Applications and Applied Mathematics: An International Journal (AAM)

Volume 10 | Issue 1 Article 3

6-2015

Combinatorial Identities for Incomplete Tribonacci Polynomials

Mark Shattuck
University of Tennessee

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 10, Issue 1 (June 2015), pp. 40 – 49

Applications and Applied
Mathematics:
An International Journal
(AAM)

Combinatorial Identities for Incomplete Tribonacci Polynomials

Mark Shattuck

Department of Mathematics, University of Tennessee, Knoxville, TN 37996
shattuck@math.utk.edu

Received: September 28, 2014; Accepted: February 13, 2015

Abstract

The incomplete tribonacci polynomials, denoted by $T_n^{(s)}(x)$, generalize the usual tribonacci polynomials $T_n(x)$ and have been shown to satisfy several algebraic identities. In this paper, we provide a combinatorial interpretation for $T_n^{(s)}(x)$ in terms of weighted linear tilings involving three types of tiles. This allows one not only to supply combinatorial proofs of earlier identities for $T_n^{(s)}(x)$ but also to derive new ones. In the final section, we provide a formula for the ordinary generating function of the sequence $T_n^{(s)}(x)$ for a fixed s, as previously requested. Our derivation is combinatorial in nature and makes use of an identity relating $T_n^{(s)}(x)$ to $T_n(x)$.

Keywords: Combinatorial proof; incomplete tribonacci polynomials; tribonacci numbers

MSC 2010 No.: 05A19; 05A15

1. Introduction

The *tribonacci numbers* t_n are defined by the recurrence relation $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ if $n \geq 3$, with initial values $t_0 = 0$ and $t_1 = t_2 = 1$. See sequence A000073 in (Sloane, 2010). The tribonacci numbers are given equivalently by the explicit formula

$$t_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B(n-i, i), \qquad n \geq 0, \tag{1}$$

where $B(n,i) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-j}{i}$, as shown in (Barry, 2006). The number $B(n,i)$ is the n-th row, i-th column entry of the *tribonacci triangle*, see (Alladi and V.E. Hoggatt, 1977). Note that $B(n,i) = D(n-i,i)$, where $D(a,b)$ denotes the Delannoy number sequence, and the reader is referred to A008288 in (Sloane, 2010).

The *tribonacci polynomials* $T_n(x)$ were introduced in (V.E. Hoggatt and Bicknell, 1973) and are defined by the recurrence $T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x)$ if $n \geq 3$, with initial values $T_0(x) = 0$, $T_1(x) = 1$, and $T_2(x) = x^2$. In analogy to (1), the tribonacci polynomials are given by the following explicit formula (Ramírez and Sirvent, 2014):

$$T_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}. \tag{2}$$

The *incomplete tribonacci polynomials* $T_n^{(s)}(x)$ were considered in (Ramírez and Sirvent, 2014) and are defined as

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}, \qquad 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{3}$$

Note that the incomplete tribonacci polynomials generalize the ordinary ones and reduce to them when $s = \lfloor \frac{n}{2} \rfloor$. The *incomplete tribonacci number*, denoted by $t_n^{(s)}$, is defined as the value of $T_n^{(s)}(x)$ at $x = 1$. Incomplete Fibonacci numbers and polynomials have also been considered and are defined in a comparable fashion; see, e.g., (Filipponi, 1996) and (Ramírez, 2013b). Some combinatorial identities for the incomplete Fibonacci numbers were given in (Belbachir and Belkhir, 2014) and a bi-periodic generalization was studied in (Ramírez, 2013a).

In (Ramírez and Sirvent, 2014), several identities were found for the incomplete tribonacci numbers and polynomials using various algebraic methods. In this paper, we supply combinatorial proofs of these identities using a weighted tiling interpretation of $T_n^{(s)}(x)$ (described in Theorem 0.1 below). In some cases, a further generalization of an identity can be given. In addition, using our interpretation, one also can find other relations not given in (Ramírez and Sirvent, 2014) that are satisfied by $T_n^{(s)}(x)$. In the final section, we provide an explicit formula for the generating function of $T_n^{(s)}(x)$, as requested in (Ramírez and Sirvent, 2014). Our derivation is combinatorial in nature and makes use of some identities involving $T_n(x)$.

2. Combinatorial interpretation for $T_n^{(s)}(x)$

We will use the following terminology. By a *square*, *domino*, or *tromino*, we will mean, respectively, a 1×1, 2×1, or 3×1 rectangular tile. A (linear) *tiling* of length n is a covering of the numbers $1, 2, \ldots, n$ written in a row by squares, dominos, and trominos, where tiles of the same kind are indistinguishable. Note that such tilings may be identified as *compositions* of n with parts of size 1, 2, or 3; see, e.g., (Heubach and Mansour, 2009). Let $\mathcal{T}_n$ denote the set of all tilings of length n. It is well known that $\mathcal{T}_n$ has cardinality t_{n+1}; see, e.g., (Benjamin and Quinn, 2003, p. 36). We will often represent squares, dominos, and trominos by the letters r, d, and t, respectively. Thus, a member of $\mathcal{T}_n$ may be regarded as a word in the alphabet $\{r, d, t\}$ in which there are $n-2i-3j$, i, and j occurrences of the letters r, d, and t, respectively, for some i and j.

By a *longer piece* within a member of $\mathcal{T}_n$, we will mean one that is either a domino or a tromino. Given $0 \le s \le \lfloor \frac{n}{2} \rfloor$, let $\mathcal{T}_n^{(s)}$ denote the subset of $\mathcal{T}_n$ whose members contain *at most* s longer pieces. For example, if $n = 5$ and $s = 1$, then

$$\mathcal{T}_5^{(1)} = \{r^5, dr^3, rdr^2, r^2dr, r^3d, tr^2, rtr, r^2t\}.$$

Note that $\mathcal{T}_n^{(s)}$ is all of $\mathcal{T}_n$ when $s = \lfloor \frac{n}{2} \rfloor$. By a *square-and-domino* tiling, we will mean a member of $\mathcal{T}_n$ that contains no trominos.

Given $\pi \in \mathcal{T}_n^{(s)}$, let $\delta(\pi)$ and $\nu(\pi)$ record the number of squares and dominos, respectively, in π. We now provide a combinatorial interpretation of the polynomial $T_{n+1}^{(s)}(x)$ in terms of linear tilings.

Theorem 0.1: The polynomial $T_{n+1}^{(s)}(x)$ is the distribution for the statistic $2\delta + \mu$ on $\mathcal{T}_n^{(s)}$.

Proof: First note that $T_{n+1}^{(s)}(x)$ may be written as

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} B(n-i, i)(x), \tag{4}$$

where $B(n,i)(x) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-j}{i} x^{2n-i-3j}$. We next observe that when $x = 1$, the polynomial $B(n,i)(x)$ gives the cardinality of the set $\mathcal{B}_{n,i}$ consisting of square-and-domino tilings of length n in which the squares come in two colors, black and white, and containing i dominos and white squares combined. To see this, note that members of $\mathcal{B}_{n,i}$ containing exactly j dominos are in one-to-one correspondence with words in the alphabet $\{D, W, B\}$ containing j D's, $i-j$ W's, and $n-i-j$ B's and thus have cardinality

$$\binom{n-j}{j, i-j, n-i-j} = \frac{(n-j)!}{j!(i-j)!(n-i-j)!} = \binom{n-j}{i}\binom{i}{j}.$$

Summing over j gives

$$|\mathcal{B}_{n,i}| = \sum_{j=0}^{i} \binom{i}{j}\binom{n-j}{i}.$$

Given $\pi \in \mathcal{B}_{n,i}$, let $\delta_1(\pi)$ and $\delta_2(\pi)$ record the number of black and white squares, respectively. Thus, if $\pi \in \mathcal{B}_{n,i}$ has j dominos, then

$$2\delta_1(\pi) + \delta_2(\pi) = 2(n-i-j) + i - j = 2n - i - 3j.$$

Considering all j, this implies $B(n,i)(x)$ is the distribution polynomial on $\mathcal{B}_{n,i}$ for the statistic $2\delta_1(\pi) + \delta_2(\pi)$. Suppose now $\lambda \in \mathcal{B}_{n-i,i}$ is given and contains j dominos for some j, where $0 \le i \le s$. We replace each domino of λ with a tromino and each white square with a domino. The resulting tiling λ' belongs to $\mathcal{T}_n^{(s)}$ and has j trominos, $i-j$ dominos, and $n-2i-j$ squares. Thus we have

$$2\delta(\lambda') + \nu(\lambda') = 2\delta_1(\lambda) + \delta_2(\lambda)$$

for all $\lambda \in \mathcal{B}_{n-i,i}$. By (4), it follows that $T_{n+1}^{(s)}(x)$ is the distribution on $\cup_{i=0}^{s}\mathcal{B}_{n-i,i}$ for $2\delta_1+\delta_2$, equivalently, for the distribution of $2\delta+\nu$ on $\mathcal{T}_n^{(s)}$. □

Remark: Taking $x=1$ in the prior theorem shows that the cardinality of $\mathcal{T}_n^{(s)}$ is $t_{n+1}^{(s)}$. Taking $s=\left\lfloor\frac{n}{2}\right\rfloor$ shows that $T_{n+1}(x)$ is the distribution polynomial for $2\delta+\mu$ on all of $\mathcal{T}_n$.

Using our interpretation for $T_n^{(s)}(x)$, one obtains the following recurrence formula from (Ramírez and Sirvent, 2014) as a corollary.

Corollary 0.2: If $n \geq 2s+1$, then

$$T_{n+3}^{(s)}(x) = x^2 T_{n+2}^{(s)}(x) + x T_{n+1}^{(s)}(x) + T_n^{(s)}(x) - (xB(n-s,s)(x) + B(n-1-s,s)(x)). \tag{5}$$

Proof: We will show that the right-hand side of (5) gives the weighted sum of all the members of $\mathcal{T}_{n+2}^{(s)}$ with respect to the statistic $2\delta+\nu$ by considering the final piece. The first term clearly accounts for all tilings ending in a square. On the other hand, if a member of $\mathcal{T}_{n+2}^{(s)}$ ends in a longer piece, then there can be at most $s-1$ additional longer pieces. From the proof of Theorem 0.1 above, we have for each m that $B(m-s,s)(x)$ gives the weight of all members of $\mathcal{T}_m^{(s)}$ containing exactly s longer pieces. Note that addition of a longer piece to the end of a tiling already containing s longer pieces is not allowed. Thus, by subtraction, the total weight of all members of $\mathcal{T}_{n+2}^{(s)}$ ending in a domino is given by $x(T_{n+1}^{(s)}(x) - B(n-s,s)(x))$ and the weight of those ending in a tromino by $T_n^{(s)}(x) - B(n-1-s,s)(x)$, which completes the proof. □

3. Some identities of $T_n^{(s)}(x)$

In this section, we first generalize some previous identities for $T_n^{(s)}$, which were shown by algebraic methods, using our combinatorial interpretation for $T_n^{(s)}(x)$. We also consider some further identities for $T_n^{(s)}(x)$ that can be obtained using Theorem 0.1. In this section and the next, by the *weight* of a subset S of $\mathcal{T}_n$ or $\mathcal{T}_n^{(s)}$, we will mean the sum $\sum_{\lambda\in S} x^{2\delta(\lambda)+\nu(\lambda)}$.

The $x=1$ case of the following identity was shown in (Ramírez and Sirvent, 2014) algebraically using induction.

Identity 0.3: If $h \geq 1$ and $n \geq 2s+2$, then

$$\sum_{i=0}^{h-1} x^{2(h-i-1)} T_{n+i}^{(s)}(x) = \frac{1}{1+x^3}\left(T_{n+h+2}^{(s+1)}(x) - x^{2h}T_{n+2}^{(s+1)}(x) + x^{2h+1}T_n^{(s)}(x) - xT_{n+h}^{(s)}(x)\right). \tag{6}$$

Proof: We show, equivalently,

$$T_{n+h+2}^{(s+1)}(x) = (1+x^3)\sum_{i=0}^{h-1} x^{2(h-i-1)} T_{n+i}^{(s)}(x) + x^{2h}T_{n+2}^{(s+1)}(x) + xT_{n+h}^{(s)}(x) - x^{2h+1}T_n^{(s)}(x).$$

For this, we'll argue that the right-hand side gives the total weight of all the members of $\mathcal{T}_{n+h+1}^{(s+1)}$. First note that $x^{2h}T_{n+2}^{(s+1)}(x)$ gives the weight of the members of $\mathcal{T}_{n+h+1}^{(s+1)}$ in which positions $n+2$ through $n+h+1$ are covered by squares (i.e., the right-most longer piece ends at position $n+1$ or before). On the other hand, the weight of all members of $\mathcal{T}_{n+h+1}^{(s+1)}$ whose right-most longer piece starts at position $n+i-1$ for some $0 \le i \le h-1$ is given by $\left(x^{2(h-i)+1}+x^{2(h-i-1)}\right)T_{n+i}^{(s)}(x)$ since such tilings λ are of the form $\lambda=\lambda' dr^{h-i}$ or $\lambda=\lambda' tr^{h-i-1}$ for some tiling λ' of length $n+i-1$, where r^m denotes a sequence of m squares. Note that $\lambda' \in \mathcal{T}_{n+i-1}^{(s)}$ since the number of longer pieces in λ' is limited to s. Summing over $0 \le i \le h-1$ gives the indexed sum on the right-hand side. Next, the term $xT_{n+h}^{(s)}(x)$ accounts for all members of $\mathcal{T}_{n+h+1}^{(s+1)}$ whose final piece is a domino which were missed in the sum. Finally, members of $\mathcal{T}_{n+h+1}^{(s+1)}$ of the form $\lambda' dr^h$, where λ' has length $n-1$, were accounted for by both the $x^{2h}T_{n+2}^{(s+1)}(x)$ term and by the $i=0$ term of the indexed sum; hence, we must subtract their weight, $x^{2h+1}T_n^{(s)}(x)$, to correct for this double count. Combining all of the cases above completes the proof. $\square$

The following identity from (Ramírez and Sirvent, 2014) gives a formula for the sum of all the incomplete tribonacci polynomials of a fixed order.

Identity 0.4: If $n \ge 1$, then

$$\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x)=(\ell+1)T_{n+1}(x)-\sum_{i=0}^{\ell}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j}{i}x^{2n-3(i+j)}, \tag{7}$$

where $\ell=\left\lfloor \frac{n}{2}\right\rfloor$.

Proof: Let $\lambda \in \mathcal{T}_n$ and suppose that it contains exactly k longer pieces, where $0 \le k \le \ell$. Then the weight of λ is counted by each summand of $\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x)$ such that $s \ge k$. That is, the tiling λ is counted $\ell+1-k$ times by this sum. The proof of Theorem 0.1 above shows that the total weight of all members of $\mathcal{T}_n$ containing exactly k longer pieces is given by

$$\sum_{j=0}^{k}\binom{k}{j}\binom{n-k-j}{k}x^{2n-3(k+j)},$$

upon considering the number j of dominos. Thus, the only inner sum in the double sum on the right-hand side of (7) in which λ is counted occurs when $i=k$ and here it is counted k times (due to the extra factor of $i=k$). Since λ is clearly counted $\ell+1$ times by the term $(\ell+1)T_{n+1}(x)$, we have by subtraction that λ is counted $\ell+1-k$ times by the right-hand side of (7) as well. Since λ was arbitrary, the identity follows. $\square$

The $x=1$ case of the next identity was conjectured in (Ramírez and Sirvent, 2014) and follows from the generating function proof given in (Kiliç and Prodinger, 2014).

Identity 0.5: If $n \ge 1$, then

$$\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x)=(\ell+1)T_{n+1}(x)-\sum_{j=1}^{n-1}(xT_j(x)+T_{j-1}(x))T_{n-j}(x), \tag{8}$$

where $\ell=\left\lfloor \frac{n}{2}\right\rfloor$.

Proof: Suppose $\lambda \in \mathcal{T}_n$ has exactly k longer pieces. By the proof of the preceding identity, we need only show that the weight of λ is counted k times by the sum on the right-hand side of (8). Note that $xT_j(x)T_{n-j}(x)$ gives the weight of all members of $\mathcal{T}_n$ in which a domino covers positions j and $j+1$, while $T_{j-1}(x)T_{n-j}(x)$ gives the weight of those in which a tromino covers positions $j-1$, j, and $j+1$. Thus, for each longer piece of λ, there is a term in the sum that counts the weight of λ, which implies that λ is counted k times by the sum, as desired. $\square$

Remark: Comparing the $x=1$ cases of the preceding two identities, it follows that

$$\sum_{n\geq 1} a_n z^n = \frac{z^2+z^3}{(1-z-z^2-z^3)^2},$$

where

$$a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} i \binom{i}{j} \binom{n-i-j}{i},$$

which can also be shown directly using the methods of (Wilf, 2005, Section 4.3), as was done in (Kiliç and Prodinger, 2014).

The following three identities do not occur in (Ramírez and Sirvent, 2014) but are a consequence of the combinatorial interpretation of $T_n^{(s)}(x)$ given in Theorem 0.1.

Identity 0.6: If $n \geq 2s+1$, then

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} (x^{i+2} T_{n-2i}^{(s-i)}(x) + x^i T_{n-2i-2}^{(s-i-1)}(x)). \tag{9}$$

Proof: Suppose a member of $\mathcal{T}_n^{(s)}$ ends in exactly i dominos, where $0 \leq i \leq s$. If the right-most piece that is not a domino is a square, then the tiles coming to the left of this square constitute a member of $\mathcal{T}_{n-2i-1}^{(s-i)}$ and thus the weight of the corresponding subset of $\mathcal{T}_n^{(s)}$ is $x^{i+2}T_{n-2i}^{(s-i)}(x)$. On the other hand, if the right-most non-domino piece is a tromino, then the tiles to the left of this tromino form a member of $\mathcal{T}_{n-2i-3}^{(s-i-1)}$ and thus the weight of the corresponding subset is $x^i T_{n-2i-2}^{(s-i-1)}(x)$. Considering all possible i gives (9). $\square$

Our next formula relates the incomplete tribonacci polynomials to the trinomial coefficients.

Identity 0.7: If $n \geq 3s+1$, then

$$T_n^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{s-i} \binom{s}{i,j,s-i-j} x^{2s-i-2j} T_{n-s-i-2j}^{(s-i-j)}(x). \tag{10}$$

Proof: Suppose that there are i dominos and j trominos among the final s tiles within a member of $\mathcal{T}_{n-1}^{(s)}$, where $n \geq 3s+1$. Then there are $\binom{s}{i,j,s-i-j}$ ways to arrange these tiles, which contribute $x^{2(s-i-j)+i}$ towards the weight, with the remaining tiles forming a member of $\mathcal{T}_{n-s-i-2j-1}^{(s-i-j)}$. Considering all possible i and j gives (10). $\square$

The incomplete Fibonacci polynomials introduced in (Ramírez, 2013a) are given as

$$F_n^{(s)}(x) = \sum_{r=0}^{s} \binom{n-r-1}{r} x^{n-2r-1}, \qquad 0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Our next identity relates the incomplete Fibonacci and tribonacci polynomials.

Identity 0.8: If $n \geq 2s$, then

$$T_{n+1}^{(s)}(x) = x^{n/2}F_{n+1}^{(s)}(x^{3/2}) + \sum_{i=1}^{n-2} x^{(i-1)/2} \sum_{j=0}^{s-1} (T_{n-i-1}^{(j)}(x) - T_{n-i-1}^{(j-1)}(x))F_i^{(s-j-1)}(x^{3/2}). \tag{11}$$

Proof: First note that the weight of all members of $\mathcal{T}_n^{(s)}$ that contain no trominos is given by

$$\sum_{r=0}^{s} \binom{n-r}{r} x^{2n-3r} = x^{n/2}F_{n+1}^{(s)}(x^{3/2}).$$

So assume a member of $\mathcal{T}_n^{(s)}$ contains at least one tromino and that the left-most tromino covers positions i through $i+2$. Suppose further that there are exactly r dominos to the left of the left-most tromino. Then the weight of all such members of $\mathcal{T}_n^{(s)}$ is given by $\binom{i-r-1}{r}x^{2i-3r-2}T_{n-i-1}^{(s-r-1)}(x)$. Summing over the possible i and r implies that the total weight of all the members of $\mathcal{T}_n^{(s)}$ containing at least one tromino is

$$\sum_{i=1}^{n-2}\sum_{r=0}^{s-1} \binom{i-r-1}{r} x^{2i-3r-2}T_{n-i-1}^{(s-r-1)}(x).$$

To obtain the expression in (11), we write $T_{n-i-1}^{(s-r-1)}$ as $\sum_{j=0}^{s-r-1}(T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)})$, where $T_{n-i-1}^{(-1)} = 0$. We then obtain a total weight formula of

$$\begin{aligned}
&\sum_{i=1}^{n-2}\sum_{r=0}^{s-1} \binom{i-r-1}{r} x^{2i-3r-2} \sum_{j=0}^{s-r-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) \\
&= \sum_{i=1}^{n-2}\sum_{j=0}^{s-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) \sum_{r=0}^{s-j-1} \binom{i-r-1}{r} x^{2i-3r-2} \\
&= \sum_{i=1}^{n-2}\sum_{j=0}^{s-1} (T_{n-i-1}^{(j)} - T_{n-i-1}^{(j-1)}) x^{(i-1)/2}F_i^{(s-j-1)}(x^{3/2}),
\end{aligned}$$

which gives (11). □

4. Generating function formula for $T_n^{(s)}(x)$

The generating function formula for the incomplete tribonacci numbers was found in (Ramírez and Sirvent, 2014) and a formula was requested for the corresponding polynomials. The next result provides such a formula. We remark that our method is more combinatorial than that used in (Ramírez and Sirvent, 2014) in the case $x = 1$ and thus supplies an alternate proof in that case.

Theorem 0.9: Let $Q_s(z)$ be the generating function for the incomplete tribonacci polynomials $T_n^{(s)}(x)$, where $n \geq 2s+1$. Then

$$\frac{Q_s(z)}{z^{2s+1}} = \frac{T_{2s+1}(x) + (T_{2s-1}(x) + xT_{2s}(x))z + T_{2s}(x)z^2 - z^2\left(\frac{x+z}{1-x^2z}\right)^{s+1}}{1 - x^2z - xz^2 - z^3}. \tag{12}$$

Proof: Let $r_n = r_n(x)$ be given by

$$r_n = \sum_{j=0}^{s} \binom{s}{j}\binom{n+s-j-2}{s} x^{2n+s-3j-3} + \sum_{j=0}^{s} \binom{s}{j}\binom{n+s-j-3}{s} x^{2n+s-3j-6}, \qquad n \geq 3,$$

with $r_0 = r_1 = 0$ and $r_2 = x^{s+1}$.

We claim that $r_i(x)$ gives the total weight with respect to the statistic $2\delta + \nu$ of all the members of $\mathcal{T}_{i+2s}$ containing exactly $s+1$ longer pieces and ending in a longer piece, the subset of which we will denote by $\mathcal{A}$. To show this, first note that $r_i(x)$ evaluated at $x = 1$ is seen to give the number of square-and-domino tilings of length $i+s-2$ or $i+s-3$ in which squares are black or white and having exactly s white squares and dominos combined. We then increase the length of each white square and each domino by one and add a domino to the end if the original tiling had length $i+s-2$ and add a tromino to the end if it had length $i+s-3$. This yields all members of $\mathcal{A}$ in a bijective manner and thus implies $r_i(x)$ at $x = 1$ gives the cardinality of $\mathcal{A}$. Note that members of $\mathcal{A}$ ending in a domino contain $i-j-2$ squares, $s-j+1$ dominos, and j trominos for some $0 \leq j \leq s$, while members of $\mathcal{A}$ ending is a tromino contain $i-j-3$ squares, $s-j$ dominos, and $j+1$ trominos for some j. Summing over j then implies that $r_i(x)$ is the distribution for the statistic $2\delta + \nu$ on $\mathcal{A}$, as claimed.

By the interpretation for $r_i(x)$ just described, the product $r_i(x)T_{n-2s-i}(x)$ gives the total weight of all members of $\mathcal{T}_{n-1}$ containing at least $s+1$ longer pieces in which the $(s+1)$-st longer piece ends at position $i+2s$ since the final $n-2s-i-1$ positions of such a member of $\mathcal{T}_{n-1}$ may be covered by any tiling. Summing over all possible i then gives the total weight of all members of $\mathcal{T}_{n-1}$ containing *strictly more* than s longer pieces. Subtracting from $T_n(x)$ thus gives the weight of all members of $\mathcal{T}_{n-1}$ containing *at most* s longer pieces and implies the following identity:

$$T_n^{(s)}(x) = T_n(x) - \sum_{i=0}^{n-2s-1} r_i(x)T_{n-2s-i}(x), \qquad n \geq 2s+1. \tag{13}$$

In order to find a closed form expression for $Q_s(z)$ using (13), we express $T_n = T_n(x)$ as follows:

$$T_n = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s}, \qquad n \geq 2s+1. \tag{14}$$

We provide a combinatorial proof of (14) as follows. Note that (14) is clearly true if $s = 0$ or if $n = 2s+1$ since $T_0 = T_{-1} = 0$, so we may assume $s \geq 1$ and $n \geq 2s+2$. Observe first that the $T_{n-2s}T_{2s+1}$ term gives the weight of all members of $\mathcal{T}_{n-1}$ in which there is no piece covering the boundary between positions $2s$ and $2s+1$. On the other hand, the total weight of the members of $\mathcal{T}_{n-1}$ in which a domino covers this boundary is given by $xT_{n-2s-1}T_{2s}$. Finally, if a tromino covers the boundary between positions $2s$ and $2s+1$, then that tromino covers either positions $2s-1$, $2s$, and $2s+1$ or positions $2s$, $2s+1$, and $2s+2$. In the former case, the weight of the corresponding members of $\mathcal{T}_{n-1}$ is $T_{n-2s-1}T_{2s-1}$, while in the latter it would be $T_{n-2s-2}T_{2s}$. Combining all of the cases above gives (14).

Multiplying both sides of the equation

$$T_n^{(s)} = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s} - \sum_{i=0}^{n-2s-1} r_i T_{n-2s-i}$$

by z^n and summing over $n \geq 2s+1$ yields

$$\frac{Q_s(z)}{z^{2s}} = \left(T_{2s+1} + (T_{2s-1} + xT_{2s})z + T_{2s}z^2 - \sum_{i\geq 0} r_i z^i\right) \cdot \sum_{n\geq 1} T_n z^n.$$

The proof is completed by noting

$$\sum_{i\geq 0} r_i z^i = z^2 \left(\frac{x+z}{1-x^2z}\right)^{s+1}$$

and

$$\sum_{n\geq 1} T_n z^n = \frac{z}{1-x^2z-xz^2-z^3},$$

the former being computed by the methods given in (Wilf, 2005, Section 4.3). □

Taking $x = 1$ in the prior theorem yields the following result.

Corollary 0.10: Let $q_s(z)$ be the generating function for the incomplete tribonacci numbers $t_n^{(s)}$. Then

$$\frac{q_s(z)}{z^{2s+1}} = \frac{t_{2s+1} + (t_{2s-1} + t_{2s})z + t_{2s}z^2 - z^2\left(\frac{1+z}{1-z}\right)^{s+1}}{1-z-z^2-z^3}. \tag{15}$$

Remark: Corollary 0.10 appears as (Ramírez and Sirvent, 2014, Theorem 8). We note however that there was a slight misstatement of this theorem; in particular, the $t_{2s} - 2$ factor multiplying z^2 in the numerator on the right-hand side of their formula should just be t_{2s}.

5. Conclusion

In this paper, we have provided a combinatorial interpretation for the incomplete tribonacci polynomials $T_n^{(s)}(x)$ in terms of weighted linear tilings involving three types of tiles. This not only allows one to supply combinatorial proofs of prior identities shown by algebraic methods but also to establish new ones. Furthermore, combinatorial reasoning provides a way of discovering new identities, some of which may be harder to find by purely algebraic means. We have also made use of our combinatorial interpretation for $T_n^{(s)}(x)$ in determining an explicit formula for its ordinary generating function, as previously requested.

The generalized k-Fibonacci numbers $f_n^{(k)}$ considered in (Knuth, 1973) satisfy the recurrence

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)}, \qquad n \geq k,$$

with initial values $f_i^{(k)} = 0$ for $0 \leq i \leq k-2$ and $f_{k-1}^{(k)} = 1$. Note that the Fibonacci numbers correspond to the case $k = 2$ of $f_n^{(k)}$, and the tribonacci numbers to the case $k = 3$. Perhaps the results of this paper (and prior ones) can be extended once an appropriate analogue of the incomplete tribonacci number (or polynomial) in the generalized k-Fibonacci setting has been identified.

Acknowledgement

I wish to thank Neil Sloane for pointing out to me the relation between the $B(n, i)$ coefficients and the Delannoy numbers.

References

Alladi, K. and V.E. Hoggatt, J. (1977). On tribonacci numbers and related functions. *Fibonacci Quarterly*, 15:42–45.

Barry, P. (2006). On integer-sequence-based constructions of generalized pascal triangles. *Journal of Integer Sequences*, 9(Article 06.2.4).

Belbachir, H. and Belkhir, A. (2014). Combinatorial expressions involving fibonacci, hyperfibonacci, and incomplete fibonacci numbers. *Journal of Integer Sequences*, 17(Article 14.4.3).

Benjamin, A. and Quinn, J. (2003). *Proofs that Really Count: The Art of Combinatorial Proof*. Mathematical Association of America, Washington DC.

Filipponi, P. (1996). Incomplete fibonacci and lucas numbers. *Rendiconti del Circolo Matematico di Palermo*, 45:37–56.

Heubach, S. and Mansour, T. (2009). *Combinatorics of Compositions and Words*. Chapman & Hall/CRC, Boca Raton, FL.

Kiliç, E. and Prodinger, H. (2014). A note on the conjecture of ramírez and sirvent. *Journal of Integer Sequences*, 17(Article 14.5.8).

Knuth, D. (1973). *The Art of Computer Programming: Sorting and Searching, Volume 3*. Addison-Wesley, Reading, MA.

Ramírez, J. (2013a). Bi-periodic incomplete fibonacci sequences. *Annales Mathematicae et Informaticae*, 42:83–92.

Ramírez, J. (2013b). Incomplete generalized fibonacci and lucas polynomials. *http://arxiv.org/abs/1308.4192*, pre-print.

Ramírez, J. and Sirvent, V. (2014). Incomplete tribonacci numbers and polynomials. *Journal of Integer Sequences*, 17(Article 14.4.2).

Sloane, N. (2010). On-line encyclopedia of integer sequences. *http://oeis.org*.

V.E. Hoggatt, J. and Bicknell, M. (1973). Generalized fibonacci polynomials. *Fibonacci Quarterly*, 11:457–465.

Wilf, H. (2005). *generatingfunctionology, third edition*. CRC Press, Boca Raton, FL.