Solution of the SIR models of epidemics using MSGDTM

Asad A. Freihat
Pioneer Center for Gifted Students, Ministry of Education

Ali H. Handam
Al al-Bayt University

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Asad A. Freihat

Pioneer Center for Gifted Students, Ministry of Education
PO box 311, Jerash 26110, Jordan
asadfreihat@yahoo.com

Ali H. Handam

Department of Mathematics
Al al-Bayt University
P.O.Box: 130095, Al Mafraq, Jordan
ali.handam@windowslive.com

Abstract

Stochastic compartmental (e.g., SIR) models have proven useful for studying the epidemics of childhood diseases while taking into account the variability of the epidemic dynamics. Here, we use the multi-step generalized differential transform method (MSGDTM) to approximate the numerical solution of the SIR model and numerical simulations are presented graphically.

Keywords: Fractional differential equations; Caputo fractional derivative; multi-step generalized differential transform; SIR model

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1. Introduction

Over the past one hundred years, mathematics has been used to understand and predict the spread of diseases, relating important public-health questions to basic transmission parameters. From prehistory to the present day, diseases have been a source of fear and superstition. A comprehensive picture of disease dynamics requires a variety of mathematical tools, from model creation to solving differential equations to statistical analysis. Although mathematics has so far done quite well in dealing with epidemiology, there is no denying that there are certain factors

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which still lack proper mathematization. Epidemic models are used to understand the spread of infectious diseases in populations (Kermack and McKendrick, 1927), (Roumagnac et al., 2006). The practical use of epidemic models relies heavily on the realism put into the models. This does not mean that a reasonable model could include all possible effects but rather incorporate the mechanisms in the simplest possible fashion so as to maintain the major components that influence disease propagation. Great care should, however, exercised in using epidemic models to predict real phenomena (Shulgin et al., 1998). Eventhough, the SIR model is a standard compartmental model used to describe many of aspects of epidemiological diseases (Hethcote, 2000), (Lu et al., 2002), (Piccolo and Billings, 2005), (Smith, 1983) such as the dynamics of measles, chickenpox, mumps, or rubella (Olsen et al., 1988), (Anderson and May, 1991).

On another hand, Fractional calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing see for example (Miller and Ross, 1993), (Ertürk et al., 2011), (Lin, 2007).

In this paper, we use the multi-step generalized differential transform method to approximate the numerical solution of the SIR model and we compare our numerical results with a nonstandard numerical method and the fourth order Runge-Kutta method.

2. Model description

The Kermak-McKendrick model (Kermack and McKendrick, 1927) is one of the earliest triumphs in mathematical epidemiology (Brauer, 2006). We assume the population consists of three types of individuals, denoted by the letters S, I, and R (which is why this is called an SIR model). All these are functions of time.

$S(t)$ is the number of susceptible, who do not have the disease but could get it.

$I(t)$ is the number of infectives, who have the disease and can transmit it to others.

$R(t)$ is the number of removed, who cannot get the disease or transmit it: either they have a natural immunity, or they have recovered from the disease and are immune from getting it again, or they have been placed in isolation, or they have died.

The mathematical model does not distinguish between these possibilities. Schematically, the individual goes through consecutive states $I \rightarrow S \rightarrow R$, and is given by the following system of ordinary differential equations

$$\frac{dS(t)}{dt} = -P_1 SI, \tag{2.1}$$

$$\frac{dI(t)}{dt} = P_1 SI - P_2 I, \tag{2.2}$$

$$\frac{dR(t)}{dt} = P_2 I. \tag{2.3}$$

$P_2 > 0$ is called the removal rate and $P_1 > 0$ is called the infection rate.

3. Fractional Calculus

There are several prespectives of the derivatives in fractional calculus, e.g., the Riemann-Liouville, Grünwald-Letnikov, Caputo, and Generalized Functions news. Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduces an alternative approach, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 3.1.

A function $f(x)$ $(x > 0)$ is said to be in the space C_α $(\alpha \in \square)$ if it can be written as $f(x) = x^p f_1(x)$ for some $p > \alpha$ where $f_1(x)$ is continuous in $[0, \infty)$, and it is said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in \square$.

Definition 3.2.

The Riemann–Liouville integral operator of order α with $a \geq 0$ is defined as

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \tag{3.1}$$

$$(J_a^0 f)(x) = f(x). \tag{3.2}$$

Properties of the operator can be found in Caputo (1967), Miller and Ross (1993) and Podlubny (1999). Here, we only need the following: For $f \in C_\alpha$, $\alpha, \beta > 0$, $a \geq 0$, $c \in R$, $\gamma > -1$, we have

$$(J_a^\alpha J_a^\beta f)(x) = (J_a^\beta J_a^\alpha f)(x) = (J_a^{\alpha+\beta} f)(x), \tag{3.3}$$

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$$J_a^\alpha x^\gamma = \frac{x^{\gamma+\alpha}}{\Gamma(\alpha)} B_{\frac{x-a}{x}}(\alpha, \gamma+1), \tag{3.4}$$

where $B_\tau(\alpha, \gamma+1)$ is the incomplete beta function which is defined as

$$B_\tau(\alpha, \gamma+1) = \int_0^\tau t^{\alpha-1}(1-t)^\gamma \, dt, \tag{3.5}$$

$$J_a^\alpha e^{cx} = e^{ac}(x-a)^\alpha \sum_{k=0}^\infty \frac{[c(x-a)]^k}{\Gamma(\alpha+k+1)}. \tag{3.6}$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_a^α proposed by Caputo in his work on the theory of viscoelasticity.

Definition 3.3.

The Caputo fractional derivative of $f(x)$ of order $\alpha > 0$ with $a \geq 0$ is defined as

$$(D_a^\alpha f)(x) = (J_a^{m-\alpha} f^{(m)})(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} \, dt, \tag{3.7}$$

for $m-1 < \alpha \leq m, \ m \in \square, \ x \geq a, \ f \in C_{-1}^m$.

The Caputo fractional derivative was investigated by many authors, for $m-1 < \alpha \leq m$, $f(x) \in C_\alpha^m$ and $\alpha \geq -1$; we have

$$(J_a^\alpha D_a^\alpha f)(x) = J^m D^m f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(x-a)^k}{k!}. \tag{3.8}$$

For the mathematical properties of fractional derivatives and integrals one may consult the mentioned references.

4. Multi-step Generalized Differential Transform Method (MSGDTM)

Although the generalized differential transform method (GDTM) is used to provide approximate

solutions for nonlinear problems in terms of convergent series with easily computable components, it has been shown that the approximated solutions obtained are not valid for large t for some systems (Ertürk et al., 2008), (Momani and Odibat, 2008), (Odibat et al., 2008), (Odibat and Momani, 2008). Therefore we use the MSGDTM, which offers accurate solutions over a longer time frame compared to the standard generalized differential transform method [(Ertürk et al., 2011), (Freihat and Momani, 2012), (Freihat and AL-Smadi, 2013), (Momani et al., 2014), (Odibat et al., 2010), (Zeb et al., 2013)].

For this purpose, we consider the following initial value problem for systems of the fractional differential equations

$$D_*^{\alpha_1} y_1(t) = f_1(t, y_1, y_2, \ldots, y_n),$$
$$D_*^{\alpha_2} y_2(t) = f_1(t, y_1, y_2, \ldots, y_n), \tag{4.1}$$
$$\vdots$$
$$D_*^{\alpha_n} y_n(t) = f_1(t, y_1, y_2, \ldots, y_n),$$

subject to the initial conditions

$$y_i(t_0) = c_i, \, i = 1, 2, \ldots, n, \tag{4.2}$$

where $D_*^{\alpha_i}$ is the Caputo fractional derivative of order α_i, where $0 < \alpha_i \leq 1$, for $i = 1, 2, \ldots, n$. Let $[t_0, T]$ be an interval over which we wish to determine the solution of the initial value problem (4.1)-(4.2). In actual applications of the GDTM, the Kth -order approximate solution of the initial value problem (4.1)-(4.2) may be expressed by the finite series

$$y_i(t) = \sum_{i=0}^{K} Y_i(k)(t - t_0)^{k \alpha_i}, \, t \in [t_0, T], \tag{4.3}$$

where $Y_i(k)$ satisfies the recurrence relation

$$\frac{\Gamma((k+1)\alpha_i + 1)}{\Gamma(k\alpha_i + 1)} Y_i(k+1) = F_i(k, Y_1, Y_2, \ldots, Y_n), \tag{4.4}$$

$Y_i(0) = c_i$ and $F_i(k, Y_1, Y_2, \ldots, Y_n)$ are the differential transforms of the functions $f_i(t, y_1, y_2, \ldots, y_n)$ for $i = 1, 2, \ldots, n$. The basic steps of the GDTM can be found in [Chongxin and Junjie, 2010), (Momani and Odibat, 2008), (Odibat et al., 2008)].

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Assume that the interval $[t_0,T]$ is divided into M subintervals $[t_{m-1},t_m]$, $m=1,2,\ldots,M$ of equal step size $h=(T-t_0)/M$ by using the nodes $t_m=t_0+mh$. The main ideas of the MSGDTM are as follows:

First, we apply the GDTM to the initial value problem (4.1)-(4.2) over the interval $[t_0,t_1]$; to obtain the approximate solution $y_{i,1}(t)$, $t\in[t_0,t_1]$, using the initial condition $y_i(t_0)=c_i$, for $i=1,2,\ldots,n$. For $m\geq 2$ and at each subinterval $[t_{m-1},t_m]$, we use the initial condition $y_{i,m}(t_{m-1})=y_{i,m-1}(t_{m-1})$ and apply the GDTM to the initial value problem (4.1)-(4.2) over the interval $[t_{m-1},t_m]$. The process is repeated and generates a sequence of approximate solutions $y_{i,m}(t)$, $m=1,2,\ldots,M$, for $i=1,2,\ldots,n$. Finally, the MSGDTM assumes the following solution

$$
y_i(t)=\begin{cases}
y_{i,1}(t), & t\in\left[t_0,t_1\right], \\
y_{i,2}(t), & t\in\left[t_1,t_2\right], \\
\vdots & \vdots \\
y_{i,M}(t), & t\in\left[t_{M-1},t_M\right].
\end{cases}
\tag{4.5}
$$

The new algorithm, MSGDTM, is simple for computational performance for all values of h. As we will see in the next section, the main advantage of the new algorithm is that the obtained solution converges for wide time regions.

5. Solving the fractional-order SIR models of epidemics using the MSGDTM algorithm

To demonstrate the effectiveness of this scheme, we consider the fractional-order SIR models of epidemics. This example is researched because approximate numerical solutions are available for them using other numerical schemes. This allows one to compare the results obtained using this scheme with solutions obtained using other schemes.

Now we introduce the fractional-order disease model of the system described by (5.1)-(5.3). In this system, the integer-order derivatives are replaced by the fractional-order derivatives, as follows

$$
D^{\alpha_1}S(t)=-P_1SI,
\tag{5.1}
$$

$$
D^{\alpha_2}I(t)=P_1SI-P_2I,
\tag{5.2}
$$

$$
D^{\alpha_3}R(t)=P_2I,
\tag{5.3}
$$

where (S, I, R) are the state variables, P_1 and P_2 are nonnegative constants, α_i, $i = 1, 2, 3$, are parameters describing the order of the fractional time-derivatives in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. Obviously, the classical integer-order SIR models of epidemics can be viewed as a special case from the fractional-order system by setting $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $P_1 = 0.001$ and $P_2 = 0.072$ for the fractional case, the parameter α is allowed to vary. In other words, the ultimate behavior of the fractional system response must converge to the response of the integer order version of the equation.

Applying the MSGDTM Algorithm to (5.1)-(5.3) gives

$$
\begin{cases}
S(k+1) = -P_1 \dfrac{\Gamma(\alpha_1 k + 1)}{\Gamma(\alpha_1(k+1) + 1)} \left[\displaystyle\sum_{l=0}^{k} S(l) I(k-l) \right], \\[4mm]
I(k+1) = \dfrac{\Gamma(\alpha_2 k + 1)}{\Gamma(\alpha_2(k+1) + 1)} \left[P_1 \left(\displaystyle\sum_{l=0}^{k} S(l) I(k-l) \right) - P_2 I(k) \right], \\[4mm]
R(k+1) = P_2 \dfrac{\Gamma(\alpha_3 k + 1)}{\Gamma(\alpha_3(k+1) + 1)} I(k),
\end{cases} \tag{5.4}
$$

where $S(k), I(k)$ and $R(k)$ are the differential transforms of $S(t), I(t)$ and $R(t)$ respectively. The differential transform of the initial conditions are given by $S(0) = c_1, I(0) = c_2$ and $R(0) = c_3$. In view of the differential inverse transform, the differential transform series solution for the system (5.1)-(5.3) can be obtained as

$$
\begin{cases}
s(t) = \displaystyle\sum_{n=0}^{N} S(n) t^{\alpha_1 n}, \\[4mm]
i(t) = \displaystyle\sum_{n=0}^{N} I(n) t^{\alpha_2 n}, \\[4mm]
r(t) = \displaystyle\sum_{n=0}^{N} R(n) t^{\alpha_3 n}.
\end{cases} \tag{5.5}
$$

According to the multi-step generalized differential transform method, the series solution for the system (5.1)-(5.3) is

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$$
s(t) = \begin{cases} \sum_{n=0}^{K} S_1(n) t^{\alpha_1 n}, & t \in [0, t_1], \\[2ex] \sum_{n=0}^{K} S_2(n)(t-t_1)^{\alpha_1 n}, & t \in [t_1, t_2], \\[1ex] \quad \vdots & \quad \vdots \\[1ex] \sum_{n=0}^{K} S_M(n)(t-t_{M-1})^{\alpha_1 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{5.6}
$$

$$
i(t) = \begin{cases} \sum_{n=0}^{K} I_1(n) t^{\alpha_2 n}, & t \in [0, t_1], \\[2ex] \sum_{n=0}^{K} I_2(n)(t-t_1)^{\alpha_2 n}, & t \in [t_1, t_2], \\[1ex] \quad \vdots & \quad \vdots \\[1ex] \sum_{n=0}^{K} I_M(n)(t-t_{M-1})^{\alpha_2 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{5.7}
$$

and

$$
r(t) = \begin{cases} \sum_{n=0}^{K} R_1(n) t^{\alpha_3 n}, & t \in [0, t_1], \\[2ex] \sum_{n=0}^{K} R_2(n)(t-t_1)^{\alpha_3 n}, & t \in [t_1, t_2], \\[1ex] \quad \vdots & \quad \vdots \\[1ex] \sum_{n=0}^{K} R_M(n)(t-t_{M-1})^{\alpha_3 n}, & t \in [t_{M-1}, t_M], \end{cases} \tag{5.8}
$$

where $S_i(n)$, $I_i(n)$ and $R_i(n)$ for $i = 1, 2, \ldots, M$ satisfy the following recurrence relations

$$
\begin{cases}
S_i(k+1) = -P_1 \dfrac{\Gamma(\alpha_1 k+1)}{\Gamma(\alpha_1(k+1)+1)}\left[\sum_{l=0}^{k} S_i(l)I_i(k-l)\right], \\[3mm]
I_i(k+1) = \dfrac{\Gamma(\alpha_2 k+1)}{\Gamma(\alpha_2(k+1)+1)}\left[P_1\left(\sum_{l=0}^{k} S_i(l)I_i(k-l)\right) - P_2 I_i(k)\right], \\[3mm]
R_i(k+1) = P_1 \dfrac{\Gamma(\alpha_3 k+1)}{\Gamma(\alpha_3(k+1)+1)} I_i(k),
\end{cases}
\qquad (5.9)
$$

such that

$$
S_i(0) = s_i(t_{i-1}) = s_{i-1}(t_{i-1}), \; I_i(0) = i_i(t_{i-1}) = i_{i-1}(t_{i-1})
$$

and

$$
R_i(0) = r_i(t_{i-1}) = r_{i-1}(t_{i-1}).
$$

Finally, we start with $S_0(0) = c_1, I_0(0) = c_2$ and $R_0(0) = c_3$ and using the recurrence relation given in (5.9), we obtain the multi-step solution given in (5.6)-(5.8).

6. Nonnegative Solutions

Let $R_+^3 = (S(t), I(t), R(t))^T$ For the proof of the theorem about non-negative solutions we will need the following Lemma

Lemma 6.1. Generalized Mean Value Theorem (Lin, 2007)

Let

$$
f(x) \in C[a,b] \text{ and } D^\alpha f(x) \in C[a,b] \text{ for } 0 < \alpha \le 1.
$$

Then we have,

$$
f(x) = f(a) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\zeta)(x-a)^\alpha,
$$

with $0 \le \zeta < x,$ for all $x \in (a,b].$

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Remark 6.2. (Zeb et al., 2013)

Suppose $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0 < \alpha \le 1$. It is clear from the above Lemma that if $D^\alpha f(x) \ge 0$, for all $x \in (0,b)$, then the function f is non-decreasing, and if $D^\alpha f(x) \le 0$, for all $x \in (0,b)$, then the function f is non-increasing.

Theorem 6.3.

There is a unique solution for the initial value problem given by (5.1)-(5.3), and the solution remains in R_+^3.

Proof:

The existence and uniqueness of the solution of (5.1)-(5.3), in $(0,\infty)$ can be obtained from ((Lin, 2007), Theorem 3.1 and Remark 3.2). We need to show that the domain R_+^3 is positively invariant. Since $D^{\alpha_1} S \mid_{S=0} = 0$, $D^{\alpha_2} I \mid_{I=0} = 0$ and $D^{\alpha_3} R \mid_{R=0} = P_2 I \ge 0$. On each hyper-plane bounding the nonnegative orthant, the vector field points into R_+^3.

7. Numerical Results

The MSGDTM is coded in the computer algebra package Mathematica. The Mathematica environment variable digits controlling the number of significant digits are set to 20 in all the calculations. The time range studied in this work is $[0,50]$ days with time step $\Delta t = 0.05$, and we get GDTM series solution of order $K = 10$ at each subinterval. We take the initial condition for the SIR model as $S(0) = 620$, $I(0) = 10$, $R(0) = 70$.

Figure 1 shows the phase portrait for the classical SIR model using the fourth-order Runge–Kutta method ($RK4$). Figure 2 shows the phase portrait for the classical SIR model using the multi-step generalized differential transform method. From the graphical results in Figure 1 and Figure 2, it can be seen the results obtained using the multi-step generalized differential transform method match the results of the $RK4$ very well, implying that the multi-step generalized differential transform method can predict the behavior of these variables accurately for the region under consideration. Figures $3-6$ show the phase portraits for the fractional SIR models of epidemic systems using the multi-step generalized differential transform method. From the numerical results in Figures $3-6$ it is clear that the approximate solutions depend continuously on the time-fractional derivative α_i, $i = 1,2,3$. The effective dimension $\sum$ of equations (5.1)-(5.3) is defined as the sum of orders $\alpha_1 + \alpha_2 + \alpha_3 = \sum$. Also in Figure 6 we can see that the numerical results exist in the fractional-order SIR model of epidemic systems with order as low as 0.3.
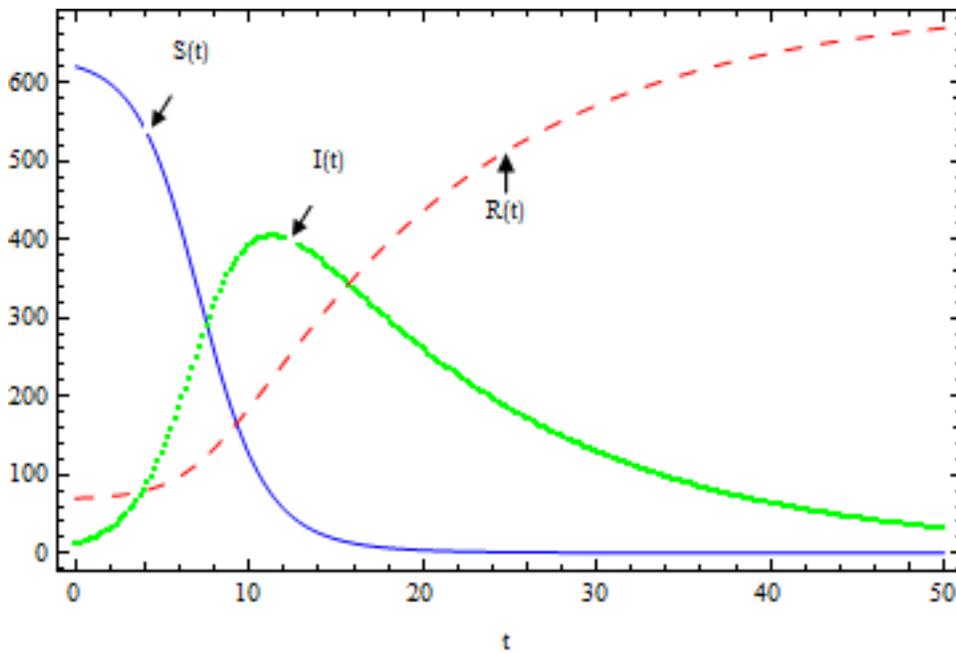
Figure 1: Phase plot of $S(t)$, $I(t)$ and $R(t)$ versus time, $\alpha_1 = \alpha_2 = \alpha_3 = 1,$
         (RK4 solution)
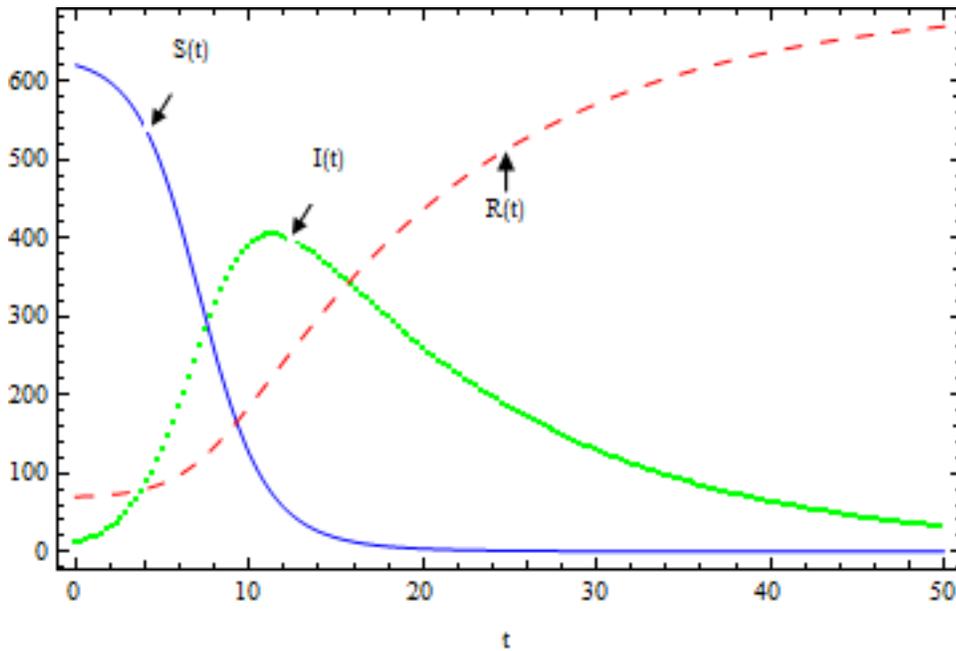
Figure 2: Phase plot of $S(t)$, $I(t)$ and $R(t)$ versus time, with $\alpha_1 = \alpha_2 = \alpha_3 = 1,$
         (MSGDTM solution)

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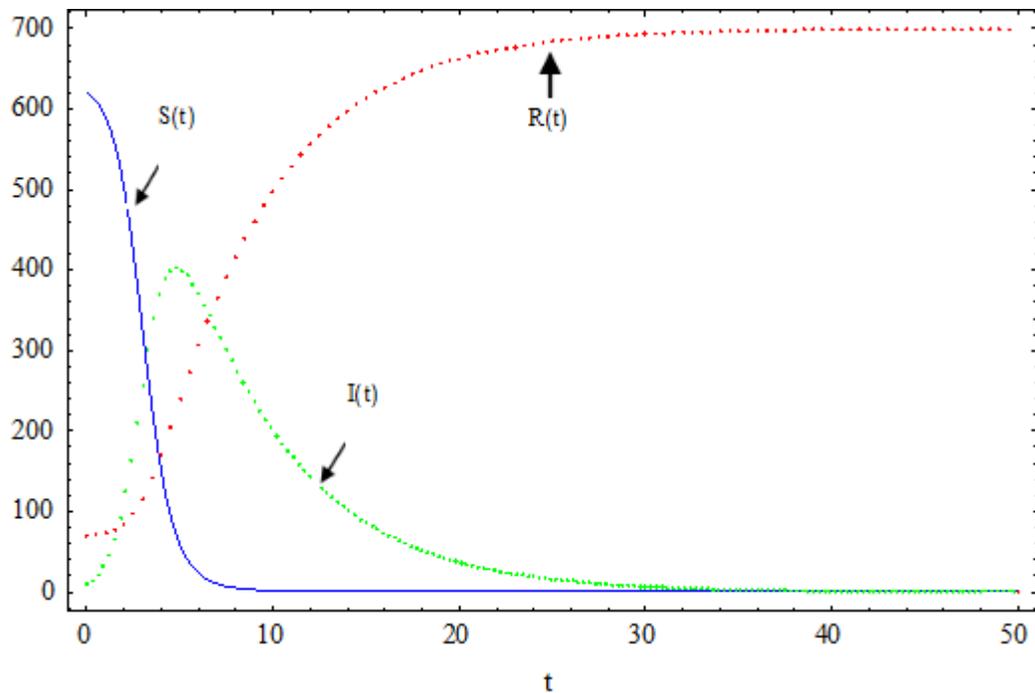
Figure 3: Phase plot of $S(t)$, $I(t)$ and $R(t)$ versus time, with $\alpha_1 = \alpha_2 = \alpha_3 = 0.8$, (MSGDTM solution)
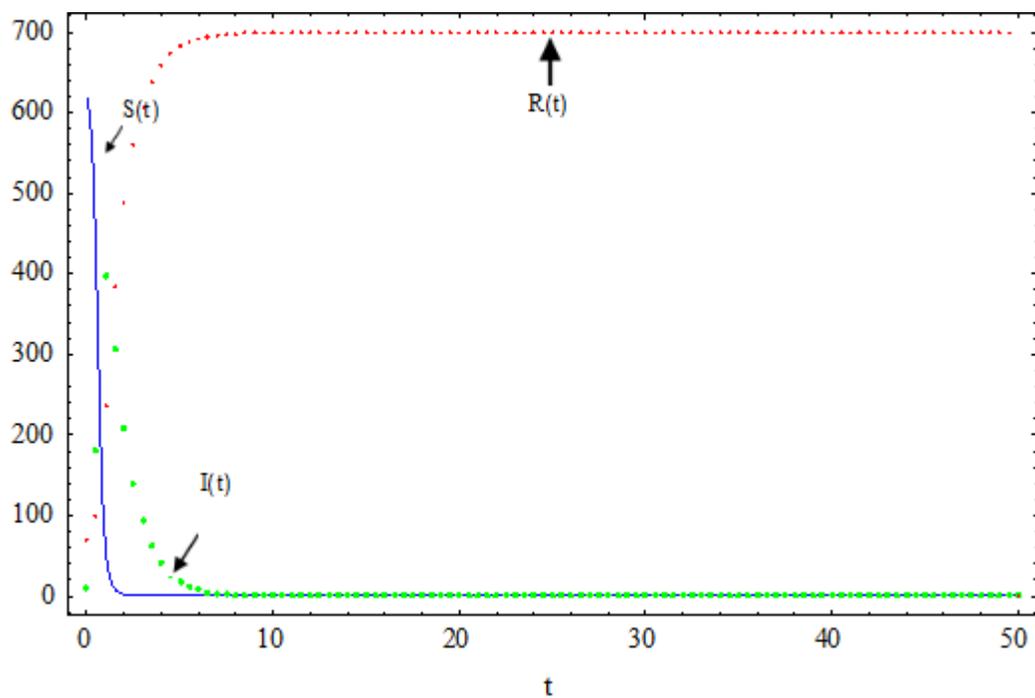
Figure 4: Phase plot of $S(t)$, $I(t)$ and $R(t)$ versus time, with $\alpha_1 = \alpha_2 = \alpha_3 = 0.6$, (MSGDTM solution)
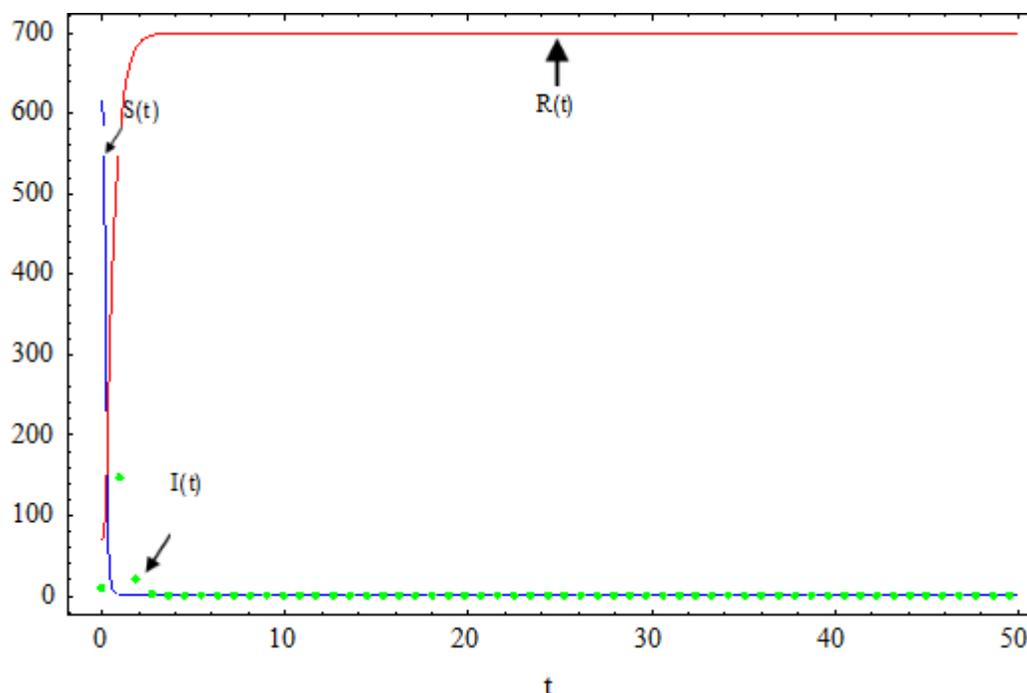
Figure 5: Phase plot of $S(t)$, $I(t)$ and $R(t)$ versus time, with $\alpha_1 = \alpha_2 = \alpha_3 = 0.4$, (MSGDTM solution)

8. Conclusions

The analytical approximations to the solutions of the epidemic models are reliable and confirm the power and ability of the MSGDTM as an easy method for computing the solution of nonlinear problems. In this paper, a fractional order differential SIR model is studied and its approximate solution is presented using a MSGDTM. The approximate solutions obtained by MSGDTM are highly accurate and valid for a long time. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Finally, the recent appearance of nonlinear fractional differential equations as models in some fields such as science and engineering makes it necessary to investigate the method of solutions for such equations. and we hope that this work is a step in this direction.

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