[Applications and Applied Mathematics: An International Journal (AAM)](https://digitalcommons.pvamu.edu/aam)

Volume 9 | Issue 1 Article 27

6-2014

Existence of Solutions for Multi-Points Fractional Evolution Equations

Soumia Belarbi
USTHB

Zoubir Dahmani
University of Mostaganem

Follow this and additional works at: https://digitalcommons.pvamu.edu/aam

Part of the Analysis Commons, and the Ordinary Differential Equations and Applied Dynamics Commons

Recommended Citation

Belarbi, Soumia and Dahmani, Zoubir (2014). Existence of Solutions for Multi-Points Fractional Evolution Equations, Applications and Applied Mathematics: An International Journal (AAM), Vol. 9, Iss. 1, Article 27.
Available at: https://digitalcommons.pvamu.edu/aam/vol9/iss1/27

This Article is brought to you for free and open access by Digital Commons @PVAMU. It has been accepted for inclusion in Applications and Applied Mathematics: An International Journal (AAM) by an authorized editor of Digital Commons @PVAMU. For more information, please contact hvkoshy@pvamu.edu.

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 9, Issue 1 (June 2014), pp. 416 – 427

Applications and Applied Mathematics:
An International Journal
(AAM)

Existence of Solutions for Multi-Points Fractional Evolution Equations

Soumia Belarbi and Zoubir Dahmani

USTHB, Algiers, Algeria
soumia-math@hotmail.fr
&
Laboratory LPAM, Faculty SEI
UMAB, University of Mostaganem, Algeria
zzdahmani@yahoo.fr

Received: March 2, 2013; Accepted: February 6, 2014

Abstract

In this paper we study an impulsive fractional evolution equation with nonlinear boundary conditions. Sufficient conditions for the existence and uniqueness of solutions are established. To illustrate our results, an example is presented.

Keywords: Caputo fractional derivative; fixed point; fractional integral.

MSC 2010 No.: 26A33; 34A12

1. Introduction

During the last decades, the theory of fractional differential equations has attracted many authors since it is much richer than the theory of differential equations of integer order (Atangana and Secer, 2013), (Balachandran et al., 2011), (Benchohra et al., 2008), (Bonila et al., 2007), (Fu, 2013), (Kosmatov, 2009), (Mirshafaei and Toroqi, 2012), (Wu and Baleanu, 2013). This fractional theory has many applications in physics, chemistry, biology, blood flow problems, signal and image processing, biophysics, aerodynamics, see for instance (Atangana and Alabaraoye, 2013), (Baneanu et al., 2012), (Bonila et al., 2007), (He, 1999), (He, 1998), (Luchko et al., 2010), (Samko et al., 1993) and the reference therein. Moreover, the fractional impulsive differential equations have played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, optimal control, etc. (Ahmed, 2007), (Lakshmikantham et al., 1989), (Samolenko and Perestyuk, 1995). This impulsive theory has been addressed by several researchers: in (Benchohra and Slimani, 2009), Benchohra et al. established sufficient conditions for the existence of solutions for some initial value problems for impulsive fractional differential equations involving the Caputo fractional derivative. In (Wang et al., 2010), J.R. Wang et al. studied nonlocal impulsive problems for fractional differential equations with time-varying generating operators in Banach spaces. In (Zhang et al., 2012), the authors investigated the existence of solutions for nonlinear impulsive fractional differential equations of order $\alpha \in (2, 3]$ with nonlocal boundary conditions.

(Ergoren and Kilicman, 2012) established some sufficient conditions for the existence results for impulsive nonlinear fractional differential equations with closed boundary conditions. Balachandran et al. (Balachandran et al., 2011) discussed the existence of solutions of first order nonlinear impulsive fractional integro-differential equations in Banach spaces, while Mallika Arjunane et al. (Arjunana et al., 2012) investigated the study of existence results for impulsive differential equations with nonlocal conditions via measures of non-compactness. In (Dahmani and Belarbi, 2013), the authors studied an impulsive problem using a bounded linear operator and some lipschitzian functions. Other research papers related to the fractional impulsive problems can be found in (Ahmad and Sivasundaram, 2009), (Dabas and Gautam, 2013), (Mahto et al., 2013), (Nieto and ORegan, 2009).

In this paper, we are concerned with the existence of solutions for the following nonlinear impulsive fractional differential equation with nonlinear boundary conditions:

$$\begin{cases} D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right),x\left(\alpha_{1}\left(t\right)\right),...,x\left(\alpha_{n}\left(t\right)\right)\right), \quad t\neq t_{i}, \quad t\in J, \quad 0<\alpha<1, \\ \Delta x|_{t=t_{i}}=I_{i}(x(t_{i}))), \quad i=1,2,...,m, \\ x\left(0\right)=g\left(x\left(s_{0}\right),...,x\left(s_{r}\right)\right), \end{cases} \tag{1.1}$$

where D^{α} is the Caputo derivative, $J=[0,b]$, $0=s_0<s_1<...<s_r=b$, and $0<t_1<t_2<...<t_i<...<t_m<t_{m+1}=b$ are constants for $r,m\in\mathbb{N}^*$, and $\Delta x|_{t=ti}=x(t_i^+)-x\left(t_i^-\right)$, such that $x(t_i^+)$ and $x\left(t_i^-\right)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t=t_i$ respectively, f is an impulsive Carathéodory function, $(\alpha_k)_{k=1,...n}\in C^1(J,J)$, g and $(I_i)_{i=1,...m}$ are appropriate functions that will be specified later.

The rest of the paper is organized as follows: In Section 2, some preliminaries are presented. Section 3 is devoted to the study of the existence and the uniqueness of solutions for the impulsive fractional problem (1.1). At the end, an illustrative example is discussed and a conclusion is given.

2. Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper (Goreno and Mainardi, 1997), (Kilbas et al., 2006), (Podlubny, 1999), (Samko et al., 1993).

Definition 1: A real valued function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty))$.

Definition 2: A function $f(t), t > 0$ is said to be in the space C_μ^n, $n \in \mathbb{N}$, if $f^{(n)} \in C_\mu$.

Definition 3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, \mu \geq -1$, is defined as

$$\begin{aligned} &J^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \\ &J^0 f(t) = f(t). \end{aligned} \tag{2.1}$$

Definition 4: The fractional derivative of $f \in C_{-1}^n$ in the sense of Caputo is defined as:

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in N^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.2}$$

In order to define solutions of (1.1), we will consider the following Banach space:

Let

$$PC\left(J, \mathbb{R}\right) = \left\{ \begin{array}{c} x : J \to \mathbb{R}, \text{is continuous at } t \neq t_i, \ x \text{ is left continuous at } t = t_i, \\ \text{and has right hand limits at } t_i, \ i = 1, 2, ..., m \end{array} \right\}, \tag{2.3}$$

endowed with the norm $\|x\|_{PC} = \sup_{t \in J} |x(t)|$.

We also give the following auxiliary result (Kilbas et al., 2006):

Lemma 1: For $\alpha > 0$, the general solution of the problem $D^\alpha x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

where c_i are arbitrary real constants for $i = 0, 1, 2, ..., n$, $n - 1 = [\alpha]$.

We also prove the following lemma:

Lemma 2: Let $0 < \alpha < 1$ and let $F(t, \varkappa) \in PC^{n+1}(J, \mathbb{R})$. A solution of the problem

$$\begin{cases} D^\alpha x(t) = F, (t, \varkappa); t \in J \diagdown \{t_1, ..., t_m\}, 0 < \alpha < 1, \\ \Delta x|_{t=t_i} = I_i(x(t_i)), i = 1, 2, ..., m, \\ x(0) = g(x(s_0), ..., x(s_r)) \end{cases} \tag{2.4}$$

is given by

$$x(t) = \begin{cases} g(x(s_0), ..., x(s_r)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau, \varkappa) d\tau, t \in [0, t_1], \\ g(x(s_0), ..., x(s_r)) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^i \int_{t_{j-1}}^{t_j} (t_j - \tau)^{\alpha-1} F(\tau, \varkappa) d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_i}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa) d\tau + \sum_{j=1}^i I_j(x(t_j)), t \in [t_i, t_{i+1}], i = 1, ...m. \end{cases} \tag{2.5}$$

Proof: Assume x satisfies (2.4).
If $t \in [0, t_1]$, then $D^{\alpha} x(t) = F(t, \varkappa)$. Hence by Lemma 5, it holds

$$x(t) = g(x(s_0), ..., x(s_r)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau.$$

If $t \in [t_1, t_2]$, then using Lemma 5 again, we get

$$\begin{aligned} x(t) &= x\left(t_1^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &= \Delta x \mid_{t=t_1} + x\left(t_1^-\right) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &= I_1\left(x\left(t_1^-\right)\right) + g(x(s_0), ..., x(s_r)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau. \end{aligned}$$

If $t \in [t_2, t_3]$, then by the same lemma, we have

$$\begin{aligned} x(t) &= x\left(t_2^+\right) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &= \Delta x \mid_{t=t_2} + x\left(t_2^-\right) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &= I_2\left(x\left(t_2^-\right)\right) + I_1\left(x\left(t_1^-\right)\right) + g(x(s_0), ..., x(s_r)) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-\tau)^{\alpha-1} F(\tau, \varkappa)\, d\tau. \end{aligned}$$

By repeating the same procedure for $t \in [t_i, t_{i+1}]$, $i = 1, ..m$, we obtain the second quantity in (2.5). The proof of Lemma 6 is complete . □

To end this section, we give the following assumptions:

$(\mathbf{H}_1)$: The nonlinear function $f : J \times PC^{n+1}(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is an h_v- impulsive Carathéodory function and there exist constants $\beta, \Im$, and $(\theta_k)_{k=1,..n}$ such that for each $k = 1, 2, ..., n$

$$\begin{aligned} &\|f(t, x, x_1, ..., x_n) - f(t, y, y_1, ..., y_n)\| \\ &\leq \beta\left[\|x-y\| + \|x_1-y_1\| + ... + \|x_n-y_n\|\right];\ x, y, x_k, y_k \in PC(J, \mathbb{R}), k = 1, ..n, \\ &\|x_k - y_k\| \leq \theta_k \|x-y\|; x, y, x_k, y_k \in PC(J, \mathbb{R}), \\ &\text{and } \Im := \max_{t \in J} \|h_v(t)\|. \end{aligned}$$

$(\mathbf{H}_2)$: The functions $I_i : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ are continuous for every $i = 1, ...m$ and there exist constants $(\varpi_i)_{i=1,...m}$, such that

$$\| I_i(x) - I_i(y) \| \leq \varpi_i \| x - y \|; \|I_i(0)\| \leq \omega;\ x, y \in PC(J, \mathbb{R}), i = 1, ..m,$$

$(\mathbf{H}_3)$: The function $g : PC^{r+1}(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous and there exist two positive constants ϱ and $\hat{\varrho}$, such that for each, $x, y \in PC(J,\mathbb{R})$, $s_l \in J$, $l = 0, ..., r$ we have

$$\| [g(x(s_0), ..., x(s_r)] - [g(y(s_0), ..., y(s_r)] \| \leq \varrho \| x - y \|,$$
$$\| [g(x(s_0), ..., x(s_r)] \| \leq \hat{\varrho}.$$

$(\mathbf{H}_4)$: There exists a positive constant ρ such that

$$(m+1)\gamma[\beta(n+1)\rho + \Im] + \rho\sum_{i=1}^{m}\varpi_i + m\omega + \hat{\varrho} \leq \rho; \gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}.$$

3. Main Results

In this section, we will derive some existence and uniqueness results concerning the solution for the system (1.1) under the assumptions $(\mathbf{H}_j)_{j=\overline{1,4}}$:

Theorem 1: If the hypotheses $(\mathbf{H}_j)_{j=\overline{1,4}}$ and

$$0 \leq \Lambda := (m+1)\gamma\beta\left(1 + \sum_{k=1}^{n}\theta_k\right) + \sum_{i=1}^{m}\varpi_i + \varrho < 1 \tag{3.1}$$

are satisfied, then the problem (1.1) has a unique solution on J.

Proof: The hypothesis $(\mathbf{H}_4)$ allows us to consider the set $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\| \leq \rho\}$. On B_ρ we define an operator $T : PC(J, B_\rho) \to PC(J, B_\rho)$ by

$$\begin{aligned} Tx(t) &= \tfrac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \textstyle\int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau))) d\tau \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_i}^{t} (t-\tau)^{\alpha-1} f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau))) d\tau \\ &+ \sum_{0<t_i<t} I_i(x(t_i)) + g(x(s_0), ..., x(s_r)). \end{aligned} \tag{3.2}$$

We shall prove that the operator T has a unique fixed point. The proof will be given in two steps:

Step1: We show that $TB_\rho \subset B_\rho$. Let $x \in B_\rho$, then we have:

$$\begin{aligned} \|Tx(t)\| \leq{} & \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} \|f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau)))\| d\tau \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} \|f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau)))\| d\tau \\ & + \sum_{0<t_i<t} \|I_i(x(t_i))\| + \|g(x(s_0), ..., x(s_r))\|. \end{aligned} \tag{3.3}$$

Consequently,

$$\begin{aligned}\|Tx(t)\| &\le \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}[(t_i-\tau)^{\alpha-1} \\ &\|f(\tau,x(\tau),x(\alpha_1(\tau)),...,x(\alpha_n(\tau)))-f(\tau,0,0,...,0)\|\,d\tau] \\ &+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\|f(\tau,0,0,...,0)\|\,d\tau \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}[(t-\tau)^{\alpha-1}, \\ &\|f(\tau,x(\tau),x(\alpha_1(\tau)),...,x(\alpha_n(\tau)))-f(\tau,0,0,...,0)\|\,d\tau] \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\|f(\tau,0,0,...,0)\|\,d\tau \\ &+\sum_{0<t_i<t}\|I_i(x(t_i))-I_i(0)\|+\sum_{0<t_i<t}\|I_i(0)\|+\|g(x(s_0),...,x(s_r))\|\,. \end{aligned} \tag{3.4}$$

By (H_2) and (H_3), and using the fact that f is an h_v- impulsive carathéodory function, we can write

$$\|Tx(t)\| \le (m+1)\gamma\beta(n+1)\rho+(m+1)\gamma\|h_v(t)\|+\rho\sum_{i=1}^{m}\varpi_i+m\omega+\hat{\varrho}. \tag{3.5}$$

Since $\|h_v(t)\| \le \Im, t\in J$, then we obtain

$$\|Tx(t)\| \le (m+1)\gamma[\beta(n+1)\rho+\Im]+\rho\sum_{i=1}^{m}\varpi_i+m\omega+\hat{\varrho}. \tag{3.6}$$

And, by (H_4), we have

$$\|Tx(t)\| \le \rho, \tag{3.7}$$

which implies that $TB_\rho \subset B_\rho$.

Step2: Now we prove that T is a contraction mapping on B_ρ : let x and $y \in B_\rho$, then for any

$t \in J$, we have

$$\|Tx(t) - Ty(t)\| \leq \|[\frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau))) \, d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha-1} f(\tau, x(\tau), x(\alpha_1(\tau)), ..., x(\alpha_n(\tau))) \, d\tau \tag{3.8}$$

$$+ \sum_{0<t_i<t} I_i(x(t_i)) + g(x(s_0), ..., x(s_r))]$$

$$- [\frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\alpha-1} f(\tau, y(\tau), y(\alpha_1(\tau)), ..., y(\alpha_n(\tau))) \, d\tau$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t - \tau)^{\alpha-1} f(\tau, y(\tau), y(\alpha_1(\tau)), ..., y(\alpha_n(\tau))) \, d\tau$$

$$+ \sum_{0<t_i<t} I_i(y(t_i)) + g(y(s_0), ..., y(s_r))]\|.$$

Hence,

$$\|Tx(t) - Ty(t)\| \leq (m+1)\gamma \left[\beta \left(\|x - y\| + \sum_{k=1}^{n} \theta_k \|x - y\| \right)\right] + \left(\sum_{i=1}^{m} \varpi_i + \varrho \right) \|x - y\|. \tag{3.9}$$

Then by (3.1), we have

$$\|Tx(t) - Ty(t)\| \leq \Lambda \|x - y\|. \tag{3.10}$$

By (3.1), we can state that T is a contraction mapping on B_ρ. Combining the Steps1-2, together with the Banach fixed point theorem, we conclude that there exists a unique fixed point $x \in PC(J, B_\rho)$ such that $(Tx) = x$. Theorem 7 is thus proved. □

Using the Krasnoselskii's fixed point theorem in (Krasnoselskii, 1964), we prove the following result.

Theorem 2: Suppose that the hypotheses $(\mathbf{H}_j)_{j=\overline{1,4}}$ are satisfied. If

$$(m+1)\gamma\beta \left(1 + \sum_{k=1}^{n} \theta_k\right) + \varrho < 1, \tag{3.11}$$

then the problem (1.1) has a solution on J .

Proof: On B_ρ, we define the operators R and S by the following expression

$$\begin{cases} Rx\left(t\right)=\frac{1}{\Gamma(\alpha)}\sum\limits_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)d\tau \\ \qquad +\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)d\tau \\ \qquad +g\left(x\left(s_0\right),...,x\left(s_r\right)\right) \end{cases} \tag{3.12}$$

and

$$Sx\left(t\right)=\sum_{0<t_i<t}I_i(x(t_i)). \tag{3.13}$$

Let $x,\ y\in B_\rho$. Then, for any $t\in J$, we have

$$\left\|Rx\left(t\right)+Sy\left(t\right)\right\|\leq\left\|Rx\left(t\right)\right\|+\left\|Sx\left(t\right)\right\|. \tag{3.14}$$

That is,

$$\begin{aligned}\left\|Rx\left(t\right)+Sy\left(t\right)\right\|\leq{}&\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}\left\|f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)\right\|d\tau\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}\left\|f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)\right\|d\tau \quad .\\ &+\sum_{0<t_i<t}\left\|I_i(y(t_i))\right\|+\left\|g\left(x\left(s_0\right),...,x\left(s_r\right)\right)\right\|.\end{aligned} \tag{3.15}$$

By (H_1,H_2) and (H_3), it follows that

$$\left\|Rx\left(t\right)+Sy\left(t\right)\right\|\leq(m+1)\,\gamma\left[\beta\left(n+1\right)\rho+\Im\right]+\rho\sum_{i=1}^{m}\varpi_i+m\omega+\hat{\varrho}. \tag{3.16}$$

Using (H_4), we obtain

$$\left\|Rx\left(t\right)+Sy\left(t\right)\right\|\leq\rho. \tag{3.17}$$

Hence,

$$Rx+Sy\in B_\rho.$$

Let us now prove the contraction of $R:$ We have

$$\begin{aligned}\left\|Rx\left(t\right)-Ry\left(t\right)\right\|={}&\Big\|\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)d\tau\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f\left(\tau,x\left(\tau\right),x\left(\alpha_1\left(\tau\right)\right),...,x\left(\alpha_n\left(\tau\right)\right)\right)d\tau+g\left(x\left(s_0\right),...,x\left(s_r\right)\right)\\ &-\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-\tau)^{\alpha-1}f\left(\tau,y\left(\tau\right),y\left(\alpha_1\left(\tau\right)\right),...,y\left(\alpha_n\left(\tau\right)\right)\right)d\tau\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f\left(\tau,y\left(\tau\right),y\left(\alpha_1\left(\tau\right)\right),...,y\left(\alpha_n\left(\tau\right)\right)\right)d\tau+g\left(y\left(s_0\right),...,y\left(s_r\right)\right)\Big\|.\end{aligned} \tag{3.18}$$

With the same arguments as before, we get

$$\|Rx(t) - Ry(t)\| \leq (m+1)\gamma\beta \left(\|x-y\| + \sum_{k=1}^{n} \theta_k \|x-y\| \right) + \varrho \|x-y\|,$$

By (3.1) we can state that R is a contraction on B_ρ.

Now, we shall prove that the operator S is completely continuous from B_ρ to B_ρ.

Since $I_i \in C(J, \mathbb{R})$, then S is continuous on B_ρ.

Now, we prove that S is relatively compact as well as equicontinuous on $PC(J, \mathbb{R})$ for every $t \in J$.

To prove the compactness of S, we shall prove that $S(B_\rho) \subseteq PC(J, \mathbb{R})$ is equicontinuous and $S(B_\rho)(t)$ is precompact for any $\rho > 0$, $t \in J$. Let $x \in B_\rho$ and $t + h \in J$, then we can write

$$\|Sx(t+h) - Sx(t)\| \leq \left\| \sum_{0<t_i<t+h} I_i(x(t_i)) - \sum_{0<t_i<t} I_i(x(t_i)) \right\|. \tag{3.20}$$

The quantity (3.20) is independent of x, thus S is equicontinous and as $h \to 0$ the right hand side of the above inequality tends to zero, so $S(B_\rho)$ is relatively compact, and S is compact. Finally by Krasnosellkii theorem, there exists a fixed point $x(.)$ in B_ρ such that $(Tx)(t) = x(t)$, and this point $x(.)$ is a solution of (1.1). This ends the proof of Theorem 8. □

Example 1: Consider the following fractional differential equation:

$$D^\alpha x(t) = \frac{\exp(-t) \left| x(t) + \sum_{k=1}^{n} x\left(\frac{t}{2^{k+1}}\right) \right|}{(25+\exp(t)) \left(1 + x(t) + \sum_{k=1}^{n} x\left(\frac{t}{2^{k+1}}\right) \right)}$$

$$t \in J = [0,1], t \neq \tfrac{1}{2}, k = 1, ..., n, 0 < \alpha < 1, \tag{3.21}$$

$$\Delta x(t)|_{t=t_i} = b_i x(t_i), \ b_i \in \left(\tfrac{1}{\sqrt{2}}, 1\right], \ i = 1, 2, ..., m,$$

$$x(0) = \tfrac{1}{3} x(\xi) + \cos x(0) - \sqrt[n]{a} \sin x(\eta), \quad \xi, \eta \in [0,1], a \in \mathbb{R}^+.$$

Corresponding to (1.1), we have

$$f\left(t,x\left(t\right),x\left(\alpha_k\left(t\right)\right)\right)=\frac{\exp\left(-t\right)\left|x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right|}{\left(25+\exp\left(t\right)\right)\left(1+x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right)},$$

$$\alpha_k\left(t\right)=\frac{t}{2^{k+1}},$$

$$I_i\left(x\left(t_i\right)\right)=b_ix\left(t_i\right),\ b_i\in\left(\frac{1}{\sqrt{2}},1\right].$$

It is easy to see that

$$s_0=0,\ s_1=\xi,\ s_2=\eta,$$

and

$$g\left(x\left(s_0\right),x\left(s_1\right),x\left(s_2\right)\right)=\frac{1}{3}x\left(\xi\right)+\cos x\left(0\right)-\sqrt[n]{a}\sin x\left(\eta\right).$$

Now, for $x,y\in PC\left(\left[0,1\right];\mathbb{R}\right)$, we have

$$\begin{aligned}
&\left|f(t,x\left(t\right),x\left(\alpha_k\left(t\right)\right))-f(t,y\left(t\right),y\left(\alpha_k\left(t\right)\right))\right|\\
&\leq\left|\frac{\exp\left(-t\right)\left|x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right|}{\left(25+\exp\left(t\right)\right)\left(1+x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right)}-\frac{\exp\left(-t\right)\left|y\left(t\right)+\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right|}{\left(25+\exp\left(t\right)\right)\left(1+y\left(t\right)+\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right)}\right|,\\
&\leq\frac{\exp\left(-t\right)}{\left(25+\exp\left(t\right)\right)}\left|\frac{\left(x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right)-\left(y\left(t\right)+\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right)}{\left(1+x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right)\left(1+y\left(t\right)+\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right)}\right|,\\
&\leq\frac{\exp\left(-t\right)}{\left(25+\exp\left(t\right)\right)\left|\left(1+x\left(t\right)+\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)\right)\left(1+y\left(t\right)+\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right)\right|}\\
&\times\left(\left|\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)-\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right|+\left|x-y\right|\right),\\
&\leq\frac{\exp\left(-t\right)}{\left(25+\exp\left(t\right)\right)}\left(\left|\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)-\sum_{k=1}^{n}y\left(\frac{t}{2^{k+1}}\right)\right|+\left|x-y\right|\right),\\
&\leq\frac{1}{26}\left(\left|\sum_{k=1}^{n}x\left(\frac{t}{2^{k+1}}\right)-y\left(\frac{t}{2^{k+1}}\right)\right|+\left|x-y\right|\right),
\end{aligned}$$

Further we can easily show that the conditions $\left(H_i\right)_{i=\overline{1,4}}$ are satisfied and it is possible to choose β, ω, ϖ , n and ϱ in such a way that the constant $\Lambda<1$. Hence, by Theorem 7, the system (3.21) has a unique solution defined on $[0,1]$.

4. Conclusion

In this paper, we have studied an impulsive fractional differential equation. Using Banach fixed point theorem, we have established new sufficient conditions for the existence of a unique solution for the problem (1.1). To illustrate this result, we have presented an example. Another result for the existence of at least a solution for (1.1) is also given using Krasnosellkii theorem.

REFERENCES

Ahmad, B. and Sivasundaram, S. (2009). Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. *Nonlinear Anal.*, 3:251–258.

Ahmed, N. (2007). Optimal feedback control for impulsive systems on the space of finitely additive measures. *Publ. Math. Debrecen.*, 70:371–393.

Arjunana, M. M., Kavithab, V., and Selvic, S. (2012). Existence results for impulsive differential equations with nonlocal conditions via measures of noncompactness. *J. Nonlinear Sci. Appl.*, 5:195–205.

Atangana, A. and Alabaraoye, E. (2013). Solving a system of fractional partial differential equations arising in the model of hiv infection of cd4+ cells and attractor one-dimensional keller-segel equations. *Advances in Difference Equations*, 2013(94).

Atangana, A. and Secer, A. (2013). A note on fractional order derivatives and table of fractional derivatives of some special functions. *Abstract and Applied Analysis*, 2013(279681).

Balachandran, K., S.Kiruthika, and Trujillo, J. (2011). Existence results for fractional impulsive integrodifferential equations in banach spaces. *Communications in Nonlinear Science and Numerical Simulation*, 16(4):1970–1977.

Baneanu, D., Diethelm, K., Scalas, E., and Trujillo, J. (2012). *Fcactional calculus: Models and numirical methods, Vol. 3 of Series on Complexity, Nonlinearity and Chaos*. World Scientific, Hackensack, NJ, USA.

Benchohra, M., Henderson, J., Ntouyas, S., and Ouahab, A. (2008). Existence results for fractional order functional differential equations with in nite delay. *J. Math. Anal. Appl.*, 338:1340–1350.

Benchohra, M. and Slimani, B. (2009). Existence and uniqueness of solutions to impulsive fractional differential equations. *Electronic Journal of Differential Equations*, 2009(10):1–11.

Bonila, B., Rivero, M., Rodriquez-Germa, L., and Trujilio, J. (2007). Fractional differential equations as alternative models to nonlinear differential equations. *Appl. Math. Comput.*, 187:79–88.

Dabas, J. and Gautam, G. (2013). Impulsive neutral fractional integro-differential equations with state dependent delays and integral condition. *Electronic Journal of Differential Equations*, 2013(273):1–13.

Dahmani, Z. and Belarbi, S. (2013). New results for fractional evolution equations using banach fixed point theorem. *Int. J. Nonlinear Anal. Appl.*, 4(1):40–48.

Ergoren, H. and Kilicman, A. (2012). Some existence results for impulsive nonlinear fractional differential equations with closed boundary conditions. *Abstract and Applied Analysis*, 2012(38).

Fu, X. (2013). Existence results for fractional differential equations with three-point boundary conditions. *Advances in Difference Equations*, 2013(257):1–15.

Goreno, R. and Mainardi, F. (1997). Fractional calculus: integral and differential equations of fractional order. *Springer Verlag, Wien.*, pages 223–276.

He, J. (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Computer Methods in Applied Mechanics and Engineering*, 167(1-2):57–68.

He, J. (1999). Some applications of nonlinear fractional differential equations and their approximations. *Bull. Sci. Technol.*, 15(2):86–90.

Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies. 204.* Elsevier Science B.V., Amsterdam.

Kosmatov, N. (2009). Integral equations and initial value problems for nonlinear differential equations of fractional order. *Nonlinear Analysis*, 70:2521–2529.

Krasnoselskii, M. (1964). *Positive Solutions of Operator Equations*. Nordho Groningen Netherland.

Lakshmikantham, V., Bainov, D., and Simeonov, P. (1989). *Theory of Impulsive Differential Equations*. World Scientific, Singapore.

Luchko, Y., Rivero, M., Trujillo, J., and Velasco, M. (2010). Fractional models, nonlocality and complex systems. *Comp. Math. Appl.*, 59:1048–1056.

Mahto, L., Abbas, S., and Favini, A. (2013). Analysis of caputo impulsive fractional order differential equations with applications. *International Journal of Differential Equations*, 2013(707457).

Mirshafaei, S. and Toroqi, E. (2012). An approximate solution of the mathieu fractional equation by using the generalized differential transform method. *Applications and Applied Mathematics*, 7(1):374–384.

Nieto, J. and ORegan, D. (2009). Variational approach to impulsive differential equations. *Nonlinear Anal.*, 10:680–690.

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego.

Samko, S. G., Kilbas, A. A., and Marichev, O. I. (1993). *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach, Yverdon, Switzerland.

Samolenko, A. and Perestyuk, N. (1995). *Impulsive Differential Equations*. World Scientic, Singapore.

Wang, J., Yang, Y., and Wei, W. (2010). Nonlocal impulsive problems for fractional differential equations with time-varying generating operatos in banach space. *Opuscula Mathematica*, 30(3):361–381.

Wu, G. and Baleanu, D. (2013). Variational iteration method for fractional calculus, a universal approach by laplace transform. *Advances in Difference Equations*, 2013(18).

Zhang, L., Wang, G., and Song, G. (2012). Existence of solutions for nonlinear impulsive fractional differential equations of order $\alpha \in (2,3]$ with nonlocal boundary conditions. *Abstract and Applied Analysis*, 2012(717235).