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Optimal Error Estimate of Upwind Scheme on Adaptive grid for Two Parameter Singular Perturbation Problem

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Abstract

A singularly perturbed convection-diffusion problem with two small parameters is considered. The problem is solved by an upwind finite difference operator on an appropriate non-uniform mesh constructed adaptively by equi-distributing a monitor function based on the solution. An error bound in the maximum norm is established theoretically with the error constants shown to be independent of both singular perturbation parameters. The normalized flux obtained via interpolating the polynomial from the numerical solution is also uniformly convergent. A numerical experiment illustrates in practice the result of convergence proved theoretically.

Keywords: Singular perturbation; Two parameter problems; Boundary layer; Upwind scheme; Adaptive mesh; Normalized flux; Uniform convergence

AMS 2010 Subject Classification: 65L10, 65L12

1. Introduction

In this article, we consider the following singularly perturbed two parameters boundary value problem:

$$\begin{aligned} Lu(x) &\equiv -\varepsilon u''(x) - \mu p(x)u'(x) + q(x)u(x) \\ &= f(x), \qquad x \in \Omega = (0,1), \quad u(0) = A, \quad \text{and} \quad u(1) = B, \end{aligned} \tag{1.1}$$

where $0<\varepsilon, \mu<<1$ and p, q, f are sufficiently smooth functions such that $0<a\le p(x)$, $0<b\le q(x)$ on $\overline{\Omega}=[0,1]$ and A and B are constants. In general the BVP (1.1) possesses two boundary layer regions of different widths at $x=0$ and $x=1$. It is significant to observe that the two-parameter problem arises in the field of engineering, mathematical physics and applied mathematics. Such equation plays a crucial role in semiconductor modelling, financial modelling, population dynamics and in many other applications [Chen and O'Malley (1974); O'Malley (1974)].

When the parameter $\mu=1$, the problem is a one-dimensional convection diffusion problem. In this case, a boundary layer of width $O(\varepsilon)$ appears in the neighbourhood of the point $x=0$. When the parameter $\mu=0$, the problem is called one-dimensional reaction-diffusion problem and boundary layers of width $O\left(\sqrt{\varepsilon}\right)$ may appear in the neighbourhood of both the points $x=0$ and $x=1$. The asymptotic nature of the solution of the continuous problem (1.1) was studied by O'Malley (1967 and 1974), where the ratio of μ to $\sqrt{\varepsilon}$ was identified as significant. Hence the analysis for the two parameter problem splits into two cases: $\mu\le C_1\sqrt{\varepsilon}$ and $\mu\ge C_2\sqrt{\varepsilon}$. In the former case, the problem is close to single parameter reaction-diffusion case, while the latter case is more obscured.

The solution of (1.1) has steep layers which are difficult to approximate efficiently by most numerical methods using uniform grid which is shown by Farrell et al. (2000) and Miller et al. (2012). So, it is informative to establish an asymptotic pointwise error bound of the form

$$\left\|u-U^N\right\|_{\Omega^N}\le C_p N^{-p}, \qquad p>0,$$

where u is the exact solution, U^N is the numerical approximations, $N\ (\ge N_0)$ is the number of mesh elements and p is the $\varepsilon,\mu-$uniform rate of convergence. A numerical method is said to be parameter-uniform if the error constant C_p is independent of perturbation parameters ε,μ and the mesh parameter N. To tackle these problems, there are possibly two strategies.

The first idea is to use simple discretization in conjunction with a suitably chosen non-uniform grid. If the presence, location, and thickness of a boundary layer are known a priori, then highly appropriate non-uniform grids can be generated. Simpler piecewise uniform grids especially Shishkin mesh have been considered in Gracia et al. (2006) and O'Riordan et al. (2003), where parameter robust numerical methods are established. In Shanthi et al. (2006), a robust numerical method for a singularly perturbed two parameter problem with a discontinuous source term is discussed. The main disadvantage of this kind of approach is that it relies heavily on knowing a considerable amount about the exact solution before one attempts to solve the differential equation. Often this information is not available, especially for nonlinear problems.

A more widely applicable idea is to use an adaptive non-uniform grid where adaptivity is governed by the numerical solution. This approach has the advantage that it can be applied using little or no a priori information. The adaptive grids approach have become extremely popular and been successfully used in Mohapatra et al. (2010, 2010) for widespread applications. With solution-adaptive methods, a commonly used technique for determining the grid points is that they equi-distribute a positive monitor function of the numerical solution over the domain. For singular perturbation problems, the aim is to cluster automatically grid points within a boundary layer and an obvious choice of adaptivity criterion is therefore the solution gradient. Mackenzie (1999) and Qiu et al. (1999) consider a simple first-order upwind scheme applied to the homogeneous version of (1.1) with $\mu = 1$ (one parameter problems) on a non-uniform grid formed by equi-distribution of the monitor function $|u'(x)|^{1/m}$, where $m \geq 2$. Their analysis and numerical experiments show that the resulting approximation is indeed first-order uniformly convergent.

The objective of this paper is to show adaptivity may be used for two parameter problems to generate mesh for which $\varepsilon, \mu-$uniform convergence is achieved. We use upwind finite differences analysis on adaptively generated grid which involves truncation error bound, discrete comparison principle and appropriate discrete barrier function. Here both the cases ($\mu \leq C_1\sqrt{\varepsilon}$ and $\mu \geq C_2\sqrt{\varepsilon}$) are dealt and the transition from convection-diffusion to reaction-diffusion is examined. Also we have calculated the normalized flux that is to calculate the spatial derivative of $u(x)$ and it is shown to be uniformly convergent and the error constant is independent of the small perturbation parameters.

The layout of the rest of the paper is as follows. In Section 2, we remember a comparison principle, stability result and some a priori estimates on the solution and its derivatives. Section 3 presents upwind finite difference discretization and generation of the non-uniform grids through equi-distribution principle. We introduce a bound of the local truncation error and bound on the maximum pointwise error in Section 4 and carry out the truncation error analysis. The analysis leads us to the main theoretical result namely the $\varepsilon, \mu-$uniform convergence in the maximum norm. We have also shown the uniform convergence of normalized flux obtained via Lagrange interpolating polynomial. Finally, two numerical examples are provided in Section 5 to illustrate the applicability of the present method with maximum point-wise error and the rate of convergence is shown in terms of tables and figures.

Throughout this paper, C (sometime subscripted) will denote the generic positive constant independent of the mesh size and the perturbation parameters ε, μ and N (the dimension of the discrete problem) which can take different values at different places, even in the same argument. Here, $\|.\|$ denotes the supremum norm over $\overline{\Omega}$.

2. A priori bounds on the solution and its derivatives

Lemma 2.1. (Comparison principle)

Let $v \in C^2(\overline{\Omega})$. If $v(0) \geq 0, v(1) \geq 0$ and $Lv(x) \geq 0, \forall x \in \Omega,$ then $v(x) \geq 0, \forall x \in \overline{\Omega}.$

***Proof**:*

The proof can be done by extending the techniques given in Miller et al. (2012) for two parameter problems.

An immediate consequence of this comparison principle is the following parameter uniform bound on the solution u.

Lemma 2.2.

If u is the solution of the boundary value problem (1.1), then

$$\|u\|_{\overline{\Omega}} \leq \max\left\{|u(0)|,|u(1)|\right\}+\frac{1}{b}\|f\|. \tag{2.1}$$

Proof:

The proof follows from Miller et al. (2012).

Lemma 2.3.

The derivatives $u^{(k)}$ of the solution u of (1.1) satisfy the following bounds:

$$\|u^{(k)}\|_{\overline{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^k}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\left\{\|u\|,\|f\|\right\},\ k=1,2 \tag{2.2}$$

$$\|u^{(3)}\|_{\overline{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^3}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\left\{\|u\|,\|f\|,\|f'\|\right\}, \tag{2.3}$$

where C depends only on $\|p\|$, $\|p'\|$, $\|q\|$, $\|q'\|$.

Proof:

For the detailed proof of the above Lemma, one may refer Kellogg and Tsan (1978).

3. Discretization

3.1. Difference Scheme

We will consider difference approximations of (1.1) on a non-uniform partition

$$\Omega^N = \{0 = x_0 < x_1 < x_2 < ... < x_{N-1} < x_N = 1\}$$

and denote

$$h_j = x_j - x_{j-1},\ j = 1,2,..., N.$$

Without loss of generality, we will assume that N is even. Given a mesh function ϕ_j, we define the following difference operators:

$$D^+\phi_j = \frac{\phi_{j+1} - \phi_j}{h_{j+1}}, \qquad D^+D^-\phi_j = \frac{2}{h_j + h_{j+1}}\left(\frac{\phi_{j+1} - \phi_j}{h_{j+1}} - \frac{\phi_j - \phi_{j-1}}{h_j}\right).$$

The upwind finite difference discretization of (1.1) takes the form

$$\begin{cases} L^N U_j \equiv -D^+D^-U_j - \mu\, p_j D^+U_j + q_j U_j = f_j,\ \ 1 \le j \le N-1, \\ U_0 = A, \quad U_N = B, \end{cases} \tag{3.1}$$

where U_j denotes the approximation to $u(x_j)$, $p_j = p(x_j)$ and $q_{j,}\ f_j$ are defined in similar fashion. The above equation can be expressed in the form

$$\begin{cases} -r_j^- U_{j-1} + r_j^c U_j - r_j^+ U_{j+1} = f_j, \quad j = 1, 2,..., N-1, \\ U_0 = A, \quad U_N = B, \end{cases} \tag{3.2}$$

where

$$r_j^c = \frac{2\varepsilon}{h_j h_{j+1}} + \frac{\mu p_j}{h_{j+1}} + q_j,\ \ r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})} + \frac{\mu p_j}{h_{j+1}},\ \ r_j^- = \frac{2\varepsilon}{h_j(h_j + h_{j+1})}.$$

One can easily see that

$$r_j^- > 0,\ r_j^+ > 0 \text{ and } r_j^c \ge r_j^- + r_j^+, \quad \text{for } j = 1,2,..., N-1, \tag{3.3}$$

which makes the matrix an $M-$matrix.

Lemma 3.1.

Let $\Re$ be an irreducible matrix with $r_{ij} \le 0$ for $i \ne j$. Then, the following conditions are equivalent to '$\Re$ is a non-singular $M-$matrix':

(i) $\Re^{-1} > 0$,

(ii) There exists a vector $e > 0$ such that $\Re e > 0$ and $\left\|\Re^{-1}\right\| \le \dfrac{\| e \|}{\min_j (\Re e)_j}$.

***Proof*:**

The proof is given in Farrel et al. (2000) and Roos et al. (2008).

3.2. Grid equi-distribution

Since the solution $u(x)$ of the BVP (1.1) exhibits boundary layer, one has to use layer-adapted non-uniform spatial grids, which are fine inside the boundary layer region and coarse in the outer region. To obtain such a grid, we use the idea of equi-distribution of a positive monitor function given depending upon the solution and its derivatives. A grid Ω^N is said to be equi-distributing if

$$\int_{x_{j-1}}^{x_j} M(u(s),s)ds = \int_{x_j}^{x_{j+1}} M(u(s),s)ds, \qquad j = 1,2,\ldots,N-1, \tag{3.4}$$

where $M(u(x),x) > 0$ is called a monitor function. Equivalently, (3.4) can be expressed as

$$\int_{x_j}^{x_{j+1}} M(u(s),s)ds = \frac{1}{N}\int_0^1 M(u(s),s)ds, \qquad j = 1,2,\ldots,N-1. \tag{3.5}$$

Equi-distribution can also be thought of as giving rise to a mapping $x = x(\xi)$ relating a computational coordinate $\xi \in [0,1]$ to the physical coordinate $x \in [0,1]$ defined by

$$\int_0^{x(\xi)} M(u(s),s)ds = \xi\int_0^1 M(u(s),s)ds.$$

In practice, the monitor function is often based on a simple function of the derivatives of the unknown solution. In this paper, we consider the monitor function

$$M(u(x),x) = \left|\frac{du}{dx}\right|^{\frac{1}{m}}, \quad m \ge 2\ . \tag{3.6}$$

One can refer Mackenzie (1999), Qiu and Sloan (1999) for more details on this argument where one parameter problem is considered. The monitor function should ideally be bounded away from zero. Such a monitor function is considered in Wu et al. (2013) but by considering the above monitor function (3.6) gives satisfactory result with much simpler analysis than Wu et al. (2013).

To simplify the treatment of the variable coefficient case, we construct the monitor function (3.6) in terms of the exact solution of (1.1) with $p(x)$ set to the constant lower bound a. This yields the mapping

$$x(\xi) = -\frac{m\varepsilon}{a\mu}\ln(1-\hat{L}\xi), \quad j = 0,1,\ldots,N, \tag{3.7}$$

where

$$\hat{L} = 1-\exp(-a\mu/m\varepsilon).$$

This mapping will be a reasonable approximation to the equi-distribution (3.6) as long as $p(x)$ does not vary excessively from a. A non-uniform grid in physical space $\{x_j\}_{j=0}^{N}$, corresponds to the evenly distributed nodes $\xi_j = \frac{j}{N}$, $j = 0,1,\ldots,N$ in the computational space. This identity gives

$$x_j = -\frac{m\varepsilon}{a\mu}\ln(1-\frac{\hat{L}_j}{N}), \quad j = 0,1,\ldots,N.$$

(3.8)

From a practical point of view, the monitor function has to be approximated from the numerical solution. For example, approximating (3.4) numerically will result the set of nonlinear algebraic equations

$$M_{j+\frac{1}{2}}(x_{j+1}-x_j) = M_{j-\frac{1}{2}}(x_j - x_{j-1}), \tag{3.9}$$

where $M_{j+\frac{1}{2}}$ is an approximation of $M\left(u(x_{j+\frac{1}{2}}),\ x_{j+\frac{1}{2}}\right)$. For M given by (3.6), an obvious choice is

$$M_{j+\frac{1}{2}} = \left(\frac{U_{j+1}-U_j}{x_{j+1}-x_j}\right)^{\frac{1}{m}}, \quad j = 0,1,\ldots,N-1.$$

The system of equations (3.1), (3.9) should be solved simultaneously to obtain the solution U_j and the grids x_j. If the variation of $p(x)$ to a is small on $[0,1]$, then the nodes formed by (3.8) are close to those formed by (3.1) and (3.9). The convergence behaviour of the approximately discretized scheme (3.8) is likely to be very close to the fully adaptive scheme

(3.9). Hence we shall be concerned with the approximate solution of (1.1) by means of (3.8). But for computational purpose, we will take the fully adaptive scheme (3.9).

Now we will state some conditions that will be assumed throughout the rest of the paper.

Assumptions

(i) Since we are interested in the limiting case that is as $\varepsilon \to 0$ and $\mu = O(\sqrt{\varepsilon}),$ we assume there exist a constant d such that

$$\frac{m\varepsilon}{a\mu} < d << 1, \tag{3.10}$$

where a is defined earlier and hence there exist C_1 for which $1 > \hat{L} > C_1 = 1 - \exp(-1/d).$

(ii) We assume that

$$N\varepsilon << 1. \tag{3.11}$$

As we are interested in adaptive approaches to the solution, the above assumptions are sensible. Now we have the following bound on the size of the mesh formed by the non-uniform grid (3.8).

Lemma 3.2.

We have the following bound:

$$h_j < \frac{m\varepsilon}{a\mu}, \quad j = 1,2,\ldots, N-1.$$

Proof:

From (3.8) and using mean value theorem, we have for $j = 1,2,\ldots, N-1,$

$$h_j = x_j - x_{j-1} = -\frac{m\varepsilon}{a\mu}\left[\ln(1-\hat{L}\xi_j) - \ln(1-\hat{L}\xi_{j-1})\right]$$

$$= \frac{m\varepsilon\hat{L}}{a\mu N}\left[\frac{1}{1-\hat{L}\eta_j}\right], \quad \text{where} \quad \eta_j \in (\xi_{j-1}, \xi_j).$$

Similarly,

$$h_{j+1} = \frac{m\varepsilon\hat{L}}{a\mu N}\left[\frac{1}{1-\hat{L}\eta_{j+1}}\right], \quad \text{where} \quad \eta_{j+1} \in (\xi_j, \xi_{j+1})$$

and since

$$\frac{1}{1-\hat{L}\eta_j} < \frac{1}{1-\hat{L}\eta_{j+1}} \ ,$$

it follows that $h_j < h_{j+1}$, $j = 1,2,..., N-1$. Also,

$$\frac{1}{1-\hat{L}\eta_j} < \frac{1}{1-\hat{L}\xi_j} \ .$$

Using the assumptions (3.10) and (3.11), we have

$$h_j < \frac{m\varepsilon\hat{L}}{a\mu N}\left[\frac{1}{1-\hat{L}_j / N}\right] = \frac{m\varepsilon}{a\mu}\left[\frac{1}{N/\hat{L}-j}\right] < \frac{m\varepsilon}{a\mu}\left[\frac{1}{N-j}\right].$$

Thus, $h_j < m\varepsilon / a\mu$ and we have the desired result.

4. Convergence Analysis

4.1. Local Truncation Error

The local truncation error at the node x_j of (3.1) is given by $\tau_j = L^N U_j - (Lu)(x_j)$, where u denote the set of exact solution values at the nodal points. Using Peano-kernel theorem, the truncation error can be expressed as

$$\tau_j = -\frac{\varepsilon}{h_j + h_{j+1}}\left[\frac{1}{h_{j+1}}\int_{x_j}^{x_{j+1}}(s-x_{j+1})^2 u'''(s)ds - \frac{1}{h_j}\int_{x_{j-1}}^{x_j}(s-x_{j-1})^2 u'''(s)ds\right] + \frac{\mu p_j}{h_{j+1}}\int_{x_j}^{x_{j+1}}(s-x_{j+1})^2 u''(s)ds, \tag{4.1}$$

from which we obtain the bound

$$\left|\tau_j\right| \le C\varepsilon\int_{x_{j-1}}^{x_{j+1}}\left|u'''(s)\right|ds + C\mu\int_{x_{j-1}}^{x_{j+1}}\left|u''(s)\right|ds. \tag{4.2}$$

For more details see Lemma 3.2 of Linβ and Roos (2004). If we invoke the derivative bounds of the continuous solution (2.3), the above expression may be simplified to

$$\left|\tau_j\right| < C\varepsilon\int_{x_{j-1}}^{x_{j+1}}\left|u'''(s)\right|ds, \tag{4.3}$$

where C is independent of ε. To initiate the construction of an appropriate bound for the local truncation error, we replace $u'''(s)$ by the bound given in (2.3) to obtain

$$\left|\tau_j\right| < C\varepsilon^{-2}\int_{x_{j-1}}^{x_{j+1}}\exp\left(\frac{-a\mu s}{\varepsilon}\right)ds.$$

Now from (3.7) we have

$$\frac{dx}{d\xi} = \frac{m\varepsilon\hat{L}}{a\mu(1-\hat{L}\xi)}.$$

Hence,

$$\left|\tau_j\right| < C\varepsilon^{-2}\int_{x_{j-1}}^{x_{j+1}}\exp(-a\mu x/\varepsilon)dx = C\frac{m\hat{L}}{a\mu}\varepsilon^{-1}\int_{\xi_{j-1}}^{\xi_{j+1}}(1-\hat{L}\xi)^{m-1}d\xi, \quad \text{by using (3.7)}$$

$$< 2C\frac{m\hat{L}}{a\mu}\varepsilon^{-1}\int_{\xi_{j-1}}^{\xi_j}(1-\hat{L}\xi)^{m-1}d\xi, \quad \text{bisecting the range of integration}$$

$$= \frac{2C}{\varepsilon N}(1-\hat{L}\bar{\xi})^{m-1}, \quad \text{using Lemma 3.2 and (3.10) where } \bar{\xi}\in(\xi_{j-1},\xi_j).$$

Now using the fact that $1-\hat{L}\bar{\xi} < 1-\hat{L}\xi_{j-1} < 2(1-\hat{L}\xi_j)$, we have

$$\frac{C}{\varepsilon N}(1-\hat{L}\xi_j)^{m-1} < \frac{C}{\varepsilon N}(1-\hat{L}\xi_j),$$

which yields

$$\left|\tau_j\right| < \frac{C}{\varepsilon N}\exp(-a\mu x_j/m\varepsilon), \quad j=1,2,...,N-2. \tag{4.4}$$

Now we need to find a bound for $\left|\tau_{N-1}\right|$. Suppose σ is the smallest integer that satisfies $10^{-\sigma} \le \varepsilon << 1$. Select m such that $m \ge \sigma+2$. We write $\left|\tau_{N-1}\right|$ as

$$\left|\tau_{N-1}\right| < \frac{C}{N}\left(1-\hat{L}\xi_{N-1}\right)\left[\frac{1}{\varepsilon}\left(1-\hat{L}\xi_{N-1}\right)^{\sigma}\right].$$

We have to show that

$$\left[\frac{1}{\varepsilon}\left(1-\hat{L}\xi_{N-1}\right)^{\sigma}\right] \le 1.$$

Now the above expression can be written as

$$\frac{1}{\varepsilon}\left(\frac{1}{N}+\left(\frac{N-1}{N}\right)\exp(-a\mu/m\varepsilon)\right)^{\sigma}\leq 1.$$

The above inequality is satisfied for $\varepsilon<\varepsilon_0$ provided $N\geq 2\varepsilon^{-1/\sigma}$, where $0<\varepsilon_0\leq 1$ to be chosen later. We require that

$$\left(\frac{1}{N}+\left(\frac{N-1}{N}\right)\exp(-a\mu/m\varepsilon)\right)\leq\varepsilon^{1/\sigma},$$

and for $N\geq 2\varepsilon^{-1/\sigma}$ this reduces to

$$\left(\frac{N-1}{N}\right)\exp(-a\mu/m\varepsilon)\leq\frac{1}{2}\varepsilon^{1/\sigma}.$$

Now,

$$\left(\frac{N-1}{N}\right)\exp(-a\mu/m\varepsilon)<\exp(-a\mu/m\varepsilon)<\left(\frac{m\varepsilon}{a\mu}\right)^2,\quad \text{if } \varepsilon<\frac{a\mu}{m}.$$

Hence,

$$\left(\frac{N-1}{N}\right)\exp(-a\mu/m\varepsilon)<\frac{\varepsilon}{2}\left(\frac{2m^2\varepsilon}{a^2\mu^2}\right)<\frac{1}{2}\varepsilon<\frac{1}{2}\varepsilon^{1/\sigma},$$

provided that $\varepsilon<1$ and $\varepsilon<a^2\mu^2/2m^2$. If we define

$$\varepsilon_0=\min\left\{1,a\mu/m,a^2\mu^2/m^2\right\},$$

the above result follows. Numerical experiment suggests that the restrictions $10^{-\sigma}\leq\varepsilon$ and $m\geq\sigma+2$ can be removed. We may identify $2\varepsilon^{-1/\sigma}$ with N_0 and conclude that

$$|\tau_{N-1}|<\frac{C}{N}\left(1-\hat{L}\xi_{N-1}\right)=\frac{C}{N}\exp(-a\mu x_{N-1}/m\varepsilon) \tag{4.5}$$

provided $N\geq N_0$ and $\varepsilon<\varepsilon_0$.

Lemma 4.1.

The required bound on the local truncation errors is given by

$$\left|\tau_j\right| < \frac{C}{\varepsilon N}\exp(-a\mu x_j / m\varepsilon), \quad j = 1,2,\dots, N-1. \tag{4.6}$$

***Proof**:*

Combining the bounds (4.4) and (4.5) gives us the required result.

4.1. Bound on Maximum Pointwise Error

Before deriving the error estimate for the numerical scheme (3.1), we provide here some lemmas which are prerequisites for the main result.

Lemma 4.2. (Discrete comparison principle)

The system

$$L^N V_j = F_j$$

with V_0 and V_N specified has a unique solution. If

$$L^N V_j < L^N Z_j \text{ for } 1 \le j \le N-1 \text{ with } V_0 < Z_0 \text{ and } V_N < Z_N,$$

then

$$V_j < Z_j \text{ for } 1 \le j \le N.$$

***Proof**:*

It is easy to verify that the matrix associated with L^N is an irreducible $M-$matrix and therefore has a positive inverse. Hence, the result follows.

Lemma 4.3.

We define a mesh function S_j such that

$$S_0 = 1, \qquad S_j = \prod_{k=1}^{j}\left(1+\frac{a\mu h_k}{m\varepsilon}\right)^{-1}, \quad j = 1,2,\dots, N. \tag{4.7}$$

Then, for $j = 1,2,\dots, N-1,$ and for some constant C, we have:

$$L^N S_j \geq \frac{C}{\max\left\{\frac{m\varepsilon}{a\mu}, h_{j+1}\right\}} S_j .$$

Proof:

We have

$$\frac{S_j - S_{j-1}}{h_j} = -\frac{a\mu}{m\varepsilon} S_j .$$

Now with $m \geq 2$ and $p_j > a$, we have

$$\begin{aligned} L^N S_j &= \frac{2\varepsilon}{h_j + h_{j+1}} \left[\frac{S_{j+1} - S_j}{h_{j+1}} - \frac{S_j - S_{j-1}}{h_j} \right] - \mu a \left[\frac{S_{j+1} - S_j}{h_{j+1}} \right] + bS_j \\ &= \frac{2a^2\mu^2 h_{j+1}}{m^2\varepsilon(h_j + h_{j+1})} \left[\frac{m\varepsilon}{m\varepsilon + a\mu h_{j+1}} S_j \right] + \frac{a^2\mu^2}{m\varepsilon} \left(\frac{m\varepsilon}{m\varepsilon + a\mu h_{j+1}} S_j \right) + bS_j \\ &\geq \left(\frac{\mu a}{m\varepsilon + a\mu h_{j+1}} \right) \left[a\mu - \frac{2a\mu h_{j+1}}{m(h_j + h_{j+1})} \right] S_j \\ &\geq \frac{C}{\max\left\{\frac{m\varepsilon}{a\mu}, h_{j+1}\right\}} S_j . \end{aligned}$$

Remark 4.4.

The function S_j is the piecewise (0,1) Padé approximation of $\exp(-a\mu x_j / m\varepsilon)$. A similar comparison function was used originally by Kellogg and Tsan (1978) to analyse the methods on uniform grids and more recently, it is used by many authors like Kopteva et al. [Kopteva and Stynes (2001); Kopteva et al. (2005)], Mohapatra and Natesan (2010) and Qiu and Sloan (1999) for analysis on nonuniform grids.

In the following lemma, we provide a two-sided bound for S_j which will be used later.

Lemma 4.5.

The grid function S_j defined in Lemma 4.3 satisfy

$$\exp(-a\mu x_j / m\varepsilon) < S_j < C \exp(-a\mu x_j / m\varepsilon), \quad j = 1, 2, ..., N-1. \tag{4.8}$$

Proof:

We have

$$x_j = \sum_{k=1}^{j} h_k .$$

Therefore,

$$\exp(-a\mu x_j / m\varepsilon) = \exp\left(\sum_{k=1}^{j} \frac{-a\mu h_k}{m\varepsilon}\right) = \prod_{k=1}^{j} \exp(-a\mu h_k / m\varepsilon).$$

Again we know that for any value of $\theta > 0$, we have

$$\exp(-\theta) < (1+\theta)^{-1}.$$

Therefore,

$$\exp(-a\mu x_j / m\varepsilon) = \prod_{k=1}^{j} \left(1 + \frac{a\mu h_k}{m\varepsilon}\right)^{-1} < S_j .$$

Now we need to find the upper bound for S_j.

$$\ln\left[\prod_{k=1}^{j}\left(1+\frac{-a\mu h_k}{m\varepsilon}\right)\right] = \sum_{k=1}^{j}\ln\left(1+\frac{a\mu h_k}{m\varepsilon}\right) > \sum_{k=1}^{j}\ln\left[\frac{a\mu h_k}{m\varepsilon} - \frac{1}{2}\left(\frac{-a\mu h_k}{m\varepsilon}\right)^2\right]$$
$$> \frac{a\mu x_j}{m\varepsilon} - \frac{1}{2}\sum_{k=1}^{N-1}\frac{1}{k^2} > \frac{a\mu x_j}{m\varepsilon} - 1.$$

Hence,

$$\prod_{k=1}^{j}\left(1+\frac{-a\mu h_k}{m\varepsilon}\right)^{-1} < \exp(1 - a\mu x_j / m\varepsilon) < C\exp(-a\mu x_j / m\varepsilon).$$

We will now state the main result of this paper.

Theorem 4.6.

Let $u(x)$ be the exact solution of (1.1) and let U_j be the discrete solution of (3.1) on the grid defined by (3.8). Then there exists a constant C, independent of N, ε and μ such that

$$\max_{0\le j\le N}\left|u(x_j) - U_j\right| \le CN^{-1}, \qquad j = 0,1,\ldots,N. \tag{4.9}$$

***Proof*:**

We already know from (4.6) and Lemma 4.5 that

$$\left|\tau_j\right| < \frac{C}{\varepsilon N}\exp(-a\mu x_j / m\varepsilon) < \frac{C}{\varepsilon N} S_j .$$

As we know that the discrete maximum principle holds in $[0,1]$. Now we will apply this principle in $[0, x_{N-1}]$ for the barrier function W_j defined by

$$W_j = CN^{-1}(1+S_j), \;\; j = 0,1,..., N-1.$$

The local truncation error and the nodal error are related by

$$L^N e_j = \tau_j .$$

Using (4.5) and Lemma 4.3, we have

$$L^N e_j = \tau_j \le \frac{C}{\varepsilon N} S_j \le CN^{-1} L^N S_j \le L^N W_j , \qquad j = 1, 2,..., N-2.$$

Since $e_0 \le W_0$ and $e_{N-1} \le W_{N-1}$, we conclude that

$$e_j \le W_j \le CN^{-1}, \quad j = 1, 2,..., N-1.$$

Now the same argument can be repeated with e_j being replaced by $-e_j$ and, hence,

$$\left|e_j\right| \le CN^{-1}, \quad j = 1, 2,..., N-1$$

and this completes the proof.

The estimate given in Theorem 4.6 shows that the first-order upwind scheme applied on the equi-distributed grids is uniformly first-order accurate at all of the mesh points. We can obtain a global approximation to the exact solution by interpolating the numerical solution at the mesh points using piecewise constant or piecewise linear functions. We now show that these global approximations are uniformly first-order accurate throughout the domain.

Theorem 4.7.

Let $\bar{u}(x)$ be the piecewise constant or piecewise linear interpolant of the first order upwind solution $u(x)$ of (1.1) obtained on the grid (3.8). Then $\bar{u}(x)$ satisfies the $\varepsilon, \mu-$uniform estimate

$$\|u(x)-\bar{u}(x)\| \le CN^{-1}. \tag{4.10}$$

***Proof**:*

Let $\bar{u}(x)$ denote the piecewise polynomial interpolant of degree k with either $k=0$ or $k=1$, where $x \in (x_{j-1}, x_j)$. Then,

$$\bar{u}(x) = \begin{cases} U_j \chi_j(x), & k=0, \\ U_{j-1}\phi_{j-1}(x) + U_j \phi_j(x), & k=1, \end{cases} \tag{4.11}$$

where

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in (x_{j-1}, x_j), \\ 0, & \text{otherwise,} \end{cases}$$

and $\phi_{j-1}(x)$, $\phi_j(x)$ are Lagrange's interpolating polynomials of first degree given by

$$\phi_{j-1}(x) = \frac{x_j - x}{x_j - x_{j-1}}, \qquad \phi_j(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}}.$$

For $k=0$,

$$\bar{u}(x) - u(x) = U_j \chi_j(x) - u(x_j) + u(x_j) - u(x) = U_j - u(x_j) + \int_x^{x_j} u'(s)ds.$$

For $k=1$, we have

$$\begin{aligned} \bar{u}(x) - u(x) &= U_{j-1}\phi_{j-1}(x) + U_j\phi_j(x) - u(x) \\ &= [U_{j-1} - u(x_{j-1})]\phi_{j-1}(x) + [U_j - u(x_j)]\phi_j(x) \\ &\quad - \phi_{j-1}(x)\int_{x_{j-1}}^{x} u'(s)ds + \phi_j(x)\int_x^{x_j} u'(s)ds. \end{aligned}$$

Using Theorem 4.6, we can conclude

$$|\bar{u}(x) - u(x)| \le \begin{cases} CN^{-1} + \int_x^{x_j} |u'(s)|ds, & \text{if } k=0, \\ CN^{-1} + C\int_{x_{j-1}}^{x_j} |u'(s)|ds, & \text{if } k=1. \end{cases}$$

Now we have to find the bound of the integral

$$I_j = \int_{x_{j-1}}^{x_j} |u'(s)| ds .$$

Using Lemma 2.3 and the equi-distribution principle (3.5), we may conclude that

$$\begin{aligned} I_j &\le C\varepsilon^{-1}\int_{x_{j-1}}^{x_j} \exp(-a\mu s/\varepsilon)ds < C\varepsilon^{-1}\int_{x_{j-1}}^{x_j} \exp(-a\mu s/m\varepsilon)ds \\ &\le C\varepsilon^{1/m-1}\int_{x_{j-1}}^{x_j} M(u(s),s)ds = \frac{C\varepsilon^{1/m-1}}{N}\int_0^1 \quad M(u(s),s)ds \\ &\le \frac{C\varepsilon^{1/m}}{N}\left(1-\exp(-a\mu/m\varepsilon)\right) \le CN^{-1} . \end{aligned}$$

The required result follows.

4.3. The Normalized Flux

In this section, we are interested in proving the uniform convergence for the normalized flux which is defined as $\varepsilon u'(x)$ (For more details, refer Kopteva and Stynes (2001), Mohapatra and Natesan (2010), Shishkin and Shishkina (2009). Using the numerical solution, we have found the global solution by Lagrange's interpolating polynomial defined in (4.11).

Theorem 4.8.

Let $\bar{u}(x)$ be the piecewise constant or piecewise linear interpolant of the first order upwind solution $u(x)$ of (1.1) obtained on the grid (3.8). Then, the error of the normalized flux satisfies

$$\varepsilon|u'(x)-\bar{u}'(x)| \le CN^{-1}, \qquad \text{for } x \in \Omega.$$

***Proof*.**

Let $\bar{u}(x)$ be the piecewise interpolating polynomial defined in (4.11). Now for any $x \in (x_{j-1}, x_j)$, we have

$$\begin{aligned} \bar{u}'(x)-u'(x) &= \bar{u}'(x)-u'(x_j)+u'(x_j)-u'(x) \\ &= \bar{u}'(x)-u'(x_j)+\int_x^{x_j} |u''(s)| ds, \quad \text{for } s \in (x_{j-1}, x_j). \end{aligned}$$

Using Mean Value Theorem, we have the following bound for the above expression

$$|u'(x)-\bar{u}'(x)| \le Ch_j |u''(x)| + \int_x^{x_j} |u''(s)| ds.$$

Now we have to find the bound for $\int_x^{x_j} |u''(s)| ds$. Using (4.3) and (4.6), we will have

$$|u'(x)-\bar{u}'(x)| \le \frac{C}{\varepsilon N} \exp\left(\frac{-a\mu x_j}{m\varepsilon}\right).$$

Multiplying ε on both sides, we will get the required result.

5. Numerical Experiments

To show the applicability and efficiency of the present method it has been implemented to the following test problems.

Example 5.1. Consider the singularly perturbed two parameter problem

$$\begin{cases} -\varepsilon u''(x)-\mu u'(x)+u(x)=x, \quad x\in\Omega, \\ u(0)=1, \quad u(1)=0. \end{cases} \tag{5.1}$$

The exact solution is given by $u(x)=(x+\mu)+C_1\exp(-m_1 x/2\varepsilon)-C_2\exp((1-x)m_2/2\varepsilon)$, where

$$m_{1,2}=\mu\pm\sqrt{\mu^2+4\varepsilon},$$

$$C_1=\frac{(1+\mu)\exp(m_2/2\varepsilon)+1-\mu}{1-\exp(-\sqrt{\mu^2+4\varepsilon}/\varepsilon)}, \quad C_2=\frac{1+\mu+(1-\mu)\exp(-m_1/2\varepsilon)}{1-\exp(-\sqrt{\mu^2+4\varepsilon}/\varepsilon)}$$

Example 5.2.

Consider the singularly perturbed two parameter problem

$$\varepsilon u''(x)+\mu(2+x)u'(x)=1+2x, \quad x\in\Omega, \quad u(0)=0, \quad u(1)=1. \tag{5.2}$$

The exact solution is not known for this example.

For any value of N, the maximum pointwise errors $E^N_{\varepsilon,\mu}$ are calculated by

$$E_{\varepsilon,\mu}^{N} = \left\| u(x_j) - U_j \right\|,$$

where u is the exact solution of (5.1) and U_j is the numerical solution of (5.1) obtained by the proposed method. Since the exact solution of (5.2) is not known, we use the double mesh method to calculate the maximum pointwise errors that is

$$E_{\varepsilon,\mu}^{N} = \left\| u(x_j) - \tilde{U}_j \right\|,$$

where $\tilde{U}_j$ is the interpolation of the numerical solution calculated on Ω^{2N} to Ω^{N}. The errors $F_{\varepsilon,\mu}^{N}$ associated with the normalized flux are obtained by

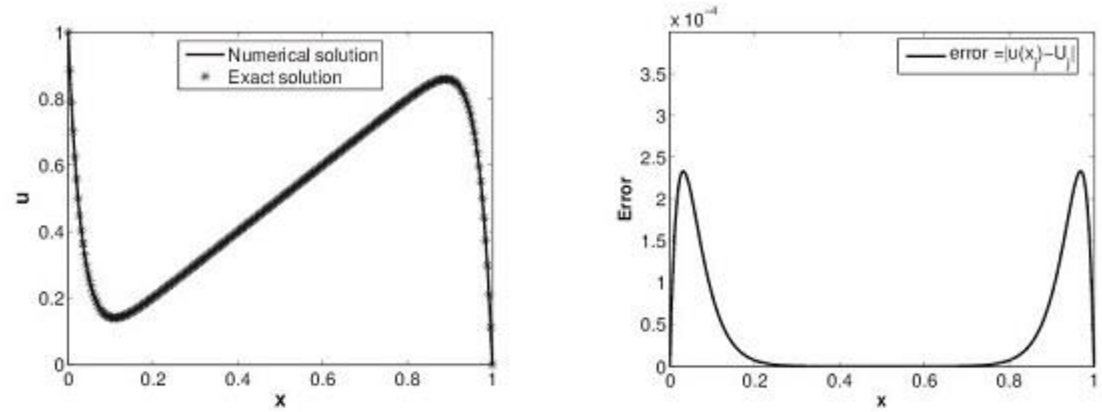
$$F_{\varepsilon,\mu}^{N} = \varepsilon \left\| u'(x) - \bar{u}'(x) \right\|,$$

where $\bar{u}(x)$ denotes the Lagrange's interpolating polynomial defined in (4.11). We use the double mesh method to compute the rate of convergence as

$$p^{N} = \log_2\left(\frac{E_{\varepsilon,\mu}^{N}}{E_{\varepsilon,\mu}^{2N}}\right), \qquad q^{N} = \log_2\left(\frac{F_{\varepsilon,\mu}^{N}}{F_{\varepsilon,\mu}^{2N}}\right).$$

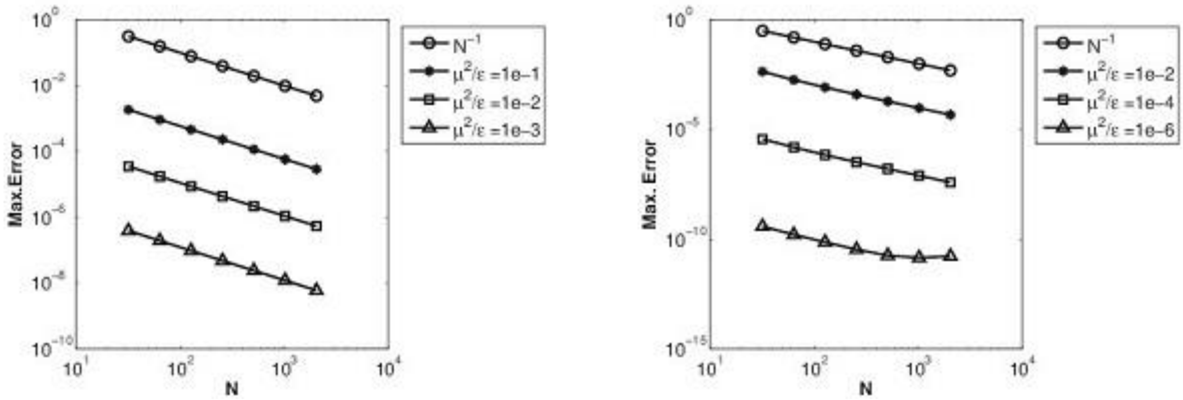
In Tables 1 and 2, we present the maximum pointwise error and the corresponding order of convergence p^N for $\mu = 1e-2$ and $N = 32, 64, ..., 2048$. Since the adaptive mesh is generated iteratively, we have presented the number of iterations required to obtain the final computed mesh in Table 1 for each value of μ and ε. This guarantees that the iterative process is convergent. Similarly the maximum pointwise errors and the corresponding order of convergence q^N associated with normalized flux are given in Tables 3 and 4 respectively. In Table 5 and Table 6, the numerical result corresponding to Example 5.2 is presented. These results clearly show the $\varepsilon, \mu-$uniform convergence of order one.

Figure 1(a) displays the numerical solution and the exact solution of Example 5.1 for $\varepsilon = 1e-3$, $\mu = 1e-7$ and $N = 256$ and Figure 1(b) represents the corresponding maximum pointwise error. To visualize the order of convergence more clearly, the loglog plots of maximum pointwise error $E_{\varepsilon,\mu}^{N}$ are shown in Figures 2 and 3 for $\mu = 1e-1$ and $\mu = 1e-2$. The loglog plots for the errors associated with normalized flux are shown in Figures 4.



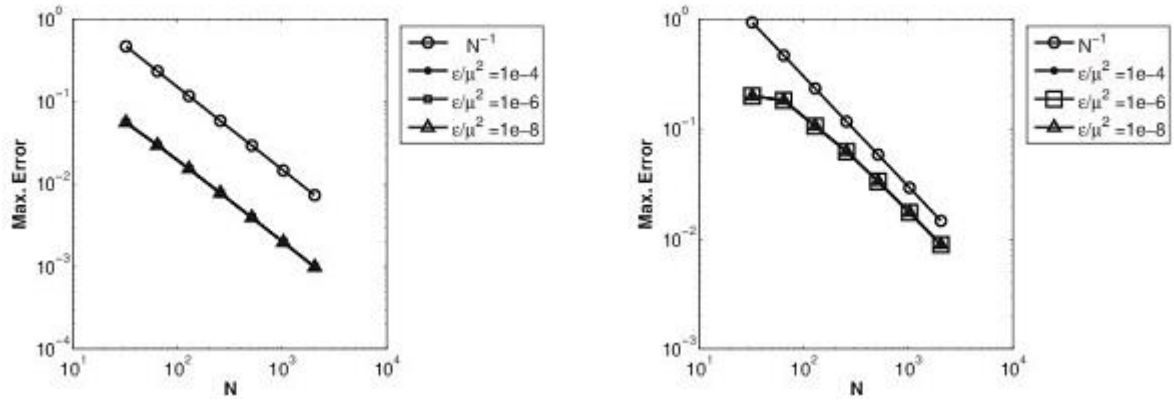
(a) Numerical solution and Exact solution. (b) Error.

Figure 1. Numerical solution with exact solution and the error of Example 5.1 for $\varepsilon = 1e-3,\ \mu = 1e-7$ and $N = 256$



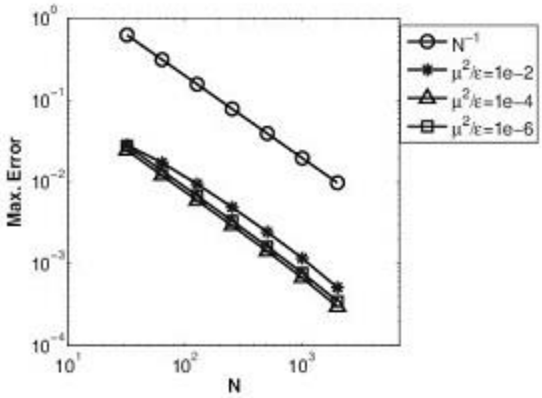
(a) $\mu = 1e-1$. (b) $\mu = 1e-2$.

Figure 2. Loglog plot of the maximum error for different values of μ^2/ε

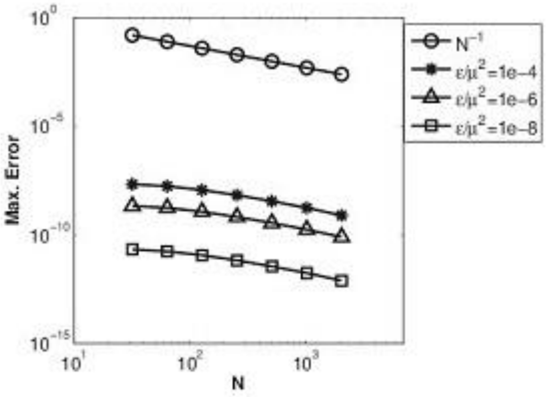


(a) $\mu = 1e-1$. (b) $\mu = 1e-2$.

Figure 3. Loglog plot of the maximum error for different values of ε/μ^2



(a) μ^2/ε.



(b) ε/μ^2.

Figure 4. Loglog plot of the maximum error for the normalized flux for $\mu = 1e-2$

Table 1. Maximum pointwise errors $E^N_{\varepsilon,\mu}$ and the rate of convergence p^N for Example 5.1

ε/μ^2	Number of intervals N						
	32	64	128	256	512	1024	2048
$1e-4$	2.0046e-1 0.1361	1.8242e-1 0.7750	1.0660e-1 0.7663	6.6273e-2 0.9019	3.3539e-2 0.9452	1.7419e-2 0.9707	8.8884e-3
No. of iterations	46	66	101	123	180	240	319
$1e-6$	2.0047e-1 0.1359	1.8244e-1 0.7496	1.0662e-1 0.7622	6.2687e-2 0.9019	3.3548e-2 0.9452	1.7424e-2 0.9706	8.8909e-3
No. of iterations	70	108	173	230	361	519	763
$1e-8$	2.0047e-1 0.1359	1.8244e-1 0.7496	1.0662e-1 0.7622	6.2687e-2 0.9019	3.3548e-2 0.9452	1.7424e-2 0.9706	8.8909e-3
No. of iterations	95	149	244	335	543	837	1069

Table 2. Maximum pointwise error $E^N_{\varepsilon,\mu}$ and the rate of convergence p^N for Example 5.1

μ^2/ε	Number of intervals N						
	32	64	128	256	512	1024	2048
$1e-1$	3.9157e-2 1.1091	1.8153e-2 1.0731	8.6281e-3 1.0264	4.2358e-3 1.0211	2.0872e-3 1.0108	1.0358e-3 1.0054	5.1596e-4
$1e-3$	2.4680-4 1.2810	1.0156e-4 1.1489	4.5800e-5 1.0795	2.1672e-5 1.0415	1.0529e-5 1.0212	5.1876e-6 1.0108	2.5745e-6
$1e-4$	3.6557e-6 1.2513	1.5357e-6 1.1433	6.9522e-7 1.0781	3.2928e-7 1.0406	1.6007e-7 1.0210	7.8881e-8 1.0103	3.9159e-8

Table 3. Maximum pointwise errors $F^N_{\varepsilon,\mu}$ and the rate of convergence q^N for Example 5.1

ε/μ^2	Number of intervals N						
	32	64	128	256	512	1024	2048

$1e-4$	2.1538e-7 0.2899	1.7617e-7 0.5910	1.1695e-7 0.7883	6.7715e-8 0.9172	3.5856e-8 1.0187	1.7698e-8 1.1177	8.1558e-9
$1e-6$	2.1538e-9 0.2899	1.7617e-9 0.5910	1.1695e-9 0.7883	6.7718e-10 0.9172	3.5858e-10 1.0187	1.7698e-10 1.1462	7.9964e-11
$1e-8$	2.1538e-11 0.2899	1.7617e-11 0.5910	1.1695e-11 0.7883	6.7718e-12 0.9172	3.5858e-12 1.0187	1.7698e-12 1.1462	7.9964e-13

Table 4. Maximum pointwise errors $F_{\varepsilon,\mu}^N$ and the rate of convergence q^N for Example 5.1

μ^2/ε	Number of intervals N						
	32	64	128	256	512	1024	2048
$1e-1$	3.9723e-2 0.1361	3.6146e-2 0.3424	3.0173e-2 0.5428	2.3840e-2 0.7865	1.3820e-2 0.9521	7.1435e-3 1.1158	3.2963e-3
$1e-3$	1.2683e-2 1.0140	6.2802e-3 1.0146	3.1084e-3 1.0218	1.5309e-3 1.047	7.4092e-4 1.0711	3.5264e-4 1.1387	1.6016e-4
$1e-4$	2.4383e-2 1.0181	1.2039e-2 1.0162	5.9525e-3 1.0223	2.9306e-3 1.0403	1.4249e-3 1.0790	6.7451e-4 1.1694	2.9988e-4

Table 5. Maximum pointwise errors $E^N_{\varepsilon,\mu}$ and the rate of convergence p^N for Example 5.2

ε/μ^2	Number of intervals N						
	32	64	128	256	512	1024	2048
$1e-4$	7.1743e-2 0.9498	3.7142e-2 0.9910	1.8687e-2 0.9682	9.5514e-3 0.9851	4.8252e-3 0.9858	2.4364e-3 0.9972	1.2206e-3
$1e-6$	7.1929e-2 0.9398	3.7496e-2 0.9708	1.9132e-2 0.9869	9.6534e-3 0.9989	4.8304e-3 0.9923	2.4281e-3 0.9961	1.2174e-3
$1e-8$	7.1930e-2 0.9397	3.7500e-2 0.9704	1.9138e-2 0.9855	9.6659e-3 0.9933	4.8554e-3 0.9988	2.4297e-3 0.9970	1.2174e-3

Table 6. Maximum pointwise errors $E^N_{\varepsilon,\mu}$ and the rate of convergence p^N for Example 5.2

μ^2/ε	Number of intervals N						
	32	64	128	256	512	1024	2048
$1e-1$	2.1363e-2 0.8484	1.1865e-2 0.9144	6.2951e-3 0.9338	3.2953e-3 0.9676	1.6851e-3 1.0831	7.9541e-4 1.0052	3.9627e-4
$1e-3$	2.2427e-4 1.0877	1.0552e-4 0.9952	5.2934e-5 1.0342	2.5847e-5 1.1091	1.1982e-5 1.0203	5.9072e-6 1.0471	2.8587e-6
$1e-4$	2.5968e-6 1.0871	1.2223e-6 0.9950	6.1328e-7 1.0309	3.0014e-7 1.1055	1.3949e-7 1.0292	6.8349e-8 1.0456	3.3112e-8

6. Concluding Remarks

In this article, we have presented the error analysis for the first order upwind difference approximation of a singularly perturbed two parameter problem. The solution is obtained on a mesh that arises from the equi-distribution of the monitor function. We have shown that the errors associated with the global solution and the normalized flux converge at a rate of first order which is optimal, *i.e.*, $O(N^{-1})$, independent of small parameters. Hence if the mesh is generated adaptively, it is possible to obtain difference solutions that converge uniformly with respect to the perturbation parameters. Whilst the model problem is quite simple, the presented analysis does give a considerable insight into the properties of methods on adaptive grids.

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