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Private Out-Domination Number of Generalized de Bruijn Digraphs

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Abstract

Dominating sets are widely applied in the design and efficient use of computer networks. They can be used to decide the placement of limited resources, so that every node has access to the resource through neighbouring node. The most efficient solution is one that avoids duplication of access to the resources. This more restricted version of minimum dominating set is called an private dominating set. A vertex v in a digraph D is called a private out-neighbor of the vertex u in S (subset of V(D)) if u is the only element in the intersection of in-neighborhood set of v and S. A subset S of the vertex set $V(D)$ of a digraph D is called a private out-dominating set of D if every vertex of $V - S$ is a private out-neighbor of some vertex of S. The minimum cardinality of a private out-dominating set is called the private out-domination number. In this paper, we investigate the private out-domination number of generalized de Bruijn digraphs. We estabilsh the bounds of private out-domination number. Finally, we present exact values and sharp upperbounds of private out-domination number of some classes of generalized de Bruijn digraphs

Keywords: Digraph, Private out-neighbor, Private out-domination number

MSC 2010 No.: 05C69, 05C20

1. Introduction

Domination in graphs has been studied extensively recently, since it has many applications. The book *"Fundamentals of domination in graphs"* by (Haynes et al., 1998) is entirely devoted to this area. Let $G = (V, E)$ be a connected graph. The open neighborhood $N(v)$ of a vertex v in a graph G consists of the set of vertices adjacent to v, that is, $N(v) = \{w \in V : vw \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V(G)$ is called a dominating set of G if every vertex of $V - S$ is adjacent to some vertex of S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. One variation of domination in graphs called perfect domination was studied by (Biggs, 1973; Livingston and Stout, 1990; Bange et al., 1998). A set $S \subseteq V(G)$ is called a perfect dominating set of G if every vertex of $V - S$ is adjacent to exactly one vertex in S. The minimum cardinality of a perfect dominating set of G is called the perfect domination number of G.

The concept of domination in undirected graphs is naturally extended to digraphs. In fact, domination in digraphs comes up more naturally in modeling real world problems.There is a survey on domination in digraphs written by (Ghosal et al., 1998).

The resource location problem in an interconnection network is one of the facility location problems. (Ghosal et al., 1998) and (Kikuchi and Shibata, 2003) found thats construction of the absorbants and the dominating sets corresponds to solving two kinds of resource location problems. For example, each vertex in an absorbant or a dominating set provides a service (file-server, and so on) for a network. In this case, every vertex has a direct access to file-servers. Since each file-server may cost a lot, the number of an absorbant or a dominating set has to be minimized.

In this paper, we introduce a new notion called private out-domination in digraphs. Our motivation for studying the private out-domination in digraphs arose from the work involving resource allocation and placement in parallel computers which was studied by (Livingston and Stout, 1990).

Let D be a digraph with vertex V and arc set A. The out-neighborhood set is defined as $N^+(u) = \{v : (u, v) \in A\}$ and the in-neighborhood set is defined as $N^-(u) = \{v : (v, u) \in A)\}$. The closed out-neighborhood of u is the set $N^+[u] = N^+(u) \cup \{u\}$ and the closed in-neighborhood of u is the set $N^-[u] = N^+(u) \cup \{u\}$. For $S \subseteq V$, the out-neighborhood of S is the set $N^+(S) = \bigcup_{s \in S} N^+(s)$ and the in-neighborhood is the set $N^-(S) = \bigcup_{s \in S} N^-(s)$. $N^+[S]$ and $N^-[S]$ are defined similarly. A subset $S \subseteq V$ is called a dominating set of D (or out-dominating set) if every vertex of $V - S$ is out-dominated by some vertex of S. The minimum cardinality of an out-dominating set D is called the out-domination number of D and is denoted by $\gamma(G)$. An *absorbant* of a digraph D is a set S of vertices of D such that for all $v \in V - S, N^+(v) \cap S \neq \emptyset$, i.e., $N^-[S] = V$. The absorbant number of D, dentoed by $\gamma_a(D)$, is defined as the minimum cardinality of an absorbant of D. A set $S \subseteq V$ is called a twin-dominating set of D if it is both a dominating set and an absorbant set of D and is denoted by $\gamma^*(D)$. The properties of domination number, absorbant number and twin domination number in generalized de Bruijn digraphs have been studied by

(Shan et al.,). (Araki, 2007; Araki, 2008) studied some domination parameters.

A vertex v in D is called a private out-neighbor of a vertex $u \in S$ in D if $N^-[v] \cap S = \{u\}$, and v is called a private in-neighbor of vertex $u \in S$ with respect to S in D if $N^+[v] \cap S = \{u\}$. The private out-neighbor set of v, $P_n^+[v,S]$ with respect to a set S in D is defined as $P_n^+[v,S] = N^+[v] - N^+[S - \{v\}]$ and private in-neighbor set of v, $P_n^-[v,S]$ with respect to a set S in D is defined as $P_n^-[v,S] = N^-[v] - N^-[S - \{v\}]$. A subset S of V is called a private out-dominating set of D if every vertex of $V - S$ is a private out-neighbor of some vertex of S. The minimum cardinality of a private out-dominating set is called the private out-domination number of D. It is denoted by $\gamma_p^+(D)$. A vertex v in D is called a private in-neighbor of the vertex u with respect to S in D if $N^+[v] \cap S = \{u\}$. A subset S of V is called a private absorbant of D if every vertex of $V - S$ is a private in-neighbor of some vertex of S. The minimum cardinality of a private absorbant is called the private absorbant number of D. It is denoted by $\gamma_p^-(D)$.

The generalized de Bruijn digraph $G_B(n,d)$ is defined in by the congruence equations as follows:

$$V(G_B(n,d)) = \{0, 1, 2, ..., n-1\}$$

$$A(G_B(n,d)) = \{(x,y) : y \equiv dx + i(modn), 0 \leq x \leq d-1\}$$

2. The private out-domination number of generalized de Bruijn digraphs

Let m, n be positive integers, $m|n$ means m divides n. In what follows, we may assume $d \geq 2$ and $n \geq d$. Now, we find private out-domination number for the generalized de Bruijn digraphs with $d = 1$ and $n = d$.

Proposition 2.1. For any $n \geq 1, \gamma_p^+(G_B(n,1)) = n$.

***P*roof.** The result is true since $G_B(n,1)$ is a graph with $A(G_B(n,d)) = \{(x,x) : x = 0, 1, \ldots, n-1\}$.

Proposition 2.2. For any $n \geq 1$, $\gamma_p^+(G_B(n,d)) = 1$, when $n = d$.

***P*roof.** The result is true since $G_B(n,d)$ is a symmetric complete graph.

Lemma 2.3. $\gamma_p^+(G_B(n,d)) \geq \left\lceil \frac{n}{d+1} \right\rceil$

***P*roof.** Let S be a private out-dominating set of $G_B(n,d)$. Then $|S| + d|S| \geq n$ and equality holds when $S \cap N^+(S) = \emptyset$. So $\gamma_P^+(G_B(n,d)) = |S| \geq \left\lceil \frac{n}{d+1} \right\rceil$.

Lemma 2.4. (Shibata et al., 1994) Every arc of $G_B(n,d)$ is a loop or a double arc if and only if $d = 1, n-1$ or n.

The private out-domination number of the following digraph $G_B(4,3)$ is 2, which is not satisfying the following theorem.

Example 2.5. Consider the graph $G_B(4,3)$.

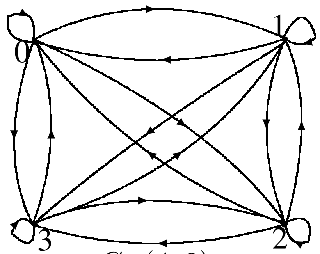

Fig. 1: The digraph $G_B(4,3)$.

$V(G_B(4,3)) = \{0,1,2,3\}$ and $A(G_B(4,3)) = \{(0,0),(0,1),(0,2),(1,3),(1,0),(1,1),(2,2),(2,3),(2,0),(3,1),(3,2),(3,3)\}$.
The set $S=\{0,2\}$ is a minimum private out-dominating set.

An interesting problem is how the private out-domination number reaches its maximum in some generalized de Bruijn digraphs. We consider $G_B(n,d)$ for the special case $n=d+1$.

Theorem 2.6. If $n=d+1$ and $n \neq 4$, then

$$\gamma_p^+(G_B(n,d)) = \begin{cases} 1, & d \text{ is even} \\ n & d \text{ is odd.} \end{cases}$$

***P*roof.** We divide the proof into two cases.

***C*ase 1:** d is even. Define $S=\{\frac{d}{2}\}$. $P_n^+\left(\frac{d}{2},S\right) = N^+(\frac{d}{2}) = \Big\{\frac{d^2}{2}(mod\ (d+1)), \frac{d^2}{2}+1(mod\ (d+1)), \ldots, \frac{d^2}{2}+d-1(mod\ (d+1))\Big\}$. By assumption that d is even, we see that $\frac{d^2}{2} \equiv \frac{d}{2}+1(mod\ (d+1))$. It follows that $P_n^+\left[\frac{d}{2},S\right] = \{0,1,2,\ldots,d-1\}$, and thus $S=\{\frac{d}{2}\}$ is an private out-dominating set of $G_B(d+1,d)$.

***C*ase 2:** d is odd.

Define $S=\{v\}$, for any $v \in V$. Clearly $N^+(v) = P_n^+(v,S)$ and by Lemma 2.4, $\{v\} \subseteq P_n^+(v,S)$. Since $n=d+1$, there is a vertex w in $V-S$ which is not a private out-neighbor of any vertex of S. Define $S=\{u,v\}$, for any two distinct vertices $u,v \in V$. By Lemma 2.4, $\{u\} \subseteq N^+(u)$. Since $n=d+1$, there is a vertex $x \in V(G_B(n,d))$ but $x \notin N^+(u)$ and also $\{v\} \subseteq N^+(v)$. Since $n=d+1$, there is a vertex $y \in V(G_B(n,d))$ but $y \notin N^+(v)$.

Suppose that $x=y$. Then $N^+(u)$ contains all the vertices of $G_B(d+1,d)$ other than x and $N^+(v)$ contains all the vertices of $G_B(d+1,d)$ other than y. Since $x=y$, $N^+(u)=N^+(v)$. We get $P_n^+(u,S)=\emptyset$ and $P_n^+(v,S)=\emptyset$. Therefore $P_n^+(z,S)=\emptyset$, for all $z \in S$. This shows that every vertex of $V-S$ is not a private in-neighbor of any vertex of S.

Suppose not, $x \in N^+(v)$, then there exists a vertex $w \in N^+(u)$ and $w \notin N^+(v)$. Clearly $N^+(u)-N^+(v)=\{w\}$ and $N^+(v)-N^+(u)=\{x\}$. Also $P_n^+(u,S)=\{w\}$ and $P_n^+(v,S)=\{x\}$.

Since $n \neq 4$, all other vertices in $V - S - \{w, x\}$ are not private out-neighbors of any vertex of S.

Now, define $S = \{v_1, v_2, \ldots, v_k\}, 3 \leq k < n$. Since $n = d + 1$ and by Lemma 2.4, the open out-neighborhood of any vertex v of S does not contain a vertex of $V(G_B(n, d))$ other than v. Let x_i be the vertex of $V(G_B(n, d))$, which is not in $N^+(v_i)$, for $i = 1, 2, \ldots, k$.

If $x_1 = x_2 = \cdots = x_k, k < n$, then x_1 is not a private out-neighbor of any vertex of S. Suppose that $x_r \neq x_s$ for some $r, s = 0, 1, \ldots, k$. Then $N^+(v_s) \cup N^+(v_r) = V(G_B(n, d))$ and for any vertex $v_k \in S, k \neq r, s$, we get $P_n^+[v_k, S] = \emptyset$. Clearly $P_n^+[v_r, S] = \{x_s\}$ and $P_n^+[v_s, S] = \{x_r\}$. Since $n \neq 4$, all other vertices in $V - S - \{v_r, v_s\}$ are not private out-neighbors of any vertex of S and hence the result follows.

For $d = 2$, by giving a method to determine private out-dominating sets of $G_B(n, d)$, we present some sufficient conditions for the private out-domination number of $G_B(n, d)$ to be the lower bound $\left\lceil \frac{n}{d+1} \right\rceil$.

Theorem 2.7. $\gamma_p^+(G_B(n, d)) = \left\lceil \frac{n}{d+1} \right\rceil$, where $d = 2$.

***P*roof.** Define $n \equiv r(mod\ (3))$.

***C*ase 1:** Suppose that $r = 0$.

Then construct $S = \bigcup_{i=0}^{\frac{n}{3}-1} \{\frac{n}{3} + i\}$. Now, we show that $P_n^+(v, S) \cap S = \emptyset$, for every $v \in S$. $P_n^+(\frac{n}{3}, S) = \{\frac{2n}{3}, \frac{2n}{3}+1\}, P_n^+(\frac{n}{3}+1, S) = \{\frac{2n}{3}+2, \frac{2n}{3}+3\}, \ldots, P_n^+(\frac{n}{3}+\frac{n}{6}, S) = \{0, 1\}, \ldots, P_n^+(\frac{2n}{3}-1, S) = \{\frac{n}{3} - 2, \frac{n}{3} - 1\}$. Since n is congruent to $0\ mod\ (n)$ and $0\ mod\ (3)$, we have,

$$P_n^+\left(\frac{n}{3} + k, S\right) = \left\{0, 1, \ldots, \frac{n}{3} - 1, \frac{2n}{3}, \ldots, n\right\}$$

for every $k = 0, \ldots, \frac{n}{3} - 1$. As desired, we have $P_n^+(v, S) \cap S = \emptyset$, for every $v \in S$.

Suppose that $P_n^+(v, S) \cap P_n^+(u, S) \neq \emptyset$, for some $u, v \in S$ and $u \neq v$. Then $1 \leq |P_n^+(v, S) \cap P_n^+(u, S)| \leq 2$.

Suppose that $|P_n^+(v, S) \cap P_n^+(u, S)| = 1$. Then $|P_n^+(v, S) \cup P_n^+(u, S)| = 3$. Let the common vertex be $z, z \in P_n^+(v, S) \cap P_n^+(u, S)$. Hence z is not a private out-neighbor of any vertex of S. Hence $z \in S$, which is a contradiction.

Suppose not, $|P_n^+(v, S) \cap P_n^+(u, S)| = 2$. Then $P_n^+(v, S) = P_n^+(u, S)$. This implies that $u = v$, which is a contradiction.

Since n is congruent to $0\ mod\ (n)$ as well as $0\ mod\ (3)$, we have, $\left(\bigcup_{v\in S} P_n^+(v, S)\right) \cup S = \{0, 1, \ldots, \frac{n}{3} - 1, \frac{2n}{3}, \ldots, n - 1\} \cup \{\frac{n}{3}, \ldots, \frac{2n-1}{3}\} = \{0, 1, 2, \ldots, n - 1\} = V$ and so S is a private out-dominating set of $G_B(n, d)$. Therefore, $\gamma_p^+(G_B(n, d)) \leq |S| = \frac{n}{3} = \left\lceil \frac{n}{d+1} \right\rceil$.

***C*ase 2:** Suppose that $r = 1$.

Then construct $S = \bigcup_{i=0}^{\frac{n-1}{3}} \left\{ \frac{n-1}{3} + i \right\}$. Define $S^* = S - \{\frac{n-1}{3}, \frac{2(n-1)}{3}\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Clearly $P_n^+(\frac{n-1}{3}, S) = \{\frac{2n+1}{3}\}$, since $\frac{2n-2}{3} \in S$, $P_n^+(\frac{2(n-1)}{3}, S) = \{\frac{n-4}{3}\}$, since $\frac{n-1}{3} \in S$, $P_n^+(\frac{n-1}{3}+1, S^*) = \{\frac{2n+1}{3}+1, \frac{2n+1}{3}+2\}$, $P_n^+(\frac{n-1}{3}+2, S^*) = \{\frac{2n+1}{3}+3, \frac{2n+1}{3}+4\}, \ldots,$
$P_n^+(\frac{n-1}{3}+\frac{n+2}{6}, S^*) = \{0, 1\}, \ldots, P_n^+(\frac{2n-5}{3}, S^*) = \{\frac{n-10}{3}, \frac{n-7}{3}\}$.

Therefore, for every $k = 1, \ldots, \frac{n-1}{3} - 1$, $P_n^+\left(\frac{n-1}{3} + k, S^*\right) = \left\{\frac{2n+6k-2}{3}(mod\ n), \frac{2n+6k+1}{3}(mod\ n)\right\} = \left\{\frac{2n+4}{3}, \frac{2n+7}{3}, \ldots, n-1, 0, 1, \ldots, \frac{n-10}{3}, \frac{n-7}{3}\right\}$. Since n is congruent to $0\ mod\ (n)$ as well as $1\ mod\ (3)$, we have, $\left(\bigcup_{v \in S} P_n^+(v, S^*)\right) \cup S^* \cup \{\frac{n-1}{3}, \frac{2n+1}{3}\} \cup \{\frac{2(n-1)}{3}, \frac{n-4}{3}\} = \{\frac{2n+4}{3}, \frac{2n+7}{3}, \ldots, n-1, 0, 1, \ldots, \frac{n-10}{3}, \frac{n-7}{3}\} \cup \{\frac{n+2}{3}, \ldots, \frac{2n-5}{3}\} \cup \{\frac{n-1}{3}, \frac{2n+1}{3}\} \cup \{\frac{2(n-1)}{3}, \frac{n-4}{3}\} - \{0, 1, 2, \ldots, n-1\} = V$ and thus S is a private out-dominating set of $\gamma_p^+(G_B(n,d))$. Therefore $\gamma_P^+(G_B(n,d)) \leq |S| = \frac{n+2}{3} = \lceil \frac{n}{d+1} \rceil$.

***C*ase 3:** Suppose that $r = 2$.

Then construct $S = \bigcup_{i=0}^{\frac{n-2}{3}} \left\{ \frac{n-2}{3} + i \right\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Clearly $P_n^+(\frac{n-2}{3}, S) = \{\frac{2n-1}{3}\}$, since $\frac{2n-4}{3} \in S$, $P_n^+(\frac{n-2}{3} + 1, S^*) = \{\frac{2n+2}{3}, \frac{2n+5}{3}\}$, $P_n^+(\frac{n-2}{3} + 2, S^*) = \{\frac{2n+8}{3}, \frac{2n+11}{3}\}, \ldots, P_n^+(\frac{n-2}{3} + \frac{n+4}{6}, S^*) = \{0, 1\}, \ldots, P_n^+(\frac{2n-4}{3}, S^*) = \{\frac{n-8}{3}, \frac{n-5}{3}\}$. Therefore, for every $k = 1, \ldots, \frac{n-2}{3} - 1$, $P_n^+(\frac{n-2}{3} + k, S^*) = \{\frac{2n-4+6k}{3}(mod\ n), \frac{2n+6k-1}{3}(mod\ n)\} = \{0, 1, \ldots, \frac{n-5}{3}, \frac{2n-5}{3}, \ldots, n-1\}$. Since n is congruent to $0\ mod\ (n)$ as well as $2\ mod\ (3)$, we have, $\left(\bigcup_{v \in S} P_n^+(v, S^*)\right) \cup S^* \cup \{\frac{n-2}{3}, \frac{2n-1}{3}\} = \{\frac{2n+2}{3}, \frac{2n+5}{3}, \ldots, n-1, 0, 1, \ldots, \frac{n-8}{3}, \frac{n-5}{3}\} \cup \{\frac{n+1}{3}, \ldots, \frac{2n-4}{3}\} \cup \{\frac{n-2}{3}, \frac{2n-1}{3}\} = \{0, 1, 2, \ldots, n-1\} = V$ and thus S is a private out-dominating set of $\gamma_p^+(G_B(n,d))$. Therefore $\gamma_p^+(G_B(n,d)) \leq |S| = \frac{n+1}{3} = \lceil \frac{n}{d+1} \rceil$. By Lemma 2.3, the theorem follows.

We consider $G_B(n,d)$ for the special case $d|n$.

Theorem 2.8. $\gamma_p^+(G_B(n,d)) \leq \frac{n}{d}$, when $d|n$.

***P*roof.** Define $S = \{0, 1, 2, \ldots, (\frac{n}{d} - 1)\}$. Then $|S| = \frac{n}{d}$.

***C*ase 1:** Suppose that $\frac{n}{d} < d$.

Define $S^* = S - \{0\}$. For each element in S^* in order, the private out-neighbor set contains d consecutive integers in order. So $P_n^+(v, S^*)$, for every $v \in S^*$ contains all elements of $V(G_B(n,d))$ other than $S^* \cup \{0\} \cup P_n^+(0, S)$. As desired, we have $P_n^+(v, S) \cap S^* = \emptyset$, for every $v \in S^*$.

Suppose that $P_n^+(v, S^*) \cap P_n^+(u, S^*) \neq \emptyset$, for some $u, v \in S^*$ and $u \neq v$. Then $1 \leq |P_n^+(v, S^*) \cap P_n^+(u, S^*)| \leq d - 1$.

Let $k = |P_n^+(v, S^*) \cap P_n^+(u, S^*)|, 1 \leq k \leq d-1$. Then $|P_n^+(v, S) \cup P_n^+(u, S)| = 2d - k$, so there exists at least one vertex $z \in P_n^+(v, S^*) \cap P_n^+(u, S^*)$ that is not a private out-neighbor of any vertex of S^*. Hence $z \in S^*$, which is a contradiction.

Clearly,

$$\begin{aligned} P_n^+(0, S) &= \left\{\frac{n}{d}, \frac{n}{d}+1, \ldots, d-1\right\}, \\ P_n^+(1, S^*) &= \{d, d+1, \ldots, 2d-1\}, \\ P_n^+(2, S^*) &= \{2d, 2d+1, \ldots, 3d-1\}, \ldots, \\ P_n^+\left(\frac{n}{d}-1, S^*\right) &= \{n-d, n-d+1, \ldots, n-1\}. \end{aligned}$$

Since n is congruent to $0 \; mod \; (n)$ as well as $0 \; mod \; (d)$, we have, $\left(\bigcup_{v \in S^*} P_n^+(v, S^*)\right) \cup S \cup \{\frac{n}{d}, \frac{n}{d}+1, \ldots, d-1\} = \{d, d+1, \ldots, n-1\} \cup \{0, 1, \ldots, \frac{n}{d}-1\} \cup \{\frac{n}{d}, \frac{n}{d}+1, \ldots, d-1\} = V$ and hence S is a private out-dominating set of $G_B(n,d)$ and $\gamma_p^+(G_B(n,d)) \leq |S| = \frac{n}{d}$.

Now we explain the steps given in the proof of the above Theorem Case 1 by giving an example. Consider the graph $G_B(12, 4)$. Here $S = \{0, 1, 2\}$.

Table 1: The vertices of $G_B(12, 4)$ and their corresponding out-neighbors.

(i)

0	1	2	3
4	5	6	7
8	9	10	11

(ii)

0	3	6	9
1	4	7	10
2	5	8	11

The set of elements in every row in Table 1(i) is exactly the out neighborhood of each vertex in the same row of Table 1(ii).

***Case* 2:** Suppose that $\frac{n}{d} = kd$, where k is a positive integer.

Define $S^* = S - \{0, 1, 2, \ldots, k-1\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Clearly $P_n^+(v, S) = \emptyset$, for every $v = 0$ to $k-1$. Clearly,

$$\begin{aligned} P_n^+\left(\frac{n}{d^2}, S^*\right) &= \left\{\frac{n}{d}, \frac{n}{d}+1, \ldots, \frac{n}{d}+d-1\right\}, \\ P_n^+\left(\frac{n}{d^2}+1, S^*\right) &= \left\{\frac{n}{d}+d, \frac{n}{d}+d+1, \ldots, \frac{n}{d}+2d-1\right\}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ P_n^+\left(\frac{n}{d}-1, S^*\right) &= \{n-d, n-d+1, \ldots, n-1\}. \end{aligned}$$

Since n is congruent to $0 \; mod \; (n)$ and $0 \; mod \; (d)$, we have, $\left(\bigcup_{v \in S^*} P_n^+(v, S^*)\right) \cup S = \{\frac{n}{d}, \frac{n}{d}+1, \ldots, n-1\} \cup \{0, 1, \ldots, \frac{n}{d}-1\} = V$ and hence S is a private out-dominating set of $G_B(n,d)$. Therefore $\gamma_p^+(G_B(n,d)) \leq |S| = \frac{n}{d}$.

Now we explain the steps given in the proof of the above Theorem Case 2 by giving an example. Consider the graph $G_B(32, 4)$. Here $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

Table 2: The vertices of $G_B(32,4)$ and their corresponding out-neighbors.

(i)

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

(ii)

0	8	16	24
1	9	17	25
2	10	18	26
3	11	19	27
4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

The set of elements in every row in Table 2(i) is exactly the out-neighborhood of each vertex in the same row of Table 2(ii).

***C*ase 3:** Suppose that $\frac{n}{d} > d$.

Then define $m = \lceil \frac{n}{d^2} \rceil$ and
$S^* = S - \{0, 1, 2, \ldots, m-1\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Clearly $P_n^+(v, S) = \emptyset$, for every $v = 0$ to $m-2$.

Since n is congruent to $0\ mod\ (n)$ as well as $0\ mod\ (d)$, we have, $\left(\bigcup_{v \in S^*} P_n^+(v, S^*) \right) \cup S = \{\frac{n}{d}+d, \frac{n}{d}+d+1, \ldots, n-1, \frac{n}{d}, \frac{n}{d}+1, \ldots, \frac{n}{d}+d-1\} \cup \{0, 1, \ldots, \frac{n}{d}-1\} = V$ and hence S is a private out-dominating set of $G_B(n,d)$ and $\gamma_p^+(G_B(n,d)) \leq |S| = \frac{n}{d}$. Therefore, $\gamma_p^+(G_B(n,d)) \leq \frac{n}{d}$.

Now we explain the steps given in the proof of the above Theorem Case 3 by giving an example. Consider the graph $G_B(20,4)$. Here $S = \{0, 1, 2, 3, 4\}$.

Table 3: The vertices of $G_B(20,4)$ and their corresponding out-neighbors.

(i)

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19

(ii)

0	5	10	15
1	6	11	16
2	7	12	17
3	8	13	18
4	9	14	19

The set of elements in every row in Table 3(i) is exactly the out-neighborhood of each vertex in the same row of Table 3(ii).

For $2 \leq d \leq 4$, by giving a method to determine private out-dominating sets of $G_B(n,d)$.

Theorem 2.9. For $d = 2, 4$ if $(d+1)|n$ and 2 does not divide n, then $\gamma_p^+(G_B(n,d)) = \frac{n}{d+1}$.

***P*roof. *C*ase 1:** Suppose that $d = 2$.

By Theorem, $\gamma_p^+(G_B(n,d)) = \lceil \frac{n}{d+1} \rceil$. As $(d+1)|n$, $\lceil \frac{n}{d+1} \rceil = \frac{n}{d+1}$.

***C*ase 2:** Suppose that $d = 4$.

Then $V(G_B(n,4)) = \bigcup_{i=0}^{\frac{n}{5}-1}\{5i, 5i+1, 5i+2, 5i+3, 5i+4\}$. Let $S = \{5i+2 | i = 0, 1, \ldots, \frac{n}{5}-1\}$. For every $v \in V$, $P_n^+(v,S) = \{20i+8,\ 20i+9,\ 20i+10,\ 20i+11\}$. The numbers $20i+8 = 5(4i+1)+3(mod\ n),\ 20i+9 = 5(4i+1)+4(mod\ n),\ 20i+10 = 5(4i+2)(mod\ n),\ 20i+11 = 5(4i+2)+1(mod\ n)$ are not equal to $5j+2$ for $0 \leq j \leq \frac{n}{5}-1$. Therefore $P_n^+(v,S)\bigcap S = \emptyset$, for all $v \in S$.

Suppose that $P_n^+(u,S)\bigcap P_n^+(v,S) \neq \emptyset$, for some $u, v \in S$. Then $1 \leq |P_n^+(u,S)\bigcap P_n^+(v,S)| \leq 3$.

Suppose that $|P_n^+(u,S)\bigcap P_n^+(v,S)| = 1$. Then $P_n^+(u,S)\bigcap P_n^+(v,S)$ consists of seven consecutive integers, so there exist atleast one $z \in P_n^+(u,S)\bigcap P_n^+(v,S)$ that is not a private out-neighbor of any vertex of S. Hence $z \in S$, which is a contradiction.

Suppose not, $|P_n^+(u,S)\bigcap P_n^+(v,S)| = 2$. Then $P_n^+(u,S)\bigcap P_n^+(v,S)$ consists of six consecutive integers, so there exist atleast two vertices $x, y \in P_n^+(u,S)\bigcap P_n^+(v,S)$ are not private out-neighbors of any vertex of S. Hence $x, y \in S$, which is a contradiction.

Suppose not, $|P_n^+(u,S)\bigcap P_n^+(v,S)| = 3$. Then $P_n^+(u,S)\bigcap P_n^+(v,S)$ consists of five consecutive integers, so there exist atleast three vertices $x, y, z \in P_n^+(u,S)\bigcap P_n^+(v,S)$ are not private out-neighbors of any vertex of S. Hence $x, y, z \in S$, which is a contradiction.

Suppose not, $|P_n^+(u,S)\bigcap P_n^+(v,S)| = 4$. Then $P_n^+(u,S) = P_n^+(v,S) = 4$ This implies that $u = v$, which is a contradiction.

Therefore S is a private out-dominating set. That is, $\gamma_p^+(G_B(n,4)) \leq |S| = \frac{n}{d+1}$. By Lemma 2.3, the theorem follows.

Theorem 2.10. For $d = 3$, $\gamma_p^+(G_B(n,3)) \leq \lceil \frac{n}{4} \rceil + 1$.

Proof. Define $n-4 \equiv r(mod\ 8)$ and $k = \lceil \frac{n-4}{8} \rceil$.

Case 1: Suppose that $r = 0$.

Then construct a set as $S = \bigcup_{i=0}^{\frac{n}{4}}\{k+i\}$ and $S^* = S - \{\frac{n-4}{8}, \frac{3n-4}{8}\}$. Now, we show that $P_n^+(v,S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Any vertex in S^* is of the form $\frac{n-4}{8}+j$, $j = 1, 2, \ldots, \frac{n}{4}-1$. $P_n^+(\frac{n-4}{8}+j, S^*) = \{3(\frac{n-4}{8}+j)(mod\ n), (3(\frac{n-4}{8}+j)+1)(mod\ n), (3(\frac{n-4}{8}+j)+2)(mod\ n)\}$. The numbers $3(\frac{n-4}{8}+j)(mod\ n)$, $(3(\frac{n-4}{8}+j)+1)(mod\ n)$ and $(3(\frac{n-4}{8}+j)+2)(mod\ n)$ are not equal to $\frac{n-4}{8}+i$, $j = 1, 2, \ldots, \frac{n}{4}-1$ and $i = 1, 2, \ldots, \frac{n}{4}-1$. Suppose not. If $3(\frac{n-4}{8}+j)+2 \equiv (\frac{n-4}{8}+i)(mod\ n)$, then by simple calculation we get $2(\frac{n-4}{8})+3j-i-1 \equiv 0(mod\ n)$, the maximum value of $3j-i$ is $3\frac{n}{4}-4$. Which is a contradiction to the fact $2(\frac{n-4}{8})+3j-k-1 \leq \frac{n}{4}+3\frac{n}{4}-3-2 = n-5 < n$. Similarly we can prove the other two terms are not equal to any number in S^*. Therefore $P_n^+(v,S^*) = \emptyset$, for every $v \in S^*$. Next we prove that $P_n^+(v,S^*) \cap P_n^+(u,S^*) = \emptyset$, for every $u, v \in S^*$ and $u \neq v$.

Suppose not. Then there exists two vertices $u, v \in S$ such that $P_n^+(v,S^*) \cap P_n^+(u,S^*) \neq \emptyset$. Also $1 \leq |P_n^+(v,S^*) \cap P_n^+(u,S^*)| \leq 3$.

Suppose that $|P_n^+(v,S^*)\cap P_n^+(u,S^*)|=1$. Then $|P_n^+(v,S^*)\cup P_n^+(u,S)|=5$. Note that $P_n^+(v,S^*)\cup P_n^+(u,S^*)$ contains five consecutive integers. Then $z\in P_n^+(v,S^*)\cap P_n^+(u,S^*)$ is not a private out-neighbor of any vertex of S and so $z\in S^*$, which is a contradiction.

Suppose not, $|P_n^+(v,S^*)\cap P_n^+(u,S^*)|=2$. Then $|P_n^+(v,S^*)\cup P_n^+(u,S^*)|=4$. Note that $P_n^+(v,S^*)\cup P_n^+(u,S^*)$ contains four consecutive integers. Then $P_n^+(v,S^*)\cap P_n^+(u,S^*)=\{x,y\}$. Both x and y are not private out-neighbor of any vertex of S^* and so $z\in S^*$, which is a contradiction.

Suppose not, $|P_n^+(v,S)\cap P_n^+(u,S)|=3$. Then $|P_n^+(v,S)\cup P_n^+(u,S)|=3$. This implies that $P_n^+(v,S)=P_n^+(u,S)$. Therefore $u=v$, which is a contradiction to the fact that $P_n^+(v,S^*)\cap S^*=\emptyset$, for every $v\in S^*$.

Clearly $P_n^+(\frac{3n+4}{8},S)=\{\frac{n-12}{8}\}$, since $\frac{n-4}{8},\frac{n+4}{8}\in S$ and $P_n^+(\frac{n-4}{8},S)=\{\frac{3n+4}{8}\}$, since $\frac{3n-12}{8},\frac{3n-4}{8}\in S$. Since n is congruent to $0\ mod\ (n)$ as well as $0\ mod\ (4)$, we have, $\left(\bigcup\limits_{v\in S^*}P_n^+(v,S^*)\right)\cup S^*\cup\{\frac{n-4}{8},\frac{3n+4}{8},\frac{3n-4}{8},\frac{n-12}{8}\}=\{\frac{n-20}{8},\ldots,\frac{3n-20}{8}\}\cup\{\frac{n+4}{8},\ldots,\frac{3n-12}{8}\}\cup\{\frac{n-4}{8},\frac{3n+4}{8},\frac{3n-4}{8},\frac{n-12}{8}\}=V$ and so S is a private out-dominating set of $G_B(n,3)$.

Therefore, $\gamma_p^+(G_B(n,3))\le|S|=\frac{3n-4}{8}=\lceil\frac{n}{4}\rceil+1$.

Case 2: Suppose that $r\neq 0$.

Then construct a set as $S=\bigcup\limits_{i=0}^{\lceil\frac{n}{4}\rceil-1}\{k+i\}$ and $S^*=S-\left\{\lceil\frac{n-4}{8}\rceil,\ \lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1\right\}$. By a similar argument as in Case 1. we can prove that $P_n^+(v,S^*)\cap S^*=\emptyset$, for every $v\in S^*$ and $P_n^+(v,S^*)\cap P_n^+(u,S^*)=\emptyset$, for every $u,v\in S^*$ and $u\neq v$.

Let x and y be variable bounds whose values are as defined as follows:

Table 4: The private out-neighborhood of $\lceil\frac{n-4}{8}\rceil$ and $\lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1$ in S which depends the value of r

r	1	2	3	4	5	6	7
x	0	0	0	0	1	1	1
y	*not defined*	0	1	2	0	1	2

Clearly $P_n^+\left(\lceil\frac{n-4}{8}\rceil,S\right)=\bigcup\limits_{i=x}^{2}\left\{\left(3\lceil\frac{n-4}{8}\rceil+i\right)(mod\ n)\right\}$ and $P_n^+\left(\lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1,S\right)=\bigcup\limits_{i=0}^{y}\left\{\left(3\left(\lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1\right)+i\right)(mod\ n)\right\}$.

Since n is congruent to $0\ mod\ (n)$ as well as $r\ mod\ (4)$, we have, $\left(\bigcup\limits_{v\in S^*}P_n^+(v,S^*)\right)\cup S^*\cup\left\{\lceil\frac{n-4}{8}\rceil,\lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1\right\}\cup P_n^+\left(\lceil\frac{n-4}{8}\rceil,S\right)\cup P_n^+\left(\lceil\frac{n-4}{8}\rceil+\lceil\frac{n}{4}\rceil-1,S\right)=V$ and so S is a private out-dominating set of $G_B(n,3)$ and $\gamma_p^+(G_B(n,3))\le|S|=\lceil\frac{n}{4}\rceil$.

Therefore, $\gamma_p^+(G_B(n,3))\le\lceil\frac{n}{4}\rceil+1$.

Example 2.11. The upper bound is sharp for the digraph $G_B(12,3)$

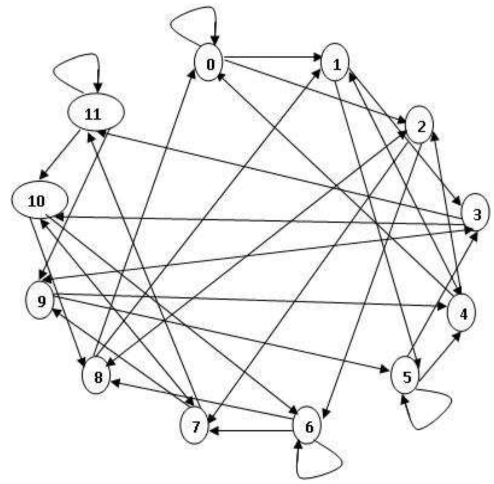

Fig. 2: The digraph $G_B(12,3)$.

$$
\begin{aligned}
V(G_B(12,3)) &= \{0,1,2,\ldots,11\}\\
A(G_B(12,3)) &= \{(0,0),(0,1),(0,2),(1,3),(1,4),(1,5),(2,6),(2,7),\\
&\quad (2,8),(3,9),(3,10),(3,11),(4,0),(4,1),(4,2),\\
&\quad (5,3),(5,4),(5,5),(6,6),(6,7),(6,8),(7,9),(7,10),\\
&\quad (7,11),(8,0),(8,1),(8,2),(9,3),(9,4),(9,5),(10,6),\\
&\quad (10,7),(10,8),(11,9),(11,10),(11,11)\}
\end{aligned}
$$

$S = \{1,2,3,4\}$ is a minimum private out-dominating set.

Theorem 2.12. If $n = dk+1$ and $d = 4$, then $\gamma_p^+(G_B(n,4)) \leq \lceil \frac{n}{5} \rceil + 1$.

***P*roof.** Define $n \equiv r(mod\ 60)$.

***C*ase 1:** Suppose that $r = 9$.

Then define $l = \frac{n-9}{60}$. Construct the sets $S = \bigcup_{i=0}^{\lceil \frac{n}{5} \rceil} \{4l+i\}$ and $S^* = S - \{4l, 4l+\lceil \frac{n}{5} \rceil\}$. Any vertex in S^* is of the form $\frac{n-9}{15}+j, j = 1,2,\ldots,\lceil \frac{n}{5} \rceil - 1$. $P_n^+(\frac{n-9+15j}{15}, S^*) = \{\frac{4n-36+60j}{15}(mod\, n), \frac{4n-21+60j}{15}(mod\, n), \frac{4n-6+60j}{15}(mod\ n), \frac{4n+9+60j}{15}(mod\ n)\}$. The numbers $\frac{4n-36+60j}{15}(mod\ n)$, $\frac{4n-21+60j}{15}(mod\, n)$, $\frac{4n-6+60j}{15}(mod\, n)$ and $\frac{4n+9+60j}{15}(mod\, n)$ are not equal to $\frac{n-9}{15}+i, i = 1,2,\ldots,\lceil \frac{n}{5} \rceil - 1$. Suppose not. If $\frac{4n+9+60j}{15} \equiv (\frac{n-9}{15}+i)(mod\, n)$, then by simple calculation we get $\frac{n+6}{5}+4j-i \equiv 0(mod\ n)$, the maximum value of $4j-i$ is $4\lceil \frac{n}{5} \rceil - 4$. Which is a contradiction to the fact $\frac{n+6}{5}+4j-i = \frac{n+6}{5}+4\frac{n}{5}+1-4 = n-\frac{9}{5} < n$.

Similarly we can prove the other three terms are not equal to any number in S^*. Therefore $P_n^+(v,S^*) \cap S^* = \emptyset$, for every $v \in S^*$. Next we prove that $P_n^+(v,S^*) \cap P_n^+(u,S^*) = \emptyset$, for every

$u, v \in S^*$ and $u \neq v$.

Suppose not, there exists two vertices $u, v \in S$ such that $P_n^+(v, S^*) \cap P_n^+(u, S^*) \neq \emptyset$, then $1 \leq |P_n^+(v, S^*) \cap P_n^+(u, S^*)| \leq 4$.

Suppose that $|P_n^+(v, S^*) \cap P_n^+(u, S^*)| = k, k = 1, 2, 3, 4$. Then $|P_n^+(v, S^*) \cup P_n^+(u, S^*)| = 2d - k$. Note that $P_n^+(v, S^*) \cup P_n^*(u, S^*)$ contains $2d - k$ consecutive integers. Then there exists at least one vertex $z \in P_n^+(v, S^*) \cap P_n^+(u, S^*)$ is not a private out-neighbor of any vertex of S and then $z \in S$. This is a contradiction.

Clearly $P_n^+(\frac{n-19}{15}, S) = \{\frac{4n+9}{15}(mod \quad n)\}$, since $\frac{4n-36}{15}(mod \quad n), \frac{4n-16}{15}(mod \quad n), \frac{4n-6}{15}(mod\ n) \in S$ and $P_n^+[(\frac{n-9}{15} + \lceil\frac{n}{5}\rceil, S)] = \{\frac{4n-36}{15} + 4\lceil\frac{n}{5}\rceil(mod\ n)\}, \frac{4n-36}{15} + 4\lceil\frac{n}{5}\rceil + i(mod\ n) \in S$, for $i = 1, 2, 3$. Since n is congruent to 0 mod (n) as well as r mod (60), we have, $\left(\bigcup_{v \in S^*} P_n^+(v, S^*)\right) \cup S^* \cup \{\frac{n-9}{15}, \frac{4n+9}{15}, \frac{n-9}{15} + \lceil\frac{n}{5}\rceil, \frac{4n-36}{15} + 4\lceil\frac{n}{5}\rceil\}(mod\ n) = \{\frac{4n+24}{15}, \frac{4n+39}{15}, \ldots, \frac{4n-66}{15} + 4\lceil\frac{n}{5}\rceil, \frac{4n-51}{15} + 4\lceil\frac{n}{5}\rceil\}(mod\ n) \cup \{\frac{n+6}{15}, \frac{n+21}{15}, \ldots, \frac{n-9}{15} + \lceil\frac{n}{5}\rceil - 1\} \cup \{\frac{n-9}{15}, \frac{4n+9}{15}, \frac{n-9}{15} + \lceil\frac{n}{5}\rceil, \frac{4n-36}{15} + 4\lceil\frac{n}{5}\rceil\}(mod\ n) = V$ and so S is a private out-dominating set of $G_B(n, 4)$.

Therefore $\gamma_p^+(G_B(n, 4)) \leq |S| = \lceil\frac{n}{5}\rceil + 1$.

Case 2: Suppose that $r \equiv 0(mod\ 5)$.

Then define $l = \frac{n-5}{20}$. Construct the set $S = \bigcup_{i=0}^{\lceil\frac{n}{5}\rceil - 1} \{8l + 2 + i\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S) \cap S = \emptyset$, for every $v \in S$ and $P_n^+(v, S) \cap P_n^+(u, S) = \emptyset$, for every $u, v \in S$ and $u \neq v$. Since n is congruent to 0 mod (n) as well as r mod (60), we have, $\left(\bigcup_{v \in S^*} P_n^+(v, S)\right) \cup S\{0, 1, 2, \ldots, n-1\} = V$ and so S is a private out-dominating set of $G_B(n, 4)$.

Therefore, $\gamma_p^+(G_B(n, 4)) \leq |S| = \lceil\frac{n}{5}\rceil$.

Case 3: Suppose that $r \neq 9$ and 5 does not divide r.

Then define $j = \frac{n-1}{20}, m = \lceil\frac{n-1}{20}\rceil$ and $k = \lfloor\frac{n-13}{60}\rfloor$. Construct the sets $S = \bigcup_{i=0}^{\lceil\frac{n}{5}\rceil - 1} \{m + k + i\}$ and $S^* = S - \{m + k, m + k + \lceil\frac{n}{5}\rceil - 1\}$. By a similar argument as in Case 1, we can prove that $P_n^+(v, S^*) \cap S^* = \emptyset$, for every $v \in S^*$ and $P_n^+(v, S^*) \cap P_n^+(u, S^*) = \emptyset$, for every $u, v \in S^*$ and $u \neq v$.

Let x and y be variable bounds whose values are as defined as follows:

Table 5: The private out-neighborhood of $m + k$ and $m + k + \lceil\frac{n}{5}\rceil - 1$

in S which depends the value of r

r	1	13	17	21	29	33	37	41	49	53	57
x	3	0	1	2	0	1	2	3	1	2	3
y	$not\ defined$	1	1	1	2	2	2	2	3	3	3

Clearly, $P_n^+(m+k,S)=\bigcup_{i=x}^{3}\{(4(m+k)+i)(mod\ n)\}$ and $P_n^+(m+k+\lceil\frac{n}{5}\rceil-1,S)=\bigcup_{i=0}^{y}\{(4(m+k+\lceil\frac{n}{5}\rceil-1)+i)(mod\ n)\}$.

Since n is congruent to $0\ mod\ (n)$ as well as $r\ mod\ (60)$, we have, $\left(\bigcup_{v\in S^*}P_n^+(v,S^*)\right)\cup S^*\cup\{m+k,m+k+\lceil\frac{n}{5}\rceil-1\}\cup P_n^+(m+k,S)\cup P_n^+(m+k+\lceil\frac{n}{5}\rceil-1,S)=V$ and so S is a private out-dominating set of $G_B(n,4)$ and $\gamma_p^+(G_B(n,4))\leq|S|=\lceil\frac{n}{5}\rceil$. Therefore, $\gamma_p^+(G_B(n,4))\leq\lceil\frac{n}{5}\rceil+1$.

Example 2.13. The upper bound is sharp. Consider the digraph $G_B(9,4)$.

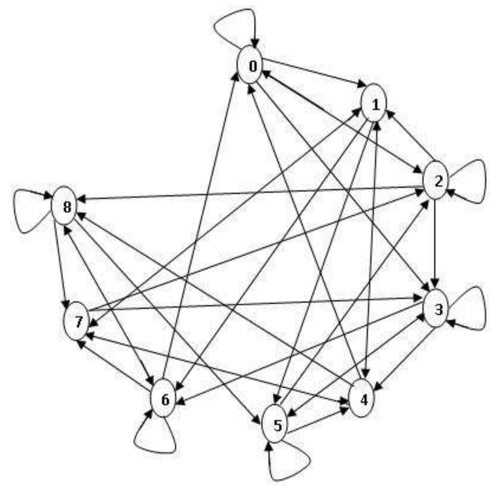

Fig. 3: The digraph $G_B(9,4)$.

$$
\begin{aligned}
V(G_B(9,4)) &= \{0,1,2,\ldots,8\}\\
A(G_B(9,4)) &= \{(0,0),(0,1),(0,2),(0,3),(1,4),(1,5),(1,6),(1,7),\\
&\quad (2,8),(2,0),(2,1),(2,2),(3,3),(3,4),(3,5),(3,6),\\
&\quad (4,7),(4,8),(4,0),(4,1),(5,2),(5,3),(5,4),(5,5),\\
&\quad (6,6),(6,7),(6,8),(6,0),(7,1),(7,2),(7,3),(7,4),\\
&\quad (8,5),(8,6),(8,7),(8,8)\}
\end{aligned}
$$

$S=\{0,1,2\}$ is a minimum private out-dominating set.

3. Conclusion

In this paper, we established the bounds for the private out-domination number of generalized de Bruijn digraphs. We gave technique for constructing private out-dominating set for some class of generalized de Bruijn digraphs (d divides n, $d=2$, $d=3$ and $n=4k+1,d=4$). There are some other class of generalized de bruijn digraphs having private out-dominating sets

($n = 5k + 1, 6k + 1$). It is interesting to characterize the extremal graphs achieving its bounds in some theorems.

Our future work will evaluate the efficiency of this technique in reality. Further more, it is also interesting to study the existence of private out-domination number of generalized Kautz digraphs.

We conclude this paper with the following open problem.

Open Problem 1. Find the private out-dominating sets for the generalized de Bruijn digraphs, when $n = dk \pm r$, $r = 0, 1, 2, ..., d - 1$.

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