the application of definition of the resolvent operator $R^H_{\lambda,M}(\cdot)$.
□

The fuzzy over-relaxed scheme 1: Step 1. Choose the arbitrary initial points $x_0 \in X$, $u_0 \in (T_{x_0})_{a(x_0)}$ and $v_0 \in (F_{x_0})_{b(x_0)}$.

Step 2. Compute the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ by the following iterative scheme:

$$g(x_{n+1}) = (1-\alpha_n)g(x_n) + \alpha_n y_n, \quad n \geq 0, \tag{4}$$

and y_n satisfies

$$\|y_n - R^H_{\lambda,M}[H(A(g(x_n)), B(g(x_n))) - \lambda S(u_n, v_n)]\| \leq \sigma_n \|y_n - g(x_n)\|, \tag{5}$$

where $x_n \in X$, $u_n \in (T_{x_n})_{a(x_n)}$ and $v_n \in (F_{x_n})_{b(x_n)}$, and $\{\alpha_n\} \subseteq [0, \infty)$ is a sequence of over-relaxed factors, $\{\sigma_n\}$ is a scalar sequence, $n \geq 0$, $\lambda > 0$, $\sum_{n=0}^{\infty} \sigma_n < \infty$, $\sigma_n \to 0$ and $\alpha = \lim_{n\to\infty} \sup \alpha_n < 1$.

Step 3. Obtain the estimates

$$\begin{aligned} \|u_n - u\| &\leq \mathcal{D}\left((T_{x_n})_{a(x_n)}, (T_x)_{a(x)}\right), \\ \|v_n - v\| &\leq \mathcal{D}\left((F_{x_n})_{b(x_n)}, (F_x)_{b(x)}\right), \end{aligned} \tag{6}$$

where $u_n \in (T_{x_n})_{a(x_n)}$, $u \in (T_x)_{a(x)}$, $v_n \in (F_{x_n})_{b(x_n)}$ and $v \in (F_x)_{b(x)}$.

Step 4. If $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ satisfy (4), (5) and (6), respectively, to an amount of accuracy, stop. Otherwise, set $n = n+1$ and repeat Steps 2 and 3.

Remark 1: For suitable choices of operators involved in scheme 1, one can obtain over-relaxed proximal point algorithm studied by Verma (2009).

Theorem 2.

Let X be a real Hilbert space. Let $A, B, g : X \longrightarrow X$ and $S, H : X \times X \longrightarrow X$ be single-valued mappings such that $H(A, B)$ is μ-cocoercive with respect to A and γ-relaxed cocoercive with respect to B, A is α-expansive, B is β-Lipschitz continuous, $\mu > \gamma$, and $\alpha \geq \beta$. Also, let H be r_1-Lipschitz continuous with respect to A, r_2-Lipschitz continuous with respect to B, and g be λ_g-Lipschitz continuous and ξ-strongly monotone mapping. Let the multi-valued mapping $M : X \longrightarrow 2^X$ be $H(\cdot,\cdot)$-cocoercive. Let $T, F : X \longrightarrow \mathcal{F}(X)$ be closed fuzzy mappings satisfying $condition(*)$ and $\widetilde{T}, \widetilde{F} : X \longrightarrow CB(X)$ be multi-valued mappings induced by the fuzzy mappings T and F, respectively. Suppose that $\widetilde{T}$ and $\widetilde{F}$ are $\mathcal{D}$-Lipschitz continuous mappings with constants δ_T and δ_F, respectively. If for some $\lambda > 0$, the following condition holds:

$$(1-\alpha)\lambda_g + \alpha\theta\lambda_g(r_1 + r_2) + \alpha\theta\lambda\{\lambda_{S_2}\delta_F + \lambda_{S_1}\delta_T\} < 1,$$

for $\theta = \dfrac{1}{\mu\alpha^2 - \gamma\beta^2}$, $\{\alpha_n\} \subseteq [0,\infty)$ is a sequence of over-relaxed factors, $\{\sigma_n\}$ is a scalar sequence such that $\sum_{n=0}^{\infty} \sigma_n < \infty$, $\sigma_n \to 0$, and $\alpha = \limsup_{n\to\infty} \alpha_n < 1$. Then, the generalized variational inclusion problem (1) is solvable and (x^*, u^*, v^*), where $x^* \in X$, $u^* \in (T_{x^*})_{a(x^*)}$, $v^* \in (F_{x^*})_{b(x^*)}$, is the solution of Problem (1), and the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ defined in fuzzy over-relaxed scheme (1) converge linearly to x, u and v, respectively.

Proof:

Let x^* be a solution of generalized variational inclusion problem (1). Then by Lemma , it follows that

$$g(x^*) = (1-\alpha_n)g(x^*) + \alpha_n R^H_{\lambda,M}[H(A(g(x^*)), B(g(x^*))) - \lambda S(u^*, v^*)], \tag{7}$$

where $x^* \in X$, $u^* \in (T_{x^*})_{a(x^*)}$, $v^* \in (F_{x^*})_{b(x^*)}$, and $\lambda > 0$ is a constant.

Let

$$g(z_{n+1}) = (1-\alpha_n)g(x_n) + \alpha_n R^H_{\lambda,M}[H(A(g(x_n)), B(g(x_n))) - \lambda S(u_n, v_n)], \tag{8}$$

for all $n \geq 0$, $x_n \in X$, $u_n \in (T_{x_n})_{a(x_n)}$ and $v_n \in (F_{x_n})_{b(x_n)}$.

Using Cauchy-Schwartz inequality and ξ-strongly monotonicity of g, we have

$$\begin{aligned} \|g(z_{n+1}) - g(x^*)\|\|z_{n+1} - x^*\| &\geq \langle g(z_{n+1}) - g(x^*), z_{n+1} - x^*\rangle \\ &\geq \xi\|z_{n+1} - x^*\|^2. \end{aligned} \tag{9}$$

It follows from (9) that

$$\|z_{n+1} - x^*\| \leq \frac{1}{\xi}\|g(z_{n+1}) - g(x^*)\|. \tag{10}$$

Using the Lipschitz continuity of the resolvent operator $R^H_{\lambda,M}$, Lipschitz continuity of H in both the arguments with respect to A and B, respectively, Lipschitz continuity of S in both the arguments, $\mathcal{D}$-Lipschitz continuity of $\widetilde{T}$ and $\widetilde{F}$, Lipschitz continuity and strongly monotonicity of g, we obtain

$$\begin{aligned}
& \|g(z_{n+1})-g(x^*)\| \\
= & \left\|(1-\alpha_n)(g(x_n)-g(x^*))+\alpha_n R^{H}_{\lambda,M}[H(A(g(x_n)),B(g(x_n)))-\lambda S(u_n,v_n)]\right. \\
& \left.-\alpha_n R^{H}_{\lambda,M}[H(A(g(x^*)),B(g(x^*)))-\lambda S(u^*,v^*)]\right\| \\
\leq & (1-\alpha_n)\|g(x_n)-g(x^*)\|+\alpha_n\theta\lambda\|S(u_n,v_n)-S(u^*,v^*)\| \\
& +\alpha_n\theta\|H(A(g(x_n)),B(g(x_n)))-H(A(g(x^*)),B(g(x^*)))\| \\
= & (1-\alpha_n)\|g(x_n)-g(x^*)\|+\alpha_n\theta\lambda\|S(u_n,v_n)-S(u_n,v^*)+S(u_n,v^*) \\
& -S(u^*,v^*)\|+\alpha_n\theta\|H(A(g(x_n)),B(g(x_n)))-H(A(g(x_n)),B(g(x^*))) \\
& +H(A(g(x_n)),B(g(x^*)))-H(A(g(x^*)),B(g(x^*)))\| \\
\leq & (1-\alpha_n)\|g(x_n)-g(x^*)\|+\alpha_n\theta\lambda\|S(u_n,v_n)-S(u_n,v^*)\|+\alpha_n\theta\lambda\|S(u_n,v^*) \\
& -S(u^*,v^*)\|+\alpha_n\theta\|H(A(g(x_n)),B(g(x_n)))-H(A(g(x_n)),B(g(x^*)))\| \\
& +\alpha_n\theta\|H(A(g(x_n)),B(g(x^*)))-H(A(g(x^*)),B(g(x^*)))\| \\
\leq & (1-\alpha_n)\|g(x_n)-g(x^*)\|+\alpha_n\theta\lambda\lambda_{S_2}\|v_n-v^*\|+\alpha_n\theta\lambda\lambda_{S_1}\|u_n-u^*\| \\
& +\alpha_n\theta r_2\|g(x_n)-g(x^*)\|+\alpha_n\theta r_1\|g(x_n)-g(x^*)\| \\
\leq & (1-\alpha_n)\lambda_g\|x_n-x^*\|+\alpha_n\theta\lambda\lambda_{S_2}\mathcal{D}\left((F_{x_n})_{b(x_n)},(F_{x^*})_{b(x^*)}\right) \\
& +\alpha_n\theta\lambda\lambda_{S_1}\mathcal{D}\left((T_{x_n})_{a(x_n)},(T_{x^*})_{a(x^*)}\right)+\alpha_n\theta(r_1+r_2)\lambda_g\|x_n-x^*\| \\
\leq & (1-\alpha_n)\lambda_g\|x_n-x^*\|+\alpha_n\theta\lambda\lambda_{S_2}\delta_F\|x_n-x^*\|+\alpha_n\theta\lambda\lambda_{S_1}\delta_T\|x_n-x^*\| \\
& +\alpha_n\theta(r_1+r_2)\lambda_g\|x_n-x^*\|. \qquad (11)
\end{aligned}$$

It follows from (11) that

$$\|g(z_{n+1})-g(x^*)\|\leq P(\theta_n)\|x_n-x^*\|, \qquad (12)$$

where

$$P(\theta_n)=\left[(1-\alpha_n)\lambda_g+\alpha_n\theta(r_1+r_2)\lambda_g+\alpha_n\theta\lambda\{\lambda_{S_1}\delta_T+\lambda_{S_2}\delta_F\}\right], \qquad (13)$$

and $\theta=\dfrac{1}{\mu\alpha^2-\gamma\beta^2}$, for $\mu>\gamma$ and $\alpha\geq\beta$.

Using (12), (10) becomes

$$\|z_{n+1}-x^*\|\leq\frac{1}{\xi}P(\theta_n)\|x_n-x^*\|, \qquad (14)$$

where $P(\theta_n)$ is defined by (13).

From (4), we have $g(x_{n+1})=(1-\alpha_n)g(x_n)+\alpha_n y_n$, which implies that

$$g(x_{n+1})-g(x_n)=\alpha_n(y_n-g(x_n)). \qquad (15)$$

Using the same arguments as for (10), we obtain

$$\|x_{n+1}-z_{n+1}\|\leq\frac{1}{\xi}\|g(x_{n+1})-g(z_{n+1})\|. \qquad (16)$$

By applying (5), it follows that

$$\begin{aligned}\|g(x_{n+1})-g(z_{n+1})\| &= \Big\|(1-\alpha_n)g(x_n)+\alpha_n y_n-\big\{(1-\alpha_n)g(x_n)\\ &\quad +\alpha_n R^{H}_{\lambda,M}[H(A(g(x_n)),B(g(x_n)))-\lambda S(u_n,v_n)]\big\}\Big\|\\ &= \Big\|\alpha_n\big\{y_n-R^{H}_{\lambda,M}[H(A(g(x_n)),B(g(x_n)))-\lambda S(u_n,v_n)]\big\}\Big\|\\ &\le \alpha_n\sigma_n\|y_n-g(x_n)\|. \end{aligned} \tag{17}$$

Making use of (15), (17) and Lipschitz continuity of g, (16) becomes

$$\begin{aligned}\|x_{n+1}-z_{n+1}\| &\le \frac{1}{\xi}\alpha_n\sigma_n\|y_n-g(x_n)\|\\ &= \frac{1}{\xi}\sigma_n\|\alpha_n(y_n-g(x_n))\|\\ &= \frac{1}{\xi}\sigma_n\|g(x_{n+1})-g(x_n)\|\\ &\le \frac{1}{\xi}\sigma_n\lambda_g\|x_{n+1}-x_n\|. \end{aligned} \tag{18}$$

Using the above-discussed arguments, we estimate

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|x_{n+1}-z_{n+1}+z_{n+1}-x^*\|\\ &\le \|x_{n+1}-z_{n+1}\|+\|z_{n+1}-x^*\|\\ &\le \frac{1}{\xi}\sigma_n\lambda_g\|x_{n+1}-x_n\|+\frac{1}{\xi}P(\theta_n)\|x_n-x^*\|\\ &\le \frac{1}{\xi}\sigma_n\lambda_g\|x_{n+1}-x^*\|+\frac{1}{\xi}\sigma_n\lambda_g\|x_n-x^*\|+\frac{1}{\xi}P(\theta_n)\|x_n-x^*\|, \end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|\le\frac{\sigma_n\lambda_g+P(\theta_n)}{\xi-\sigma_n\lambda_g}\|x_n-x^*\|. \tag{19}$$

Inequality (19) implies that the sequence $\{x_n\}$ converges linearly to x^*, for

$$P(\theta_n)=\left[(1-\alpha_n)\lambda_g+\alpha_n\theta(r_1+r_2)\lambda_g+\alpha_n\theta\lambda\{\lambda_{S_1}\delta_T+\lambda_{S_2}\delta_F\}\right],$$

and $\theta=\dfrac{1}{\mu\alpha^2-\gamma\beta^2}$ for $\mu>\gamma$ and $\alpha\ge\beta$.

It follows from (6) and the $\mathcal{D}$-Lipschitz continuity for $(T_x)_{a(x)}$ and $(F_x)_{b(x)}$ that the sequences $\{u_n\}$ and $\{v_n\}$ converge linearly to u and v, respectively, as $\{x_n\}$ converges to x^* linearly. Thus, we have

$$\begin{aligned}\limsup_n\frac{\sigma_n\lambda_g+P(\theta_n)}{\xi-\sigma_n\lambda_g} &= \limsup_n P(\theta_n)\\ &= \limsup_n\left[(1-\alpha_n)\lambda_g+\alpha_n\theta(r_1+r_2)\lambda_g+\alpha_n\theta\lambda\{\lambda_{S_1}\delta_T+\lambda_{S_2}\delta_F\}\right]\\ &= (1-\alpha)\lambda_g+\alpha\theta(r_1+r_2)\lambda_g+\alpha\theta\lambda\{\lambda_{S_1}\delta_T+\alpha\theta\lambda\lambda_{S_2}\delta_F\}, \end{aligned}$$

where $\sum_{n=0}^{\infty}\sigma_n<\infty$ and $\alpha=\lim_{n\to\infty}\sup\alpha_n$. This completes the proof. □

4. Conclusion

Fuzzy sets and fuzzy logic are powerful mathematical tools for modeling and controlling uncertain systems in industry, humanity, and nature; they are expeditious for approximating reasoning in decision making in the absence of complete and precise information. Their role is significant when applied to complex phenomena, not easily described by traditional mathematics. On the other hand, it is well recognized that variational inclusions provide us fundamental tools to solve many problems in applied sciences.

In this paper, we generalized the concept of over-relaxed proximal point scheme with fuzzy mappings and apply it to solve a generalized variational inclusion problem with fuzzy mappings by using the concept of $H(\cdot,\cdot)$-cocoercive mappings which are more general than monotone mappings. We also discuss the convergence of the sequences generated by over-relaxed proximal point scheme with fuzzy mappings by using the concept of linear convergence.

REFERENCES

Ahmad, R., Dilshad, M., Wong, M.W. and Yao, J.C. (2011). $H(\cdot,\cdot)$-cocoercive Operator and an application for solving generalized variational inclusions, Abstract and Applied Analysis, Vol. 2011, Article ID 261534, 12 pages, DOI: 1155/ 2011/ 261534.

Ahmad, R. and Ansari, Q.H. (2000). An iterative algorithm for generalized nonlinear variational inclusions, Applied Mathematics Letters, Vol. 13, pp. 23-26.

Ahmad, R., Ansari, Q.H. and Irfan, S.S.(2005). Generalized variational inclusions and generalized resolvent equations in Banach spaces, Computers & Mathematics with Applications, Vol. 29, No. 11-12, pp. 1825-1835.

Aubin, J.-P. and Ekeland, I. (1984). *Applied nonlinear analysis*, Pure and Applied Mathematics (New York), John Wiley and Sons, New York.

Baiocchi, C. and Capelo, A. (1984). *Variational and quasivariational inequalities*, Wiley, New York (1984).

Chang, M.S. and Chen, H.Y. (2000). A fuzzy user-optimal route choice problem using a link based fuzzy variational inequality formulation, Fuzzy Sets and Systems, Vol. 114, No. 2, pp. 339-345.

Chang, S.S., Lee, G.M. and Lee, B.S. (1999). Vector quasi variational inequalities for fuzzy mappings (II), Fuzzy Sets and Systems, Vol. 102, pp. 333-344.

Chang, S.S. and Zhu, Y.G. (1989). On variational inequalities for fuzzy mappings, Fuzzy Sets and System, Vol. 32, pp. 359-367.

Deepmala (2014). *A study on fixed point theorems for nonlinear contractions and its applications*, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur-492010, India.

Deepmala and Pathak, H.K. (2013). A study on some problems on existence of solutions for nonlinear functional-integral equations, Acta Mathematica Scientia, Vol. 33, No. B-5, pp. 1305-1313.

Giannessi, F. and Maugeri, A. (1995). *Variational inequalities and Network equilibrium problems*, Plenum, New York.

Harker, P.T. and Pang, J.S. (1990). Finite dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, Mathematical Programming, Vol. 48, pp. 161-220.

Hassouni, A. and Moudafi, A. (1994). A pertubed algorithm for variational inclusions, Journal of Mathematical Analysis and Applications, Vol. 185, pp. 706-712.

Huang, N.J. (2001). A new class of generalized set-valued implicit variational inclusions in Banach spaces with an applications, Computers & Mathematics with Applications, Vol. 41, pp. 937-943.

Husain, S., Gupta, S. and Mishra, V.N. (2013a). An existence theorem of solutions for the systems of generalized vector quasi-variational inequalities, American Journal of Operations Reseach, Vol. 3, pp. 329-336, DOI:10.4236/ajor.2013.33029.

Husain, S., Gupta, S. and Mishra, V.N. (2013b). Generalized $H(\cdot,\cdot,\cdot)$-η-cocoercive operators and generalized set-valued variational-like inclusions, Journal of Mathematics, Vol. 2013, Article ID 738491, 10 pages, DOI:10.1155/2013/738491.

Husain, S., Gupta, S. and Mishra, V.N. (2013c). Graph convergence for the $H(\cdot,\cdot)$-mixed mapping with an application for solving the system of generalized variational inclusions, Fixed Point Theory and Applications, Vol. 2013, No. 304, DOI:10.1186/1687-1812-2013-304.

Lan, H.Y. and Verma, R.U. (2008). Iterative algorithms for nonlinear fuzzy variational inclusions with $(A,\eta,)$-accretive mappings in Banach space, Advances in Nonlinear Variational Inequalities, Vol. 11, No. 1, pp. 15-30.

Lee, G.M., Kim. D.S. and Lee, B.S. (1999). Vector variational inequality for fuzzy mappings, Nonlinear Analysis Forum, Vol. 4, pp. 119-129.

Li, F. (2012). On over-relaxed proximal point algorithm for generalized nonlinear operator equation with (A,η,m)-monotonicity framework, International Journal of Modern Nonlinear Theory and Application, Vol. 1, pp. 67-72.

Mishra, V.N. (2007). *Some problems on approximations of functions in Banach spaces*. Ph.D. Thesis, IIT(Roorkee)-247667, India.

Pan, X.B., Li, H.G. and Xu, A.J. (2011). The over-relaxed A-proximal point algorithm for general nonlinear mixed set-valued inclusion framework, Fixed Point Theory and Applications, Vol. 2011, Article ID 840978, DOI: 10.1155/2011/840978.

Park, S., Lee, B.S. and Lee, G.M. (1998). A general vector valued variational inequality and its fuzzy extension, International Journal of Mathematics and Mathematical Sciences, Vol. 21, pp. 637-642.

Pathak, H.K. and Deepmala (2013). Common fixed point theorems for PD-operator pair under relaxed conditions with applications, Journal of Computational and Applied Mathematics, Vol. 239, pp. 103-113.

Verma, R.U. (2009). A general framework for the over-relaxed A-proximal point algorithm and applications to inclusion problems, Applied Mathematics Letters, Vol. 22, No. 5, pp. 698-703.

Verma, R.U. (2008). Approximation solvability of a class of nonlinear set-valued variational inclusions involving (A,η)-monotone mappings, Journal of Mathematical Analysis and Applications, Vol. 337, pp. 969-975.

Zadeh, L.A. (1965). Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353.