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Domination Integrity of Some Path Related Graphs

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Abstract

The stability of a communication network is one of the important parameters for network designers and users. A communication network can be considered to be highly vulnerable if the destruction of a few elements cause large damage and only few members are able to communicate. In a communication network several vulnerability measures like binding number, toughness, scattering number, integrity, tenacity, edge tenacity and rupture degree are used to determine the resistance of network to the disruption after the failure of certain nodes (vertices) or communication links (edges). Domination theory also provides a model to measure the vulnerability of a graph network. The domination integrity of a simple connected graph is one such measure. Here we determine the domination integrity of square graph of path as well as the graphs obtained by composition (lexicographic product) of two paths.

Keywords: Integrity; Domination Integrity; Square graph; Composition of Graphs

MSC 2010 No.: 05C38, 05C69, 05C76

1. Introduction

A graph structure is vulnerable if `*any small damage produces large consequences*'. The vulnerability implies a lack of resistance or weakness of graph network arising from deletion of vertices or edges or both. The design of any communication network should be such that it is not easily disrupted. Moreover it should remain stable even if it is attacked. Many graph theoretic parameters have been introduced to measure the vulnerability of communication networks. They includ binding number [Woodall (1973)], toughness [Chvatal (1973)], scattering number [Jung (1978)], integrity [Barefoot et al. (1987)], tenacity [Cozzens et al. (1994)], edge tenacity [Piazza et al. (1995)] and rupture degree [Li et al. (2005)].

In the analysis of the vulnerable communication network two quantities play a vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponent's network be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities Barefoot et al. (1987) have introduced the concept of integrity and discovered many results on this newly defined concept.

Definition 1.1.

The integrity of a graph G is denoted by $I(G)$ and defined by

$$I(G) = min\{|S| + m(G-S) : S \subset V(G)\},$$

where $m(G-S)$ is the order of a maximum component of $G-S$.

Definition 1.2.

A subset S of $V(G)$ is said to be an I-set, if $I(G) = |S| + m(G-S)$.

Bagga et al. (1992) have reported many results on integrity in a survey article. Goddard (1989) has investigated many results on integrity of graphs. Some characterizations and interrelations between integrity and other graph parameters are reported in Goddard and Swart (1990) while Mamut and Vumar (2007) have determined the integrity of the middle graph of some graphs. It is also observed that the bigger the integrity of network, more reliable is the functionality of the network after any disruption caused by non-functional devices (elements). The connectivity is useful to identify local weaknesses in some respect while the integrity gives a brief account of the vulnerability of the graph network.

Throughout this work we consider simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For any undefined terminology and notation related to the concept of domination in graph we refer to Haynes et al. (1998) while for the fundamental concepts in graph theory we rely upon Balakrishnan and Ranganathan (2012). In the remaining portion of this section we will give a brief summary of the definitions and information which are related to the present work.

Definition 1.3.

A subset S of $V(G)$ is called a dominating set if for every $v \in V(G) - S$, there exists $u \in S$ such that v is adjacent to u.

The theory of domination plays a vital role in determining the decision making bodies of minimum strength or weakness of a network when certain parts of it is paralyzed. In the case of disruption of a network, the damage will be more when the vital nodes are under siege. This motivated the study of the domination integrity when the sets of non-functioning nodes are dominating sets. The concept of domination integrity of a graph was introduced by Sundareswaran and Swaminathan (2010) as a new measure of vulnerability which is defined as follows.

Definition 1.4.

The domination integrity of a connected graph G denoted by $DI(G)$ and defined as

$$DI(G) = min\{|X| + m(G-X) : X \textit{ is a dominating set}\},$$

where $m(G-X)$ is the order of a maximum connected component of $G-X$.

The domination integrity of some standard graphs has been investigated by Sundareswaran and Swaminathan (2010). In the same paper they have investigated domination integrity of Binomial trees and Complete *k*-ary trees while in (2010, p. 92) they have investigated the domination integrity of the middle graph of some standard graphs. Sundareswaran and Swaminathan (2011, 2012) also investigated the domination integrity of trees and powers of cycles. Vaidya and Kothari (2012, 2013) have discussed domination integrity in the context of some graph operations and also of the splitting graph of path P_n and cycle C_n. Vaidya and Shah (2013, 2014) and investigated the domination integrity of shadow graphs of P_n, C_n, $K_{m,n}$ and $B_{n,n}$ and of the total graphs of P_n, C_n and $K_{1,n}$.

Definition 1.5.

For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 1.6.

The composition of two graphs G and H is denoted as $G[H]$ (also known as lexicographic product) whose vertex set is $V(G)\times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either u_1 is adjacent, i.e., to u_2 in G or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

Here, it is important to mention that, unlike the union, join, Cartesian product, direct product and strong product of two graphs, the composition of two graphs is not commutative.

Many results on the integrity of graphs in the context of union, join, composition and product of two graphs have been reported by Goddard and Swart (1988). The present work is intended to investigate the domination integrity of a square graph of P_n and composition (lexicographic product) of two paths.

2. Main Results

Theorem 2.1

$$DI(P_n^2) = \begin{cases} 2, & n = 2, \\ 3, & n = 3,4, \\ 4, & n = 5,6. \end{cases}$$

Proof:

Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and P_n^2 be the square graph of P_n. Then, $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n-3.$ The proof is divided into following three cases:

Case 1: $n = 2$

P_2^2 is P_2 itself. Consider $S = \{v_1\}$ which is a dominating set of P_2^2, then $m(G-S) = 1$. Thus, $|S| + m(G-S) = 2$. If we choose $S = \{v_2\}$, then also $|S| + m(G-S) = 2$. Hence, $DI(P_2^2) = 2$.

Case 2: $n = 3,4$

For $n = 3$ consider $S = \{v_2, v_3\}$ which is a dominating set for P_3^2 and $m(G-S) = 1$. Therefore, $|S| + m(G-S) = 3$. For $S = \{v_1, v_3\}$ also $|S| + m(G-S) = 3$. If $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$ then $m(G-S) = 2$ so $|S| + m(G-S) = 3$. Hence, $DI(P_3^2) = 3$.

For $n = 4$ consider $S = \{v_2, v_3\}$ which is a dominating set of for P_4^2 and $m(G-S) = 1$. Therefore, $|S| + m(G-S) = 3$. For $S = \{v_1, v_3\}$ or $S = \{v_2, v_4\}$ or $S = \{v_1, v_2\}$ or $S = \{v_3, v_4\}$, $m(G-S) = 2$, then for these choices of S we get $|S| + m(G-S) = 4$. If $S = \{v_i\}$, $i = 1,2,3,4$, then $m(G-S) = 3$ so $|S| + m(G-S) = 4$. Hence, $DI(P_4^2) = 3$.

Case 3: $n = 5,6$

$S=\{v_3,v_4\}$ is a dominating set for P_5^2 and P_6^2. Then, $m(G-S)=2$ and $\left|S\right|+m(G-S)=4$. It is easy to observe that there does not exist a dominating set S for which $*S*+m(G-S)\leq 3$. Therefore, $DI(P_n^2)=4$ for $n=5,6$.

Hence, from above three cases, we have

$$DI(P_n^2)=\begin{cases}2, & n=2,\\ 3, & n=3,4,\\ 4, & n=5,6.\end{cases}$$

Theorem 2.2.

For $n=7$ to 15,

$$DI(P_n^2)=\begin{cases}5+2i, & \text{if } n=7+4i, \quad \text{where } i=0,1,2,\\ 6+2i, & \text{if } n=7+4i+k, \quad \text{where } k=1,2,3 \text{ and } i=0,1.\end{cases}$$

Proof:

Let $S=\{v_{3+4j},v_{4+4j} \,/\, j=0 \text{ to } i\}\cup\{v_n\}$ when $n=7+4i, i=0,1,2$ (i.e., $n=7, 11, 15$). Then, $|S|=2i+3$ and $m(G-S)=2$. If $S=\{v_{3+4j},v_{4+4j} \,/\, j=0 \text{ to } i+1\}$ when $n=7+4i+k$, $k=1,2,3$, $i=0,1$ (i.e., $n=8, 9, 10, 12, 13, 14$), then $|S|=2i+4$ and $m(G-S)=2$. In both the cases S is a dominating set of P_n^2 as $v_1,v_2\in N(v_3)$ and $v_{5+4t},v_{6+4t}\in N(v_{4+4t})$, for $t=0,1,2,\ldots,i$, or $i-1$.

Now we claim that there does not exist any dominating set S_1 such that $|S_1| = |S|$ and $m(G-S_1)< m(G-S)$. If S_1 is a dominating set and $m(G-S_1)<m(G-S)=2$, then all the components will be K_1. Consequently, $|S_1| > |S|$. Hence, for any dominating set S_1, if $|S_1| = |S|$, then

$$m(G-S)\leq m(G-S_1). \tag{1}$$

We also claim that there does not exist any dominating set S_2 such that $|S_2| <|S|$ and $m(G-S_2)=m(G-S)=2$. But if S_2 is a dominating set and $|S_2| < |S|$, then due to construction of P_n^2, $G-S_2$ will give rise to at least one component with the number of vertices more than two. This is because, each vertex of P_n^2 is adjacent to the vertices which are at the distance two apart. This implies that there does not exist any dominating set S_2 such that $|S_2| < |S|$ and, consequently,

$$m(G-S_2)=m(G-S)=2\,. \tag{2}$$

Moreover if we consider any dominating set S_3 of P_n^2 such that $m(G-S_3)>2$, then

$$|S|+m(G-S)\ \leq\ |S_3|+m(G-S_3)\,. \tag{3}$$

Therefore, from equations (1), (2) and (3) we have

$$\begin{aligned}|S|+m(G-S)&=min\{|X|+m(G-X):X\ \textit{is a dominating set}\}\\&=DI(P_n^2)\,.\end{aligned}$$

Hence, for $n=7$ to 15

$$DI(P_n^2)=\begin{cases}5+2i, & \text{if } n=7+4i, \quad \text{where } i=0,1,2,\\ 6+2i, & \text{if } n=7+4i+k, \quad \text{where } k=1,2,3 \text{ and } i=0,1.\end{cases}$$

Theorem 2.3.

$$DI(P_n^2)=\begin{cases}9, & n=16,\\ 10, & n=17,18.\end{cases}$$

.

Proof:

To prove this result we consider following two cases:

Case 1: $n=16$

If $S=\{v_3,v_4,v_8,v_9,v_{13},v_{14}\}$, then $|S|=6$ and $m(G-S)=3$. Moreover, S is a dominating set as $v_1,v_2\in N(v_3), v_5,v_6\in N(v_4)$, $v_7\in N(v_8)$, $v_{10},v_{11}\in N(v_9)$ and $v_{12}\in N(v_{13})$, $v_{15},v_{16}\in N(v_{14})$. If for some dominating set S_1 of P_{16}^2, $m(G-S_1)=2$, then clearly

$$|S_1|\ >\ |S| \text{ so } |S_1|+m(G-S_1)>\ |S|+m(G-S)\,. \tag{4}$$

It can be verified that for any other dominating set S_2 of P_{16}^2 for which $m(G-S_2)=4$. Then,

$$|S_2|+m(G-S_2)\ \geq\ |S|+m(G-S)\,. \tag{5}$$

Thus, from equations (4) and (5) among all dominating set, $|S|+m(G-S)=6+3=9$ is minimum. Hence, $DI(P_{16}^2)=9$.

Case 2: $n = 17,18$

If $S = \{v_3, v_4, v_8, v_9, v_{13}, v_{14}, v_{17}\}$, then $|S| = 7$ and $m(G-S) = 3$. Moreover, S is dominating set P_{17}^2 and P_{18}^2. If for some dominating set S_1 of P_n^2, $m(G-S_1) = 2$, then clearly

$$|S_1| > |S| \text{ so } |S_1| + m(G-S_1) > |S| + m(G-S). \tag{6}$$

It can be verified that for any other dominating set S_2 of P_n^2 for which $m(G-S_2) = 4$. Then

$$|S_2| + m(G-S_2) \geq |S| + m(G-S). \tag{7}$$

Therefore, from equations (6) and (7) , $|S| + m(G-S) = 7+3=10$ is minimum. Thus, $DI(P_n^2) = 10$ for $n = 17,18$.

Hence, from above two cases,

$$DI(P_n^2) = \begin{cases} 9, & n = 16, \\ 10, & n = 17,18. \end{cases}$$

Theorem 2.4

For $n \geq 19$,

$$DI(P_n^2) = \begin{cases} 11, & \text{if } n = 19,\ 20, \\ 11+2i, & \text{if } n = 21+6i, \quad \text{where } i \in \{0\} \cup \mathbb{N}, \\ 12+2i, & \text{if } n = 21+6i+k, \ \text{where } k = 1,2,3 \text{ and } i \in \{0\} \cup \mathbb{N}, \\ 13+2i, & \text{if } n = 21+6i+k, \ \text{where } k = 4,5 \quad \text{and } i \in \{0\} \cup \mathbb{N}. \end{cases}$$

Proof:

To prove this result we consider following two cases:

Case 1: $n = 19, 20$

Consider $S = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}, v_{19}\}$. Then, $|S| = 7$ and $m(G-S) = 4$. Clearly S is a dominating set of P_n^2, for $n = 19, 20$.

Case 2: $n \geq 21$

Let $S_1 = \{v_3, v_4, v_9, v_{10}\}$.

- If $n = 21 + 6i$, where $i = 0,1,2,\ldots$, (i.e., for $n = 21, 27, 33, \ldots$), consider

$$S = S_1 \cup \{v_{15+6j}, v_{16+6j} \ / \ j = 0 \ to \ i\} \cup \{v_n\}.$$

  Then, $|S| = 7 + 2i$.

- If $n = 21 + 6i + k$, where $k = 1,2,3$ and $i = 0,1,2,\ldots$, (i.e., for $n = 22, 23, 24, 28, 29, 30 \ldots$), consider

$$S = S_1 \cup \{v_{15+6j}, v_{16+6j} \ / \ j = 0 \ to \ i+1\}.$$

  Then, $|S| = 8 + 2i$.

- If $n = 21 + 6i + k$, where $k = 4,5$ and $i = 0,1,2,\ldots$, (i.e., for $n = 25, 26, 31, 32, \ldots$), consider

$$S = S_1 \cup \{v_{15+6j}, v_{16+6j} \ / \ j = 0 \ to \ i+1\} \cup \{v_n\}.$$

  Then $|S| = 9 + 2i$.

In all the above cases, S will be a dominating set for P_n^2 as $v_1, v_2 \in N(v_3)$, $v_{5+6t}, v_{6+6t} \in N(v_{4+6t})$ and $v_{7+6t}, v_{8+6t} \in N(v_{9+6t})$, where $t \in \mathbb{N} \cup \{0\}$. Moreover, $m(G-S) = 4$.

Thus, we have found dominating sets for P_n^2.

Now, we discuss the minimality of $|S| + m(G-S)$. If we consider any dominating set S_1 of G such that, $|S_1| < |S|$, then due to the construction of P_n^2 (i.e., to convert $G - S_1$ into disconnected graph, we must include at least two consecutive vertices in S_1), it generates large value of $m(G - S_1)$ such that,

$$|S| + m(G-S) < |S_1| + m(G-S_1). \tag{8}$$

Let S_2 be any dominating set of P_n^2 such that $m(G - S_2) = 3$. Then, for $n = 19, 20, 21, 22, 23, 28, 31$,

$$|S| + m(G-S) \leq |S_2| + m(G-S_2), \tag{9}$$

and for $n = 24, 25, 26, 27, 29, 30$, $n \geq 32$,

$$| S | + m(G - S) < \ | S_2 | + m(G - S_2) . \tag{10}$$

Moreover, if S_3 is any dominating set of P_n^2 with $m(G - S_3) = 2$ or $m(G - S_3) = 1$, then clearly,

$$| S | + m(G - S) < \ | S_3 | + m(G - S_3) . \tag{11}$$

Therefore, from equations (8) to (11) we have,

$$\begin{aligned} | S | + m(G - S) &= min\{|X| + m(G - X) : X \ is \ a \ dominating \ set\} \\ &= DI(P_n^2) . \end{aligned}$$

Hence, for $n \geq 19$,

$$DI(P_n^2) = \begin{cases} 11, & \text{if } n = 19, 20, \\ 11 + 2i, & \text{if } n = 21 + 6i, \quad \text{where } i \in \{0\} \cup \mathbb{N}, \\ 12 + 2i, & \text{if } n = 21 + 6i + k, \ \text{where } k = 1, 2, 3 \text{ and } i \in \{0\} \cup \mathbb{N}, \\ 13 + 2i, & \text{if } n = 21 + 6i + k, \ \text{where } k = 4, 5 \ \text{ and } i \in \{0\} \cup \mathbb{N}. \end{cases}$$

Theorem 2.5.

$$DI\left(P_2\left[P_n\right]\right) = n + \left\lceil 2\sqrt{n+1} \ \right\rceil - 2 .$$

Proof:

Let P_2 be a path with vertices u_1, u_2 and P_n with $v_1, v_2, \ldots, v_n$. Let G be the graph $P_2\left[P_n\right]$. Then,

$$V(G) = \left\{(u_i, v_j) / 1 \leq i \leq 2, \ 1 \leq j \leq n\right\}$$

and

$$E(G) = \left\{(u_1, v_j)(u_2, v_k) / 1 \leq j \leq n, 1 \leq k \leq n\right\} \cup \{(u_1, v_j)(u_1, v_{j+1}), (u_2, v_j)(u_2, v_{j+1}) / 1 \leq j \leq n-1\} .$$

For the sake of convenience, we denote the vertices $(u_1, v_j) = w_{1j}, 1 \leq j \leq n$ and

$$(u_2, v_j) = w_{2j}, 1 \leq j \leq n .$$

The graph of $P_2[P_5]$ is shown in Figure 1 for better understanding of the notations and arrangement of vertices. Moreover $K_{n,n}$ is a subgraph of G and $DI(K_{n,n}) = n+1$, $DI(G) > n+1$.

Consider $S_1 = \left\{ w_{2j} / 1 \le j \le n \right\}$, $|S_1| = n$. Then, S_1 is a dominating set of G and $G - S_1 = P_n$ so $m(G - S_1) = n$.

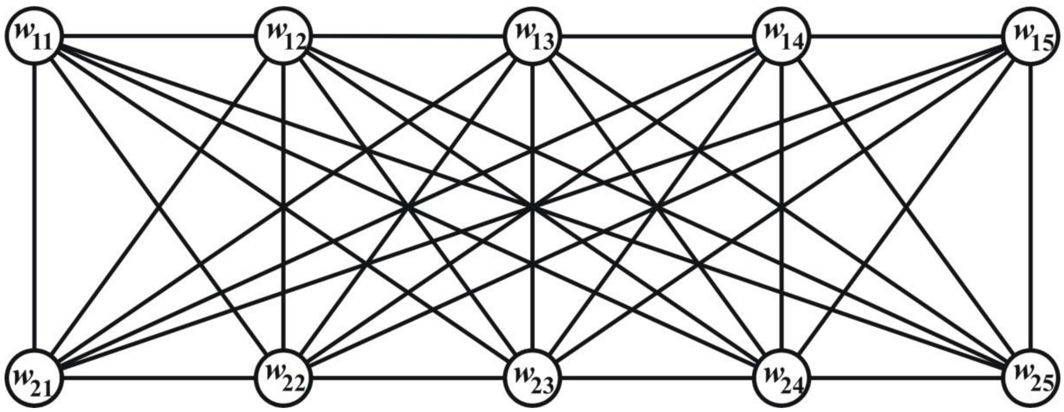

Figure 1: Arrangement of vertices in $P_2[P_5]$

Let $S_2 = \{w_{1k} = (u_1, v_k) / v_k \in I-$ set of $P_n\}$. Take $V_1 = \{v_k / v_k \in I - set\ of\ P_n\}$ so $|S_2| = |V_1|$. Consider $S = S_1 \cup S_2$. Then, S is also dominating set of G as $S_1 \subset S$. Here,

$$|S| = |S_1| + |S_2| = |S_1| + |V_1| \text{ and } G - S = P_n - V_1 \text{ so } m(G-S) = m(P_n - V_1).$$

Note that $I(P_n) = \left\lceil 2\sqrt{n+1} \right\rceil - 2$. So,

$$\begin{aligned} |S| + m(G-S) &= |S_1| + |V_1| + m(P_n - V_1) \\ &= |S_1| + I(P_n). \\ &= n + \left\lceil 2\sqrt{n+1} \right\rceil - 2 > n+1. \end{aligned}$$

Hence,

$$|S| + m(G-S) = n + \left\lceil 2\sqrt{n+1} \right\rceil - 2 > n+1. \tag{12}$$

Now we discuss the minimality of $|S| + m(G-S)$. If S_3 is any dominating set of G which is not containing S_1 or S_2 as a proper subset and $|S_3| = k < 2n$. Then, due to construction of G (w_{1j} is adjacent to w_{2k} for $1 \le i, k \le n$),

$$|S_3|+m(G-S_3)=k+2n-k=2n \;>|S|+m(G-S)\,. \tag{13}$$

Let S_5 be another dominating set of G such that $S_5=S_4\cup S_2$, where $S_4\subset S_1$ with $|S_4|<n$. In G, w_{1j} is adjacent to w_{2k} for $1\le i,k\le n$. Therefore,

$$m(G-S_5)=|S_2|+n-|S_4|\,.$$

Hence,

$$\begin{aligned}|S_5|+m(G-S_5) &= |S_2|+|S_4|+|S_2|+n-|S_4|\\ &= 2|S_2|+n\,.\\ &> |S|+m(G-S)\,.\end{aligned} \tag{14}$$

Therefore, from the above discussion and equations (13) and (14) we have $|S|+m(G-S)$ is minimum. Hence, from equation (12) and the minimality of $|S|+m(G-S)$ we have,

$$\begin{aligned}DI\left(P_2\left[P_n\right]\right) &= min\{|X|+m(G-X):X \text{ is a dominating set}\}\\ &= |S|+m(G-S)\,.\\ &= n+\left\lceil 2\sqrt{n+1}\ \right\rceil-2\,.\end{aligned}$$

Theorem 2.6.

$$DI\left(P_n\left[P_2\right]\right)=\begin{cases}4, & \text{if } n=2,3,\\ 6, & \text{if } n=4,5,\\ \dfrac{2n}{3}+4, & \text{if } n\ge 6 \ \ \& \, n\equiv 0(\text{mod } 3),\\ \dfrac{2(n-1)}{3}+4, & \text{if } n\ge 6 \ \ \& \, n\equiv 1(\text{mod } 3),\\ \dfrac{2(n+1)}{3}+4, & \text{if } n\ge 6 \ \ \& \, n\equiv 2(\text{mod } 3).\end{cases}$$

Proof:

Let P_n be a path with vertices $u_1,u_2,\ldots,u_n$ and P_2 with v_1,v_2. Let G be the graph $P_n\left[P_2\right]$. Then,

$$V(G)=\left\{(u_i,v_j)\,/\,1\le i\le n, 1\le j\le 2\right\}$$

and

$$E(G)=\left\{(u_i,v_j)(u_{i+1},v_j)/1\le i\le n-1,1\le j\le 2\right\}\cup\{(u_i,v_1)(u_{i+1},v_2)/1\le i\le n-1\}$$
$$\cup\{(u_i,v_2)(u_{i+1},v_1)/1\le i\le n-1\}.$$

Without loss of generality, we denote vertices $(u_i,v_1)=w_{i1},\ 1\le i\le n$ and $(u_i,v_2)=w_{i2},\ 1\le i\le n.$ The graph of $P_5[P_2]$ is shown in Figure 2 for better understanding of the notations and arrangement of vertices.

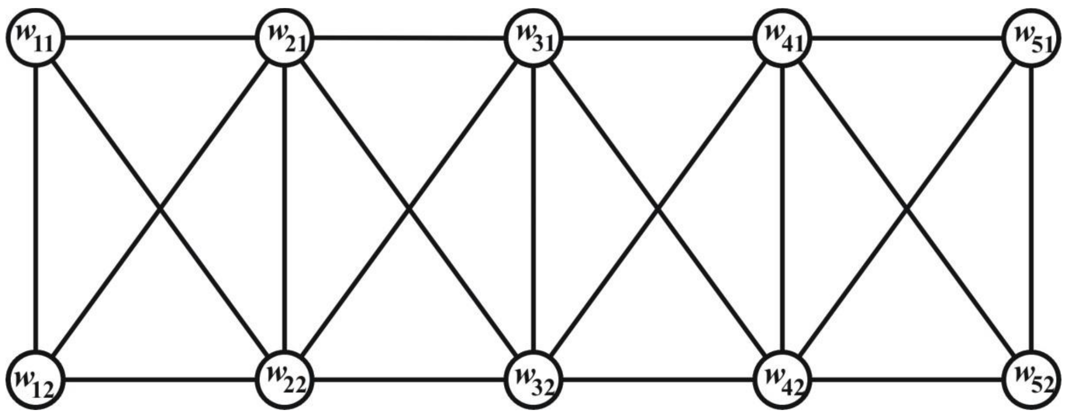

Figure 2. Arrangement of vertices in $P_5[P_2]$

To prove this result we consider following two cases:

Case 1: $n=2$ to 5

For $n=2$, $P_2[P_2]$ is isomorphic to complete graph K_4. Hence, $DI\left(P_2[P_2]\right)=4$.

For $n=3$, consider $S=\{w_{21},w_{22}\}$, which is a dominating set for $P_3[P_2]$ and $m(G-S)=2$. There, does not exist any dominating set S_1 of G such that $|S_1|+m(G-S_1)<\ |S|+m(G-S)$. Hence, $DI\left(P_3[P_2]\right)=4$.

For $n=4$, consider $S=\{w_{21},w_{22},w_{42}\}$, which is a dominating set for $P_4[P_2]$ and $m(G-S)=3$. Moreover, for any dominating set S_1 of G we have, $|S_1|+m(G-S_1)>\ |S|+m(G-S)$. Hence, $DI\left(P_4[P_2]\right)=6.$

For $n=5$, consider $S=\{w_{21},w_{22},w_{41},w_{42}\}$, which is a dominating set for $P_5[P_2]$ and $m(G-S)=2$. Moreover for any dominating set S_1 of G we have,

$$|S_1|+m(G-S_1)>\ |S|+m(G-S).$$

Hence, $DI(P_5[P_2]) = 6$.

Case 2: $n \geq 6$

Now we consider subset S of G as below:

- If $n \equiv 0(\bmod\ 3)$ (i.e., $n = 3k$) and $n \equiv 2(\bmod\ 3)$ (i.e., $n = 3k-1$), consider

$$S = \{w_{(2+3i)1} \mid 0 \leq j \leq k-1\} \cup \{w_{(2+3i)2} \mid 0 \leq j \leq k-1\} \text{ and } |S| = 2k.$$

So

$$|S| = \frac{2n}{3} \text{ for } n \equiv 0(\bmod\ 3) \text{ and } |S| = \frac{2(n+1)}{3} \text{ for } n \equiv 2(\bmod\ 3).$$

- If $n \equiv 1(\bmod\ 3)$ (i.e., $n = 3k+1$), consider

$$S = \{w_{(2+3i)1} \mid 0 \leq j \leq k-1\} \cup \{w_{(2+3i)2} \mid 0 \leq j \leq k-1\} \cup \{w_{n1}\} \text{ and } |S| = 2k+1 = \frac{2(n-1)}{3}.$$

In all the above cases S is a dominating set for G as $w_{(i-1)1}, w_{(i+1)1} \in N(w_{i1})$ and $w_{(i-1)2}, w_{(i+1)2} \in N(w_{i2})$. Moreover, $m(G-S) = 4$.

Now we discuss the minimality of $|S| + m(G-S)$. If we consider any dominating set S_1 of G such that $|S_1| < |S|$, then due to construction of G (i.e., to convert $G - S_1$ into disconnected graph we must include vertices w_{i1} and w_{i2} in S_1), It generates large value of $m(G-S_1)$ such that

$$|S| + m(G-S) < |S_1| + m(G-S_1). \tag{15}$$

Let S_2 be any dominating set of G such that $m(G-S_2) = 3$. Then, for $n \geq 6$,

$$|S| + m(G-S) < |S_2| + m(G-S_2). \tag{16}$$

Moreover, if S_3 is any dominating set of G with $m(G-S_3) = 2$ or $m(G-S_3) = 1$, then clearly,

$$|S| + m(G-S) < |S_3| + m(G-S_3). \tag{17}$$

Therefore, from above discussion and equations (15) to (17), $|S| + m(G-S)$ is minimum.

So, in both the cases we have,

$$
\begin{aligned}
|S| + m(G-S) &= min\{|X| + m(G-X) : X \text{ is a dominating set}\} \\
&= DI(G).
\end{aligned}
$$

Hence,

$$
DI\left(P_n[P_2]\right) = \begin{cases} 4, & \text{if } n = 2,3, \\ 6, & \text{if } n = 4,5, \\ \dfrac{2n}{3} + 4, & \text{if } n \geq 6 \ \ \& \, n \equiv 0 (\text{mod } 3), \\ \dfrac{2(n-1)}{3} + 4, & \text{if } n \geq 6 \ \ \& \, n \equiv 1 (\text{mod } 3), \\ \dfrac{2(n+1)}{3} + 4, & \text{if } n \geq 6 \ \ \& \, n \equiv 2 (\text{mod } 3). \end{cases}
$$

3. Conclusions

The vulnerability of a communication network is of prime importance for network designers and users. The domination integrity is one of the important parameters to measure vulnerability of graph network. We investigate the domination integrity of larger graph arising from graph operations like composition and square of a graph. Thus we have determined the vulnerability in the context of expansion of network. Investigation of the domination integrity for other graph products is an open area of research.

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