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Layachi Hadji
The University of Alabama

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Long Wavelength Analysis of a Model for the Geographic Spread of a Disease

Layachi Hadji

Department of Mathematics
The University of Alabama
Tuscaloosa, AL 35487-0350
lhadji@ua.edu

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Abstract

We investigate the temporal and spatial evolution of the spread of an infectious disease by performing a long-wavelength analysis of a classical model for the geographic spread of a rabies epidemic in a population of foxes subject to idealized boundary conditions. We consider two-dimensional and three-dimensional landscapes consisting of an infinite horizontal strip bounded by two walls a finite distance apart and a horizontal region bounded above and below by horizontal walls, respectively. A nonlinear partial differential evolution Equation for the leading order of infectives is derived. The Equation captures the space and time variations of the spread of the disease in the weakly super critical region.

Keywords: Population dynamics, Nonlinear boundary value problem, Perturbation analysis, Asymptotic analysis, Pattern formation, Evolution Equations, Landau Equation

MSC 2010 No.: 92D25, 34B15, 34E10, 15B35

1. Introduction

In recent years we have observed an increasing interest in the understanding of the spatial or geographic spread of epidemics and their interaction with landscape (Lambin et al., 2010; Ostfeld et al., 2005). In deterministic models, the spatial variations are usually incorporated into the equations for the time evolution of the species by the inclusion of diffusion terms. The latter pertain to the diffusion of the pathogens and not the species. In the majority of these models, the space region is always assumed amorphous making them free of any characteristic length scale. In order to describe the method, we consider a classical model involving the geographic spread of a rabies epidemic in a population of foxes. We complement the model by a well-defined, though idealized, set of boundary conditions. This model is considered for the sake of illustration only and the method applies also to other epidemiology models and other types of boundary conditions.

For simplicity, we consider a geographic region consisting of a horizontal strip of infinite horizontal extent bounded by two horizontal boundaries that are a distance H apart. We will also consider its extension to the three-dimensional region bounded from above and from below by horizontal boundaries. Many types of boundary conditions can be prescribed at the horizontal boundaries. For instance, new susceptible species can be introduced into the host population with a prescribed recruitment rate by imposing a Newton law of cooling type boundary conditions. And the region may be kept open in the horizontal direction so that periodic boundary conditions may be imposed. Furthermore, the presence of specific boundaries will introduce a characteristic length scale, namely H, and corresponding diffusion time scale based on H, $t_{TD} = H^2/\mathcal{D}$, where $\mathcal{D}$ is the diffusion coefficient. The model will then be characterized by two time scales which are t_{TD} and the time scale based on the contagious time of the disease. For the sake of a simple illustration of the method, we assume that the value of H is such that the two time scales are of the same order of magnitude. The prediction of stationary and time-dependent patterns in biological systems is a subject of high interest to biologists (Murray, 1983; Murray, 2003).

Recent work on spatial epidemiology has focused primarily on the numerical simulation of Turing patterns. For instance, (Liu and Jin, 2007; Sun et al., 2007) have investigated a spatial SI model with either constant or nonlinear incidence rates and were able to numerically isolate a variety of patterns such as stripes and spots. (Haque, 2012) puts forth very complex stationary and time-dependent Turing patterns obtained through the a numerical simulation of the Beddington-DeAngelis predator-prey model. These studies have considered the emergence of Turing patterns from random perturbations of the uniform state and, with the investigations being of numerical nature, they adopted a rectangular region with zero flux boundary conditions. Moreover, the question of stability of these patterns has not been addressed.

We proceed in a different way in this paper, the emphasis of which is on the application of the long wavelength expansion to population modeling. We couple the system of Equations with well defined, though idealized, boundary conditions. This allows us to investigate both the existence and stability of both two- and three-dimensional time-dependent or transients or stationary solutions through the derivation of reduced evolution Equations. The software MAPLE has been used to help with the resulting tedious algebra. These Equations are both simpler than the original set of Equations and are able to capture the full dynamics in a small region above threshold conditions. Evolution Equations have been used successfully in predicting both steady and time-dependent nonlinear thermal convection patterns, in the study of thin film instabilities and in the study of waves in reaction-diffusion systems (Proctor, 1981; Knobloch, 1986). The study of this simple model has the benefit of isolating certain interesting features of the interactions between landscape, through imposed boundary conditions, and the geographical spread of a disease. The technique can be extended to the analysis of more complex models where landscape induced patterns may evolve.

2. Model formulation and analysis

We consider a population of size N which is divided into two sub-groups, susceptibles S and infectives I and $N = S + I$ with the assumption that infectives have no immunity against reinfection when they recover. The variables S and N depend on both space and time and their dynamic evolution is well described by the reaction-diffusion epidemic model put forth by (Murray, 1993, p.651),

$$\frac{\partial \hat{S}}{\partial \hat{t}} = -r\,\hat{I}\,\hat{S} + \mathcal{D}\,\nabla^2 \hat{S}, \tag{1}$$

$$\frac{\partial \hat{I}}{\partial \hat{t}} = r\,\hat{I}\,\hat{S} - q\,\hat{I} + \mathcal{D}\,\nabla^2 \hat{I}, \tag{2}$$

where $\mathcal{D}$ is the diffusion of the pathogen associated with the susceptible species which is assumed equal to that associated with the infected species, r is quantifies the transmission efficiency of the disease from infected to susceptible and $1/q$ is a measure of the life expectancy of an infected. The system of Equations is made dimensionless by considering the following scalings,

$$I = \frac{\hat{I}}{S_0}, \quad S = \frac{\hat{S}}{S_0}, \quad x = \frac{\hat{x}}{H}, \quad t = r\,S_0\,\hat{t}, \tag{3}$$

where S_0 is some reference value of the susceptible species and where we have set the expression $\sqrt{\mathcal{D}/r\,S_0^2\,H^2} = 1$ for mathematical simplicity. Thus, the resulting dimensionless system is described by

$$\frac{\partial S}{\partial t} = -I\,S + \nabla^2 S, \tag{4}$$

$$\frac{\partial I}{\partial t} = I\,S - \lambda\,I + \nabla^2 I, \tag{5}$$

where $S(x,z,t) \geq 0$ and $I(x,z,t) \geq 0$ are the population densities of susceptible and infected individuals at the physical location (x,z) and time t, respectively. The system is characterized by a single dimensionless control parameter, namely $\lambda = q/r\,S_0$. The parameter λ represents the reciprocal of the reproduction rate. We complement Equations (1) with boundary conditions that correspond to a region consisting of an infinite strip which, for convenience, we take it to be horizontal. The region is then described by

$$\Omega = \{(x,z)| - \infty < x < \infty, 0 \leq z \leq 1\}. \tag{6}$$

In order to illustrate the method as simply as possible, we consider the following boundary conditions at the horizontal boundaries, cf. schematic diagram in Figure 1. We remind the reader that the set of Equations (1)-(2) admits traveling wave solutions with speed $\approx \sqrt{r(S+I)\mathcal{D}}$ in the unbounded region $\{(x,y)| - \infty < x, y < \infty\}$ provided $r\,(S+I)/q > 1$ (Noble, 1974). In the

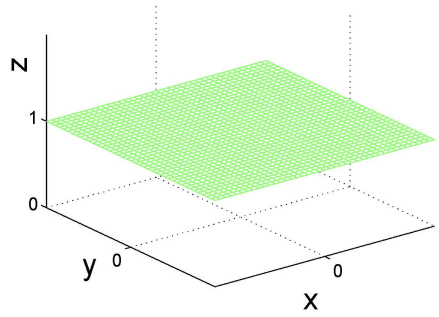

Fig. 1: A schematic diagram of a population model with idealized boundaries.

following, we show that by imposing spatial boundaries, the set of Equations (3)-(4) also admits stable stationary patterns, although the competition between these two types of patterns is not considered.

We impose non-homogeneous Dirichlet conditions on S and Neumann conditions for I,

$$S = 1, \quad \frac{\partial I}{\partial z} = 0, \quad \text{at} \quad z = 0, 1. \tag{7}$$

Other types, such as mixed boundary conditions, can also be used, though they may lead to more tedious calculations. The condition on S is equivalent to the feeding of susceptibles through the boundaries. The system of Equations (4)-(5) with boundary conditions, Equations (7), admits the base state solution, $S_B = 1$ and $I_B = 0$ for any value of λ. We introduce the perturbations ϕ and θ to the base state as $I = 0 + \phi$ and $S = 1 + \theta$ into the system of Equations (4)-(5) to obtain

$$\frac{\partial \theta}{\partial t} = -(1+\theta)\,\phi + \nabla^2\theta, \tag{8}$$

$$\frac{\partial \phi}{\partial t} = \theta\,\phi + (1-\lambda)\phi + \nabla^2\,\phi, \tag{9}$$

subject to the following homogeneous boundary conditions

$$\theta = 0, \quad \frac{\partial \phi}{\partial z} = 0, \quad \text{at} \quad z = 0, 1. \tag{10}$$

2.1 Stability threshold conditions

We consider normal modes of the form

$$[\theta, \phi] = [\Theta(z), \Phi(z)]\ \exp(\imath\,\alpha x + \sigma\,t), \tag{11}$$

into the linearized system Equations (8)-(9) to obtain,

$$\frac{d^2\Theta}{dz^2} - (\alpha^2 + \sigma)\,\Theta + \Phi = 0, \tag{12}$$

$$\frac{d^2\Phi}{dz^2} - (\alpha^2 + \sigma + \lambda - 1)\,\Phi = 0. \tag{13}$$

The marginal state is characterized by σ having zero imaginary part and zero real part. This is easily observed by multiplying both sides of Equation (13) by the complex conjugate of ϕ and then taking the average to find

$$\sigma = \frac{- < |\Phi'|^2 > -(\alpha^2 + \lambda - 1) < |\Phi|^2 >}{< |\Phi|^2 >}, \tag{14}$$

where $< . >$ and $|.|$ denote the average in the $z-$direction and magnitude, respectively. Thus, with $\sigma = 0$, the solution to Equation (13) is given by yields

$$\Phi = C_1 \cosh(mz) + C_2 \sinh(mz), \tag{15}$$

where C_1 and C_2 are arbitrary constants of integration and $m = \sqrt{(\alpha^2 + \lambda - 1)}$. The application of the boundary conditions, Equations (10), then yields the eigenvalue relation $m^2 = 0$ or $\lambda = 1 - \alpha^2$. Therefore, the instability threshold is then given by $\lambda_c = 1$ and it occurs at zero wavenumber (Busse and Riahi, 1980) or alternatively in terms of the reproduction number $R = 1/\lambda$, we have $R = 1/(1-\alpha^2)$ and $R_C = 1$. Hence, the supercritical instability region is given by either $\lambda < 1$ or $R > 1$.

2.2 Nonlinear evolution Equation

In this section we derive a single nonlinear evolution Equation which captures asymptotically the time and space variations beyond the base state. The derivation is carried out using perturbation techniques that are based on the fact that the base state is unstable to perturbations to the population densities having zero wavenumber. Thus a long wavelength approximation can be applied to Equations (8)-(9). We introduce the small perturbation parameter ϵ, $0 < \epsilon \ll 1$ and the order unity parameter $\mu > 0$ which measures the deviation from the threshold value for linear instability, namely $\lambda = 1 - \epsilon^2 \mu$ (note that in terms of the reproduction parameter, we have $R = 1 + \epsilon^2 \mu$). At the onset of instability, the characteristic horizontal dimension is much greater than the vertical dimension and so we scale the horizontal gradients and the slow time as

$$\frac{\partial}{\partial x} = \sqrt{\epsilon}\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}, \quad \tau = \epsilon^3 t, \tag{16}$$

to obtain the following scaled system

$$\epsilon^3 \theta_\tau = -\phi - \phi\theta + D^2\theta + \epsilon^2 \theta_{XX}, \tag{17}$$

$$\epsilon^3 \phi_\tau = \phi + \phi\theta - (1 - \epsilon^2 \mu)\phi + D^2\phi + \phi_{XX}, \tag{18}$$

where subscript stands for partial derivative and $D = \partial/\partial Z$. To proceed with the derivation of the evolution Equation, the variables in Equations (17)-(18) and corresponding boundary conditions, Equations (10) are expanded in powers of ϵ as follows,

$$\phi = \sum_{n=1}^{\infty} \epsilon^n \phi^{(n)}, \quad \theta = \sum_{n=1}^{\infty} \epsilon^n \theta^{(n)}, \tag{19}$$

which yields a hierarchy of boundary value problems of different orders in ϵ that are solved in a successive manner. At $O(1)$, we obtain the leading order problem,

$$D^2\phi^{(1)} = 0, \tag{20}$$
$$D^2\theta^{(1)} = \phi^{(1)}, \tag{21}$$
$$\frac{\partial\phi^{(1)}}{\partial Z} = \theta^{(1)} = 0, \tag{22}$$

the solution of which is given by

$$\phi^{(1)} = f(x,\tau), \quad \text{and} \quad \theta^{(1)} = \frac{1}{2}(Z^2 - Z)\, f(x,\tau). \tag{23}$$

Equating terms of order ϵ^2 in Eqs. (17)-(18) leads to the following problems

$$D^2\phi^{(2)} = -\phi^{(1)}\,\theta^{(1)} - \phi^{(1)}{}_{XX}, \tag{24}$$
$$D^2\theta^{(2)} = \phi^{(1)}\,\theta^{(1)} - \theta^{(1)}{}_{XX} + \phi^{(2)}, \tag{25}$$

the solutions of which are given by

$$\phi^{(2)} = -\frac{f^2}{24}(Z^4 - 2\,Z^3) - \frac{f_{XX}}{2}\,Z^2 + E, \tag{26}$$

$$\theta^{(2)} = \frac{Q}{12}(Z^4 - 2Z^3 + Z) - \frac{f^2}{720}(Z^6 - 3Z^5 + 2Z) - \frac{f_{XX}}{24}(Z^4 - Z) + \frac{E}{2}(Z^2 - Z), \tag{27}$$

where $Q = (f^2 - f_{XX})/2$ and where $E(X)$ is determined by imposing the orthogonality condition $< \theta^{(2)}\,\theta^{(1)} >= 0$. Thus, we have

$$E(X) = \frac{793}{7728}\, f_{XX} + \frac{139}{3903}\, f^2. \tag{28}$$

Finally, application of the orthogonality condition at the order ϵ^3 yields the sought evolution Equation,

$$\frac{\partial f}{\partial \tau} = -\frac{17}{168}\, f_{XXXX} - \frac{1}{56}\, f_{XX} + \mu\, f - \frac{181}{13787}\, f^3 + \frac{319}{1512}\, f\, f_{XX} + \frac{272}{1319}\,(f_X)^2. \tag{29}$$

Upon introducing the following scalings and transformations in Equation (29),

$$\zeta = e\,X, \quad \eta = \frac{168}{17\,e^4}\,\tau, \quad f = \frac{\sqrt{98718}}{36}\,\sqrt{\mu}\,F, \quad \mu = \frac{3}{3808}\,\delta^2, \tag{30}$$

we obtained the sought evolution Equation in canonical form,

$$F_\eta = -F_{\zeta\zeta\zeta\zeta} - 2\,F_{\zeta\zeta} + \delta^2\,(F - F^3) + a\,\delta F\,F_{\zeta\zeta} + b\,\delta\,(F_\zeta)^2, \tag{31}$$

where δ is the bifurcation parameter or the externally imposed control parameter, $a = \sqrt{5093/152}$ and $b = \sqrt{2881/90}$.

2.3 Analysis of the two-dimensional evolution Equation

The trivial solution $F = 0$ is rejected because any state that bifurcates from it would have to be a periodic function of ζ yielding non-positive, and thus non-physical, values of infectives. We investigate the stability of the uniform solution, $F = 1$, by deriving a Landau Equation for the amplitude of a small perturbation (Drazin and Reid, 2004). We let $F = 1 + v(x,t)$ in Equation (31) to obtain

$$\frac{\partial v}{\partial \eta} = -v_{\zeta\zeta\zeta\zeta} - 2v_{\zeta\zeta} - 2\delta^2\, v - 3\delta^2\, v^2 - \delta^2\, v^3 + a\,\delta\, v_{\zeta\zeta} + a\,\delta\, v\, v_{\zeta\zeta} + b\,\delta\,(v_\zeta)^2. \qquad (32)$$

The analysis of the linearized version of the above equation yields the dispersion relation,

$$\sigma = -[(k^2 - \frac{2 - a\,\delta}{2})^2] + (\frac{2 - a\,\delta}{2})^2 - 2\,\delta^2. \qquad (33)$$

Thus instabilities corresponding to the fastest growing unstable wavenumber occur whenever

$$\text{either} \quad \delta > \frac{2}{a - 2\sqrt{2}} = \xi \quad \text{or} \quad \delta < \frac{2}{a + 2\sqrt{2}} = \ell. \qquad (34)$$

Next we examine the effects of the nonlinear terms in the evolution equation, Equation (32), by considering the slow time $T = \gamma^3\,\eta$ and the expansions

$$\delta = \ell - \gamma\,\delta_1 - \gamma^2\,\delta_2 + \ldots, \quad v = \sum_{n=1} \gamma^n\, v^{(n)}. \qquad (35)$$

At the order $O\big(\gamma\big)$ we obtain

$$\mathcal{L}(v^1) \equiv -v^{(1)}{}_{\zeta\zeta\zeta\zeta} + (a\,\ell - 2)\,(v^{(1)})_{\zeta\zeta} - 2\,\ell^2\, v^{(1)} = 0, \qquad (36)$$

the solution of which is given by

$$v^{(1)} = \mathcal{A}(\eta)\,\cos(\omega\,\zeta), \qquad (37)$$

where $\omega^2 = \sqrt{2}\,\ell$. Proceeding to the order $O\big(\gamma^2\big)$ we have

$$\mathcal{L}(v^{(2)}) = 4\,\ell\,\delta_1\, v^{(1)} - 3\,\ell^2\,(v^{(1)})^2 - a\,\delta_1\,(v^{(1)})_{\zeta\zeta} + a\,\ell\, v^{(1)}\,(v^{(1)})_{\zeta\zeta} + b\,\ell\,(v^{(1)}{}_\zeta)^2. \qquad (38)$$

Upon setting $\delta_1 = 0$ to remove all the secular terms, Equation (38) reduces to

$$\mathcal{L}(v^{(2)}) = -3\,\ell^2\,(v^{(1)})^2 + a\,\ell\, v^{(1)}\,(v^{(1)})_{\zeta\zeta} + b\,\ell\,(v^{(1)}{}_\zeta)^2. \qquad (39)$$

and upon evaluating the right hand side of Equation (39), we obtain

$$\mathcal{L}(v^{(2)}) = \omega^4\,\mathcal{A}^2\,(\frac{b}{2\sqrt{2}} - \frac{3}{4} - \frac{a}{2\sqrt{2}}) - (\frac{b}{2\sqrt{2}} + \frac{3}{4} + \frac{a}{2\sqrt{2}})\,\cos(\omega\,\zeta), \qquad (40)$$

whose solution is given by

$$v^{(2)} = (\frac{b}{2\sqrt{2}} - \frac{3}{4} - \frac{a}{2\sqrt{2}})\,\mathcal{A}^2 - (\frac{b}{18\sqrt{2}} + \frac{1}{12} + \frac{a}{18\sqrt{2}})\,\cos(\omega\,\zeta)\,\mathcal{A}^2. \qquad (41)$$

Finally, we multiply both sides of the order $O\big(\gamma^3\big)$ of Equation (32) by $v^{(1)}$ and average over a period $2\pi/\omega$ in the ζ variable to obtain the Landau Equation for the amplitude $\mathcal{A}$,

$$\frac{d\mathcal{A}}{d\eta} = (a\,\omega^2 - 8\,\ell)\,\mathcal{A} + \nu\,\mathcal{A}^3, \qquad (42)$$

where the Landau constant ν is given by

$$-(\frac{b}{2\sqrt{2}} - \frac{3}{4} - \frac{a}{2\sqrt{2}})(6\,\ell^2 + \omega^2\,a\,\ell) + (5\,a\,\omega^2\,\ell)\,(\frac{b}{36\sqrt{2}} + \frac{1}{24} + \frac{a}{36\sqrt{2}}) - (\frac{b}{9\sqrt{2}} - \frac{1}{6} - \frac{a}{9\sqrt{2}})\,b\,\ell\,\omega^2. \qquad (43)$$

The evaluation of these constants yield

$$a\,\omega^2 - 8\,\ell = -\frac{14\sqrt{2}}{463} \approx -0.043184, \quad \text{and} \quad \nu \approx 2.48027. \tag{44}$$

Hence the Landau equation for the amplitude $\mathcal{A}$ is given by

$$\frac{d\mathcal{A}}{d\eta} = (-.043184\,\delta_2 + 2.48027\,\mathcal{A}^2)\,\mathcal{A}. \tag{45}$$

The equilibrium solution $\mathcal{A}_{eq} = \sqrt{(8\ell - a\omega^2)\delta_2/\nu}$ which bifurcates from the uniform solution $F = 1$ is thus unstable since the Landau constant, namely 2.48027, is positive (Drazin and Reid, 2004). Hence $F = 1$ is the only two-dimensional solution and is stable in the range $\ell < \delta < \xi$. However, the numerical solution of the PDE (Equation(32)) from some prescribed initial conditions will yield the transient solutions that are expected to describe the evolution of the spread of the disease. In the next section, we investigate the possibility of three dimensional solutions that bifurcate from the uniform solution $F = 1$.

3. Three-dimensional solutions and their stability

The extension of this theory to regions in $\mathbf{R}^3$, i.e. $\Omega = \{(x, y, z)| - \infty < x < \infty, -\infty < y < \infty, 0 \le z \le 1\}$, is straightforward. This may model the spread of epidemics or other diffusing species or contaminants in a region in space that may be spatially confined in one direction. Thus, we scale the variable y in the same way we have scaled x and start the analysis by considering the three-dimensional version of Equation (32) given by

$$\frac{\partial v}{\partial \eta} = -\nabla^4 v + (a\delta - 2)\,\nabla^2 v - 2\,\delta^2\,v - 3\,\delta^2 v^2 - \delta^2\,v^3 + a\,\delta\,v\,\nabla^2 v + b\,\delta\,|\nabla v|^2. \tag{46}$$

We consider periodic solutions that tessellate the whole two-dimensional plane $\mathbf{R}^2$, namely patterns in the form of either squares and hexagons, the description of which appear in Chandrasekhar text (Chandrasekhar, 1961). Furthermore, we omit the display of the results pertaining to the square patterns for the following reasons. They were found to be unstable and the results are straightforward extensions of the one-dimensional case carried out in the previous section to $\mathbf{R}^2$.

In order to derive the amplitude equation for the time evolution of the hexagonal form solution, it is sufficient to consider the scalings, (Matthews, 1988). $v = \gamma\,v^{(1)} + \gamma^2\,v^{(2)} + \ldots$; $\eta = \gamma^2\,T$ and $\delta = \ell - \gamma\,\delta_1 + \ldots$ with $\delta_1 > 0$. The linear part is defined by

$$\mathcal{L}[v^{(1)}] = -\nabla^4 v + (a\,\ell - 2)\,\nabla^2 v - 2\,\ell^2\,v, \tag{47}$$

which admits the solution with hexagonal form (Chandrasekhar, 1961),

$$v^{(1)}(\zeta, \rho) = \mathcal{A}\,[\cos(\omega\,\rho) + 2\,\cos(\sqrt{3}\,\omega\,\zeta/2)\,\cos(\omega\,\rho/2)] = \mathcal{A}\,\mathcal{H}(\zeta, \rho). \tag{48}$$

The planform function $\mathcal{H}$ in Equation (48) satisfies $\nabla^2\mathcal{H} = -\omega^2\,\mathcal{H}$ and yields the dispersion relation, $\omega^4 + (a\ell - 2)\,\omega^2 + 2\ell^2 = 0$. Proceeding to the next order in γ, we collect all terms problem. Upon collecting all terms that are of order γ^2, we obtain

$$\frac{dv^{(1)}}{dT} = (4\,\ell + a\,\omega^2)\,\delta_1\,v^{(1)} - (3\,\ell^2 + a\,\ell\,\omega^2)\,(v^{(1)})^2 + b\,\ell\,|\nabla\,v^{(1)}|^2. \tag{49}$$

Upon multiplying both sides of Equation (49) by $v^{(1)}$ and averaging over a periodic box of period $2\pi/\omega$ in both the ζ and ρ variables, we obtain an equation for the time evolution of the amplitude $\mathcal{A}$,

$$\frac{d\mathcal{A}}{dT} = \frac{8\,\pi^2\,\sqrt{3}}{\omega^2}\,(4\,\ell + a\,\omega^2)\,\delta_1\,\mathcal{A} + [\frac{14\,b\,\ell\,\sqrt{3}\,\pi^2}{3} - \frac{8\,\sqrt{3}\,\pi^2}{\omega^2}\,(3\,\ell^2 + a\,\ell\,\omega^2)]\,\mathcal{A}^2. \tag{50}$$

It is convenient to express the coefficients in Equation (50) numerically. We have

$$\frac{d\mathcal{A}}{dT} = 1178.4\,\delta_1\,\mathcal{A} - 146.3\,\mathcal{A}^2 \tag{51}$$

The equilibrium solution, $\mathcal{A}_{eq} \approx 8.05\,\delta_1$ is found to be stable since the Landau constant, namely the value -146.3, is negative. Thus, supercritical positive solution in the form of hexagons exist and are stable. At the other end of the stability region, $\delta > \xi$, we let $\delta = \xi + \gamma\,\delta_1$, with $\delta_1 > 0$. Upon repeating the previous calculations, we obtain the following amplitude equation,

$$\frac{d\mathcal{A}}{dT} = -334.38\,\delta_1\,\mathcal{A} - 340.73\,\mathcal{A}^2. \tag{52}$$

The stability analysis shows that the equilibrium solution, $\mathcal{A}_{eq} = 0.98\,\delta_1$ is stable.

4. Results and Discussion

When reverting to the original variables and parameter λ, we have found that there exists a stable uniform solution, namely

$$f = \frac{\sqrt{98718}}{36}\,\sqrt{\mu}\,F = (\frac{\sqrt{98718}}{36})\cdot(\frac{3}{3808})\cdot\frac{\sqrt{1-\lambda}}{\epsilon}\cdot(1), \quad 0 < \epsilon \ll 1 \tag{53}$$

which yields the following expressions for the population density of infectives and corresponding space-dependent population density for the susceptibles,

$$I \approx 0.25\,\sqrt{1-\lambda}, \quad S \approx 1 + 0.125\,(z^2 - z)\,\sqrt{1-\lambda}. \tag{54}$$

These are plotted in Figure (2). The dependence of I on z is such that the minimum occurs at the midpoint between the two boundaries. Furthermore, both I and S vary as $\sim\sqrt{R-1}$ for a small deviation of R from unity.

These solutions are found to be stable in the range, $1 - 8\,\epsilon^2\,\xi^2/3808 < \lambda < 1 - 8\,\epsilon^2\,\ell^2/3808$. These solutions satisfy the $O(\epsilon)$ dimensionless system of Equations Equations (4)-(5). The other space-dependent solutions which bifurcate from the uniform solution, Equation (46), and which are depicted schematically in Figure (3), are now described. For $\delta < \ell$, where $\delta = \ell - \gamma\,\delta_1$, and $\delta_1 = O(1)$, we have

$$\begin{aligned} f &= \frac{\sqrt{98718}}{36}\,\sqrt{\mu}\,F - \gamma\,v^{(1)}, & (55)\\ &= \frac{\sqrt{98718}}{36}\,\sqrt{\mu}\,(\ell - \gamma\,\delta_1) + \gamma\,v^{(1)}, & (56)\\ &\approx 0.058 + \gamma\,\delta_1\,(-0.25 + 8.05\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x, \sqrt{3\,\epsilon/34}\,y)), & (57)\end{aligned}$$

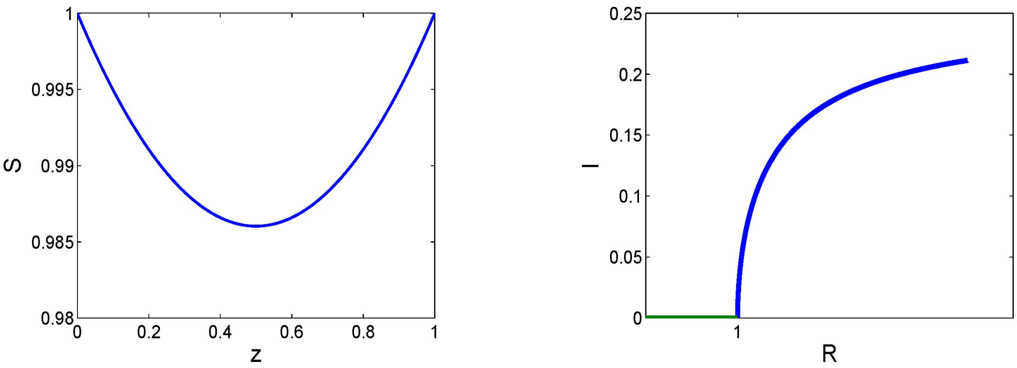

Fig. 2: Plots of the population density of susceptibles, S, versus the variable z for some super-critical parameter values (left) and of the population density of infectives, I, as function of the reproduction number R (right), respectively.

which, in terms of original variables, yields the following periodic solutions

$$I \approx \epsilon\,[0.058 + (\gamma\,\delta_1)\,(-0.25 + 8.05\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\, \sqrt{3\,\epsilon/34}\,y))], \tag{58}$$
$$S \approx 1 + \epsilon\,(z^2 - z)[0.058 + (\gamma\,\delta_1)\,(-0.25 + 8.05\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\, \sqrt{3\,\epsilon/34}\,y))]/2, \tag{59}$$

whose region of stability is $1 - (3/3808)\,\ell^2\,\epsilon^2 < \lambda < 1$. And proceeding in a similar fashion, we find that at the other end, we have

$$I \approx \epsilon\,[0.17 + (\gamma\,\delta_1)\,(0.25 - 0.98\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\, \sqrt{3\,\epsilon/34}\,y))], \tag{60}$$
$$S \approx 1 + \epsilon\,(z^2 - z)[0.17 + (\gamma\,\delta_1)\,(0.25 - 0.98\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\, \sqrt{3\,\epsilon/34}\,y))]/2. \tag{61}$$

that is table in the range $1 - (3/3808)\,\xi^2\,\epsilon^2 > \lambda$.

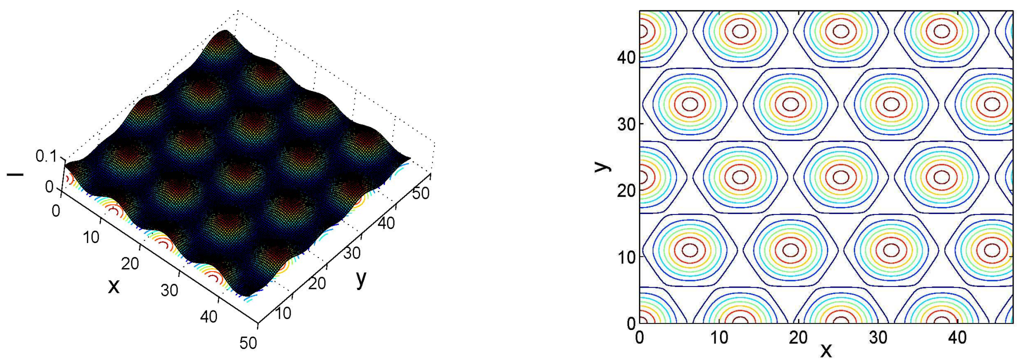

Fig. 3: Plot of the endemic state described by Equation (57) for parameter values $\delta_1 = 1$, $\epsilon = \gamma = 0.01$ (left) and its projection onto the $x - y$ plane (right).

5. Conclusion

We have considered a model for the spread of a disease which includes spatial diffusion in order to capture the effect of landscape. In the infinite domain, the model predicts stable traveling wave solutions (Murray, 1993). However, by complementing the model with horizontal boundaries, we were able to show the existence and stability of stable periodic structures. We have considered an approach that is based on the fact that the base state is unstable to disturbances of infinitely long wavelength. Thus, the nonlinear solutions and their stability are amenable to analysis by using long wavelength asymptotics which reduces the full mathematical problem into a single evolution equation. The latter describes the time and space development of the leading order quantity of infectives. The weakly non-linear solutions and their stability are determined through the analysis of the evolution equation, Equation (46), which describes the space and time evolution of infectives.

With $\mathcal{H}_L = 0.058 + (\gamma\,\delta_1)\,(-0.25 + 8.05\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\,\sqrt{3\,\epsilon/34}\,y))$, $p = 8/3808$ and $\mathcal{H}_R = 0.17 + (\gamma\,\delta_1)\,(0.25 - 0.98\,\mathcal{H}(\sqrt{3\,\epsilon/34}\,x,\,\sqrt{3\,\epsilon/34}\,y))$ and for $0 < \epsilon \ll 1$, we have summarized the findings in the following table.

Table 1. Summary of the steady stable patterns as function of the reproduction number R.

R	*Infectives (I)*	*Susceptibles (S)*
$R \leq 1$	0	1
$1 + p\,\epsilon^2\,\ell^2 < R < 1 + p\,\epsilon^2\,\xi^2$	$0.25\,\sqrt{(R-1)/R}$	$1 + 0.125\,(z^2 - z)\,\sqrt{(R-1)/R}$
$1 < R < 1 + p\,\ell^2\,\epsilon^2$	$\epsilon\,\mathcal{H}_L$	$1 + \epsilon\,(z^2 - z)\,\mathcal{H}_L/2$
$R > 1 + p\,\xi^2\,\epsilon^2$	$\epsilon\,\mathcal{H}_R$	$1 + \epsilon(z^2 - z)\,\mathcal{H}_R/2$

These steady patterns in the non-linear regime beyond the critical value of the reproduction number R owe their existence to the constant feeding into the geographical region through the boundary conditions for S. The analysis of this simple model shows, although qualitatively, how the landscape influences the spread of a disease through self-organization. And by opting for the parallel-planes geometry, we were able to quantify those patterns and get a glimpse of what more realistic geographical boundaries can induce. The numerical simulations of the full three-dimensional evolution equation, Equation (31), will reveal other types of time dependent patterns, the stability of which need also to be carried out numerically.

REFERENCES

Busse, F. H. and Riahi, N. (1980). Nonlinear convection in a layer with nearly insulating boundaries. *J. Fluid Mech.*, 96:243–256.

Chandrasekhar, S. (1961). *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

Drazin, P. G. and Reid, W. H. (2004). *Hydrodynamic Stability, Second Edition*. Cambridge University Press, United Kingdom.

Haque, M. (2012). Existence of complex patterns in the beddington-deangelis predator-prey model. *Mathematical Biosciences*, 239:179–190.

Knobloch, E. (1986). Oscillatory convection in binary mixtures. *Phys. Rev. A*, 34:1588.

Lambin, E. F., Tran, A., Vanwambeke, S. O., Linard, C., and Soti, V. (2010). Pathogenic landscapes: Interactions between land, people, disease vectors, and their hosts. *International Journal of Health Geographics*, 9:54.

Liu, Q. X. and Jin, Z. (2007). Formation of spatial patterns in epidemic model with constant removal rate of the infectives. *J. Stat. Mech.*, page P05002.

Matthews, P. C. (1988). A model for the onset of penetrative convection. *J. Fluid Mech.*, 188:571–583.

Murray, J. D. (1983). *Modelling of patterns in space and time, in: Proceedings, Heidelberg 1983*. Springer-Verlag, Germany.

Murray, J. D. (1993). *Mathematical Biology, 2nd. corrected edition*. Springer-Verlag, New-York.

Murray, J. D. (2003). *Mathematical Biology II: Spatial Models and Biomedical Applications, 3rd. edition*. Springer-Verlag, New-York.

Noble, J. V. (1974). Geographical and temporal development of plagues. *Nature*, 250:726–729.

Ostfeld, R. S., Glass, G. E., and Keesing, F. (2005). Spatial epidemiology: an emerging (or re-emerging) discipline. *Trends in Ecology & Evolution*, 20(6):328–336.

Proctor, M. R. E. (1981). Planform selection by finite-amplitude thermal convection between poorly conducting slabs. *J. Fluid Mech.*, 113:469–485.

Sun, G., Jin, Z., Liu, Q. X., and Li, L. (2007). Pattern formation in a spatial s-i model with nonlinear incidence rates. *J. Stat. Mech.*, page P11011.