Applications and Applied Mathematics: An International Journal (AAM)

Volume 9 | Issue 1 Article 17

6-2014

The Generalized Laguerre Matrix Method or Solving Linear Differential-Difference Equations with Variable Coefficients

Z. K. Bojdi
Birjand University

S. Ahmadi-Asl
Birjand University

A. Aminataei
K. N. Toosi University of Technology

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 9, Issue 1 (June 2014), pp. 272-294

Applications and Applied Mathematics:
An International Journal
(AAM)

The Generalized Laguerre Matrix Method or Solving Linear Differential-Difference Equations with Variable Coefficients

Z. Kalateh Bojdi and S. Ahmadi-Asl
Department of Mathematics
Birjand University
Birjand, Iran
salamahmadiasl@gmail.com; nahid_kb@yahoo.com

A. Aminataei*
Department of Applied Mathematics
Faculty of Mathematics
K. N. Toosi University of Technology
Tehran, Iran
ataei@kntu.ac.ir

*Corresponding Author

Received: January 14, 2013; Accepted: March 11, 2014

Abstract

In this paper, a new and efficient approach based on the generalized Laguerre matrix method for numerical approximation of the linear differential-difference equations (DDEs) with variable coefficients is introduced. Explicit formulae which express the generalized Laguerre expansion coefficients for the moments of the derivatives of any differentiable function in terms of the original expansion coefficients of the function itself are given in the matrix form. In the scheme, by using this approach we reduce solving the linear differential equations to solving a system of linear algebraic equations, thus greatly simplify the problem. In addition, several numerical experiments are given to demonstrate the validity and applicability of the method.

Keywords: Generalized Laguerre matrix method; Operational matrices; Laguerre polynomials; Linear differential-difference equations with variable coefficients

AMS-MSC 2010 No.: 33C45, 39A10

1. Introduction

Orthogonal polynomials play a prominent role in pure, applied and computational mathematics, as well as in the applied sciences and many other fields of numerical analyses such as quadratures, approximation theory and so on [Gautschi (2004), Dunkl and Xu (2001), Marcellan and Assche (2006) and Askey (1975)]. In particular, these polynomials play a significant role in the spectral methods, which have been successfully applied in the approximation of partial, differential and integral equations. Three most widely used spectral versions are the Galerkin, collocation and Tau methods. Their utility is based on the fact that if the solution sought is smooth, usually only a few terms in an expansion of global basis functions are needed to represent it to high accuracy [Gottlieb and Orszag (1977), Boyd (2000), Canuto et al. (2006) and (1984), Trefethen (2000), Hesthaven et al. (2009) and Ben-yu (1996)]. We note at this point that numerical methods for ordinary, partial and integral differential equations can be classified into the local and global categories. The finite-difference and finite-element methods are based on local arguments, whereas the spectral methods are in the global class [Shen et al. (2011) and Funaro (1992)]. Spectral methods, in the context of numerical schemes for differential equations, belong to the family of weighted residual methods, which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume and boundary element methods. Also the linear DDEs with variable coefficients and their solutions play a major role in the branch of modern mathematics and arise frequently in many applied areas. Therefore, a reliable and efficient technique for their solution is extremely important. The analytic results on the existence and uniqueness of solutions to the second order linear DDEs have been investigated by many authors [Agraval and Oregan (2009) and King et al. (2003)], however the existence and uniqueness of the solution for DDEs under these conditions is beyond the scope of this paper. We assume that the DDEs which we consider in this paper with their conditions have solutions. During the last decades, several methods have been used to solve high-order linear DDEs such as Adomian's decomposition method [Wazwaz (2010), Aminataei and Hussaini (2007) and (2010)], Taylor collocation method [Gulsu et al. (2006), Gulsu and Sezer (2006), Sezer and Gulsu (2005) and Gulsu and Sezer (2005)], Haar functions method [Maleknejad and Mirzaee (2006), Reihani and Abadi (2007)], Tau method [Ortiz and Samara (1981), Vanani and Aminataei (2011) and Ortiz(1978)], Wavelet method [Danfu and Xufeng (2007)], Hybrid function method [Hsiao (2009)], Legendre wavelet method [Razzaghi and Yousefi (2005)], collocation method based on Jacobi, Laguerre and Legendre polynomials [Imani et al. (2011), Vanani and Aminataei (2012) and Aminataei and Vanani (2013)], Taylor polynomial solutions [Sezer and Dascioglu (2006)], Boubaker polynomial approach [Akkaya and Yalcinbas (2012)], and Bernoulli polynomial approach [Erdem and Yalcinbas (2012)]. In this paper, we develop a new and efficient approach to obtain the numerical solution of the general linear DDEs with variable coefficients of the form

$$\sum_{k=1}^{d_j} A_{k,j}(x) y^{(j)}(h_{k,j}x + f_{k,j}) + \sum_{k=1}^{d_{j-1}} A_{k,j-1}(x) y^{(j-1)}(h_{k,j-1}x + f_{k,j-1}) + \dots + \sum_{k=1}^{d_0} A_{k,0}(x) y^{(0)}(h_{k,0}x + f_{k,0}) = g(x),$$
$$0 \le x \le \infty,\ j \ge 0,\ f_{k,j}, h_{k,j} \in \mathbb{R}, d_t > 0, t = 0, \dots, j,$$

(1)

with the conditions

$$\sum_{k=0}^{j} \alpha_{ik} y^{(k)}(a_i) = \mu_i , \; i = 0,1,..., j . \tag{2}$$

The main advantage of our work is its consideration of the general linear DDEs (1) with respect to (2), whereas the other papers only considered particular cases of our general problem. Also using the generalized Laguerre polynomials as the basis functions for numerical approximation whereas the classical Laguerre polynomials are particular cases of them, is another advantage. The remainder of our paper is organized as follows: In Section 2, we introduce the properties of generalized Laguerre polynomials and their basic formulation required for our subsequent development. Section 3, is devoted to the operational matrices of the generalized Laguerre polynomials (derivative and moment) with some useful theorems. Section 4, summarizes the application of the generalized Laguerre polynomials to the solution of problem (1) and (2). Thus, a set of linear equations is formed and a solution of the considered problem is introduced. Section 5, is devoted to approximations by the generalized Laguerre polynomials and a useful theorem. In Section 6, the proposed method is applied for three numerical experiments. An application of the method for a higher order linear differential equation is presented in Section 7. Finally, we make a brief conclusion in Section 8. Note that we have computed the numerical results by Matlab (version 2013) programming.

2. The Generalized Laguerre Polynomials

In this part, we define the generalized Laguerre polynomials and their properties such as their Sturm-Liouville ODEs, three-term recursion formula, etc. Let $\Lambda = (0, +\infty)$, then the Laguerre polynomials are denoted by $L_n^{\alpha}(x)(\alpha > -1)$, and they are the eigen-functions of the Sturm-Liouville problem

$$x^{-\alpha} e^{x} \left(x^{\alpha+1} e^{-x} \left(L_n^{\alpha}(x) \right)' \right)' + \lambda_n L_n^{\alpha}(x) = 0, \; x \in \Lambda,$$

with the eigenvalues $\lambda_n = n$ [Funaro (1992)].

Laguerre polynomials are orthogonal in $L^2_{w^{\alpha}}(\Lambda)$ space with the weight function $w_{\alpha}(x) = x^{\alpha} e^{-x}$, satisfying in the following relation

$$\int_0^{+\infty} L_n^{\alpha}(x) L_m^{\alpha}(x) w_{\alpha}(x) dx = \gamma_n^{\alpha} \delta_{m,n} , \gamma_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}, \tag{3}$$

where $\delta_{m,n}$ is a Kronecker delta function. The explicit form of these polynomials is in the form $L_n^{\alpha}(x)=\sum_{i=0}^{n}E_i^{\alpha}x^i$,

where

$$E_i^{\alpha}=\frac{\binom{n+\alpha}{n-i}(-1)^i}{i\,!}. \tag{4}$$

These polynomials are satisfied in the following three-term recurrence formula

$$(n+1)L_{n+1}^{\alpha}(x)=(2n+\alpha+1-x)L_n^{\alpha}(x)-(n+\alpha)L_{n-1}^{\alpha}(x), \tag{5}$$

$$L_0^{\alpha}(x)=1, L_1^{\alpha}(x)=\alpha+1-x.$$

The case $\alpha=0$ leads to the classical Laguerre polynomials, which are used most frequently in practice and will simply be denoted by $\mathrm{L_n}(\mathrm{x})$. An important property of the Laguerre polynomials is the following derivative relation [Funaro (1992)]:

$$\left(L_n^{\alpha}(x)\right)'=\sum_{i=0}^{n-1}L_i^{\alpha}(x). \tag{6}$$

Further, $(L_i^{\alpha}(x))^{(k)}$ are orthogonal with respect to the weight function $w_{\alpha+k}$, i.e.,

$$\int_0^{+\infty}(L_i^{\alpha})^{(k)}(x)(L_j^{\alpha})^{(k)}(x)w_{\alpha+k}(x)dx=\gamma_{n-k}^{\alpha+k}\delta_{i,j},$$

where $\gamma_{n-k}^{\alpha+k}$ is defined in equation (3).

A function $y(x)\in L_{w_\alpha}^2[0,\infty)$, can be expressed in terms of the generalized Laguerre polynomials as

$$y(x)=\sum_{i=0}^{\infty}a_i L_i^{\alpha}(x),$$

where the coefficients a_i are given by

$$a_i=\frac{1}{\gamma_i^{\alpha}}\int_0^{+\infty}L_i^{\alpha}(x)y(x)w^{(\alpha)}(x)dx.$$

In practice, only the first $m+1$terms of the generalized Laguerre polynomials are considered. Then we have

$$y_m(x)=\sum_{i=0}^{m} a_i L_i^{\alpha}(x)= \left(L_m^{\alpha}(x)\right)^T A,$$

where the generalized Laguerre polynomials coefficients vector A and the generalized Laguerre polynomials vector $L^{(\alpha)}(x)$ are given by

$$A=\left[a_0,a_1,...,a_m\right]^T,\ L^{(\alpha)}(x)=\left[L_0^{\alpha}(x),L_1^{\alpha}(x),...,L_m^{\alpha}(x)\right]^T.$$

Remark 1.

From equation (1), for $h_{k,j}\neq 0$, we conclude that

$$L_n^{(\alpha)}(h_{k,j}x)=\sum_{i=0}^{n} E_i^{\alpha}(h_{k,j})^i x^i;\alpha>-1. \tag{7}$$

Now, from remark 1 and the following theorem 1, we can obtain the matrix relation between the generalized Laguerre polynomials space (set) $\{L_0^{(\alpha)}(h_{k,j}x),L_1^{(\alpha)}(h_{k,j}x),...,L_n^{(\alpha)}(h_{k,j}x)\}$ and standard polynomial space (set) as the following

$$\left[L_0^{(\alpha)}(h_{k,j}x),L_1^{(\alpha)}(h_{k,j}x),...,L_n^{(\alpha)}(h_{k,j}x)\right]^T=K\left[1,x,...,x^n\right]^T, \tag{8}$$

and

$$\left[L_0^{(\alpha)}(h_{k,j}x),L_1^{(\alpha)}(h_{k,j}x),...,L_n^{(\alpha)}(h_{k,j}x)\right]^T=T\left[1,x,...,x^n\right]^T, \tag{9}$$

where K and T are lower triangular matrices

$$\left(K\right)_{i,j}=\begin{cases}0, & i>j,\\ (h_{k,j})^i E_i^{\alpha}, & i\leq j,\end{cases} \tag{10}$$

and

$$\left(T\right)_{i,j}=\begin{cases}0, & i>j,\\ E_i^{\alpha}, & i\leq j.\end{cases} \tag{11}$$

Theorem 1.

The matrices K and T are invertible if and only if $h_{k,j} \neq 0$.

Proof:

For establishing the invertibility of matrix K, it is sufficient to show that $Det(K) \neq 0$, where $Det(K)$ is a determinant of the square matrix K. But because K is a lower triangular matrix, then we have

$$Det(K) = \prod_{i=0}^{n} (h_{k,j})^i E_i^\alpha,$$

but from $h_{k,j} \neq 0$, it is sufficient to establish that

$$Det(K) = \prod_{i=0}^{n} E_i^\alpha \neq 0.$$

Now from equation (4), it is not difficult to see that

$$\prod_{i=0}^{n} E_i^\alpha \neq 0.$$

The invertibility of matrix T along similar lines of discussion of matrix K is obvious. Therefore, the proof is completed. □

Now from equations (8) and (9), we obtain the following important matrix relation

$$\left[L_0^{(\alpha)}(h_{k,j}x), L_1^{(\alpha)}(h_{k,j}x), ..., L_n^{(\alpha)}(h_{k,j}x)\right]^T = KT^{-1}\left[L_0^\alpha(x), L_1^\alpha(x), ..., L_n^\alpha(x)\right]^T, \forall \alpha > 0. \quad (12)$$

3. The Operational Matrices of the Generalized Laguerre Polynomials (Derivative and Moment)

In this section, we present the operational matrices of the generalized Laguerre polynomials (derivative and moment). To do this, first we introduce the concept of the operational matrix.

3.1. The Operational Matrix

Definition 1.

Suppose $\phi=[\phi_0,\phi_1,...,\phi_n]$, where $\phi_0,\phi_1,...,\phi_n$ are the basis functions on the given interval $[a,b]$. The matrices $E_{n\times n}$ and $F_{n\times n}$ are named as the ***operational matrices of derivatives and integrals***, respectively, if and only if

$$\frac{d}{dt}\phi(t)=E\,\phi(t), \text{ and} \int_a^x \phi(t)dt \simeq F\phi(t).$$

Further assume $g=[g_0,g_1,...,g_n]$, named as the ***operational matrix*** of the product, if and only if

$$\phi(x)\phi^T(x)\simeq G_g\phi(x). \tag{13}$$

In other words, to obtain the operational matrix of a product, it is sufficient to find $g_{i,j,k}$ in the relation

$$\phi_i(x)\phi_j(x)\simeq\sum_{k=0}^{i+j} g_{i,j,k}\phi_k(x), \tag{14}$$

which is called the linearization formula [Eslahchi and Dehghan (2011)]. Operational matrices are used in several areas of numerical analyses and continue to be important in various subjects such as integral equations [Razzaghi and Ordokhani (2001)], differential and partial differential equations [Khellat and Yousefi (2006)], etc. Also many textbooks and papers have employed the operational matrices for spectral methods.

Remark 2.

The reason for using the equalities (8), (9), (10) and (11) is that for some bases such as polynomial basis, the integral of a polynomial with degree n, is a polynomial with degree $n+1$, so it cannot be represented with a polynomial with degree n. A similar inference can be drawn for the product of two bases.

Remark 3.

The reason for using three parameters i,j,k, for the product of two functions $\phi_i(x)$ and $\phi_j(x)$ is that the coefficients $g_{i,j,k}$, completely depend on two functions $\phi_i(x)$ and $\phi_j(x)$.

Remark 4.

In the general case, the coefficient matrix G, and the coefficients $g_{i,j,k}$ of equations (13) and (14) respectively, are different for different bases, and in the following we obtain these coefficients for the generalized Laguerre polynomials. To this goal, we use the following two important formulas

$$L_n^{\alpha}(x)L_m^{\beta}(x)=\sum_{i=0}^{m+n}c_i(m,n,\alpha,\beta)L_m^{\alpha+\beta}(x), \tag{15}$$

and

$$L_n^{\alpha}(x)=\sum_{i=0}^{n}\binom{\alpha-\beta+n-i+1}{n-i}L_i^{\beta}(x), \tag{16}$$

where

$$c_i(m,n,\alpha,\beta)=\left(-1\right)^{m+n+\alpha}\sum_{k=0}^{i}\binom{i}{k}\binom{m+\alpha}{n-k+i}\binom{n+\alpha}{m-k}.$$

By combining the relations (15) and (16), we obtain

$$L_n^{\alpha}(x)L_m^{\alpha}(x)=\sum_{i=0}^{m+n}g_i(m,n,\alpha)L_k^{\alpha}(x),$$

where

$$g_i(m,n,\alpha)=\sum_{k=0}^{i}d_i(m,n,\alpha)\binom{\alpha+i-k+1}{i-k},$$

and

$$d_i(m,n,\alpha)=c_i(m,n,\alpha,\beta)\sum_{k=0}^{i}\binom{\alpha+i-k+1}{i-k}.$$

So the considered coefficients are obtained. Now we present the following theorem.

Theorem 2.

If we consider the generalized Laguerre approximation

$$y(x)\cong\sum_{i=0}^{m}a_iL_i^{(\alpha)}(x)=\left(L^{(\alpha)}(x)\right)^T A,$$

then

$$x^i y^{(j)}(x) \cong B^T L^{(\alpha)}(x) = \left(\left(G^i D^j\right)^T A\right)^T L^{(\alpha)}(x),$$

where

$$D_{i,j} = \begin{cases} 1, & i > j, \\ 0, & i \le j, \end{cases} \tag{17}$$

and

$$G_{i,j} = \begin{cases} -(i+\alpha), & j = i-1, \\ -i, & j = i, \\ -(i+\alpha), & j = i+1, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Proof:

First, we obtain the operational matrix with respect to the derivative operator. For this goal, we must obtain a matrix D which satisfy the following formula

$$\begin{bmatrix} \left(L_0^{(\alpha)}(x)\right)' \\ \left(L_1^{(\alpha)}(x)\right)' \\ \vdots \\ \left(L_n^{(\alpha)}(x)\right)' \end{bmatrix} = D \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}, \tag{19}$$

but by using equation (16), we can obtain the matrix D as the following

$$D_{i,j} = \begin{cases} 1, & i > j, \\ 0, & i \le j. \end{cases}$$

Now by j-times repeating the formula (19), we can obtain the operational matrix with respect to $y^{(j)}(x)$ as the following

$$\begin{bmatrix} \left(L_0^{(\alpha)}(x)\right)^j \\ \left(L_1^{(\alpha)}(x)\right)^j \\ \vdots \\ \left(L_n^{(\alpha)}(x)\right)^j \end{bmatrix} = D^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}. \tag{20}$$

Also for obtaining the operational matrix with respect to the moment operator we must obtain a matrix G, which satisfy the following relation

$$\begin{bmatrix} xL_0^{(\alpha)}(x) \\ xL_1^{(\alpha)}(x) \\ \vdots \\ xL_n^{(\alpha)}(x) \end{bmatrix} = G \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}, \tag{21}$$

but by using equation (5), we can obtain the matrix G as the following

$$G_{i,j} = \begin{cases} -(i+\alpha), & j = i-1, \\ -i, & j = i, \\ -(i+\alpha), & j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Now by j-times repeating the formula (21), we can obtain the operational matrix with respect to $x^j y(x)$, as the following

$$\begin{bmatrix} x^j L_0^{(\alpha)}(x) \\ x^j L_1^{(\alpha)}(x) \\ \vdots \\ x^j L_n^{(\alpha)}(x) \end{bmatrix} = G^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix}. \tag{22}$$

Now using formulae (20) and (22), yield

$$x^i y^{(j)}(x) \simeq \sum_{k=0}^{n} a_k x^i \left(L_k^{(\alpha)}(x)\right)^{(j)} = A^T x^i \begin{bmatrix} \left(L_0^{(\alpha)}(x)\right)^{(j)} \\ \left(L_1^{(\alpha)}(x)\right)^{(j)} \\ \vdots \\ \left(L_n^{(\alpha)}(x)\right)^{(j)} \end{bmatrix}$$

$$= A^T G^i D^j \begin{bmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \\ L_n^{(\alpha)}(x) \end{bmatrix} = \left(\left(G^i D^j\right)^T A\right)^T L^{(\alpha)}(x),$$

so the proof is complete. □

Theorem 3.

If $c \in R$ and $\alpha > 0,$ then

$$\left[L_0^\alpha(x+c), L_1^\alpha(x+c), \ldots, L_n^\alpha(x+c)\right]^T = W_c T^{-1}\left[L_0^\alpha(x), L_1^\alpha(x), \ldots, L_n^\alpha(x)\right]^T,$$

where W_c is a lower triangular matrix

$$\left(W_c\right)_{i,j} = \begin{cases} 0, & i < j, \\ D_i^{(\alpha,j)}, & i \geq j, \end{cases} \tag{23}$$

and $D_i^{(\alpha,j)} = \sum_{k=i}^{j} E_i^\alpha c^k \binom{k}{i}$, where E_i^α is defined in equation (4).

Proof:

From equation (7), we have

$$L_n^{(\alpha)}(x+c) = \sum_{i=0}^{n} E_i^\alpha (x+c)^i = \sum_{i=0}^{n} E_i^\alpha \sum_{j=0}^{i} \binom{i}{j} c^{i-j} x^j,$$

so if we define

$$D_i^{(\alpha,j)} = \sum_{k=i}^{j} E_i^\alpha c^k \binom{k}{i},$$

therefore we have

$$L_n^{(\alpha)}(x+c) = \sum_{i=0}^{n} D_i^{(\alpha,n)} x^i. \tag{24}$$

Using obtained result from equation (24), we have

$$\left[L_0^{(\alpha)}(x+c), L_1^{(\alpha)}(x+c), \ldots, L_n^{(\alpha)}(x+c)\right]^T = W_c\left[1, x, \ldots, x^n\right]^T, \forall \alpha > 0,$$

where W_c is given in equation (23), and using formula (12), we obtain

$$\left[L_0^{(\alpha)}(x+c), L_1^{(\alpha)}(x+c), \ldots, L_n^{(\alpha)}(x+c)\right]^T = W_c T^{-1}\left[L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \ldots, L_n^{(\alpha)}(x)\right]^T,$$

therefore $W_c T^{-1}$ is the shift operational matrix and the proof of theorem is complete. □

Now by theorem 3, we can obtain the modified version of theorem 2, for generalized Laguerre polynomials as

$$x^{i} y^{(k)}\left(h_{k,t} x+f_{k,t}\right) \cong B^{T} L^{(\alpha)}(x)=\left(\left(G^{i} D^{j}\right)^{T} A\right)^{T} W_{f_{k,t}} K T^{-1} L^{(\alpha)}(x), \forall \alpha>0.$$

4. The Method of Solution

In this section, we describe our new approach for solving the linear differential-difference equations with variable coefficients (1), with respect to the conditions (2). Our approach is based on approximating the exact solution of equation (1), by truncating the generalized Laguerre expansion as

$$y(x) \simeq \sum_{i=0}^{m} a_{i} L_{i}^{(\alpha)}(x)=\left(L^{(\alpha)}(x)\right)^{T} A, \tag{25}$$

where $A=[a_0, a_1, \ldots, a_m]^T$, and $L^{(\alpha)}(x)=\left[L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \ldots, L_m^{(\alpha)}(x)\right]^T$.

Also we assume that the coefficients $A_{k,j}(x)$ have the Taylor series expansion in the following form

$$A_{k,j}(x)=\sum_{i=0}^{m_j} e_{k,i}^{(j)} x^{i}. \tag{26}$$

Now by substituting equations (25) and (26) into equation (1), we obtain

$$\begin{aligned}&\sum_{k=1}^{s_j} \sum_{i=0}^{m_j} e_{k,i}^{(j)} x^{i} y^{(j)}(h_{k,j} x+f_{k,j})+\sum_{k=1}^{s_{j-1}} \sum_{i=0}^{m_{j-1}} e_{k,i}^{(j-1)} x^{i} y^{(j-1)}(h_{k,j-1} x+f_{k,j-1}) \\ &+\ldots+\sum_{k=1}^{s_0} \sum_{i=0}^{m_0} e_{k,i}^{(0)} x^{i} y^{(j)}(h_{k,0} x+f_{k,0}) \simeq f(x),\end{aligned} \tag{27}$$

Therefore, from equation (27), we must simplify $x^i\left(y^{(j)}(x)\right)$ as the following

$$\begin{aligned}x^{i} y^{(j)}(h_{k,j} x+f_{k,j}) &\simeq \sum_{i=0}^{m} a_{i} L_{i}^{(\alpha)}(h_{k,j} x+f_{k,j})=\left(L^{(\alpha)}(x)\right)^{T} B_{(j)}^{(i)} \\ &=\left(\left(G^{i} D^{j}\right)^{T} A\right)^{T} W_{f_{k,t}} K T^{-1} L^{(\alpha)}(x),\end{aligned} \tag{28}$$

where D and G, are defined in equations (17) and (18), respectively. Also we approximate the right hand side of equation (1), as

$$f(x)=\sum_{i=0}^{m} b_i L_i^{(\alpha)}(x)=\left(L^{(\alpha)}(x)\right)^T B, \tag{29}$$

where $B=\left[b_0,b_1,...,b_m\right]^T$, and $L^{(\alpha)}(x)=\left[L_0^{(\alpha)}(x),L_1^{(\alpha)}(x),...,L_m^{(\alpha)}(x)\right]^T$.

Using equations (28) and (29), into equation (27), we obtain

$$\left(L^{(\alpha)}(x)\right)^T\left(\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(j)}B_{(j)}^{(i)}+\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(j-1)}B_{(j-1)}^{(i)}+....+\sum_{k=1}^{s_j}\sum_{i=0}^{m_j} e_{i,k}^{(0)}B_{(0)}^{(i)}\right)$$
$$=\left(L^{(\alpha)}(x)\right)^T F\simeq\left(L^{(\alpha)}(x)\right)^T B.$$

Now, from linear independency of the generalized Laguerre polynomials, we conclude that

$$F=B, \tag{30}$$

where $F=[f_0,f_1,...,f_m]$.

Therefore, from identity (30), we have a system of $m+1$ algebraic equations of $m+1$ unknown coefficients $a_i\left(i=0,..,m\right)$. Finally, we must obtain the corresponding matrix form of the boundary conditions. For this purpose from equation (2), the values $y^{(j)}(a)$ can be written as

$$y^{(j)}(a)=\left(L^{(\alpha)}(a)\right)^T\left(D^j\right)^T A, a\in[0,+\infty). \tag{31}$$

Substituting equation (31), in the boundary conditions (2) and then simplifying it, we obtain the following matrix form

$$\sum_{i=0}^{j} b_{i,l}\, y^{(l)}(a_i)=\left(L^{(\alpha)}(a_i)\right)^T\left\{\sum_{i=0}^{j} b_{i,l}D^i A\right\}=\sigma_l, a_i\in[0,+\infty). \tag{32}$$

Now from equations (30) and (32), we have $m+j+1$ algebraic equations of $m+1$ unknown coefficients. Thus for obtaining the unknown coefficients, we must eliminate j arbitrary equations from these $m+j+1$ equations. But because of the necessity of holding the boundary conditions, we eliminate the last j equations from equality (30). Finally, replacing the last j equations of equality (30) by the j equations of equality (32), we obtain a system of $m+1$ equations of $m+1$ unknowns $a_i\left(i=0,\cdots,m\right)$.

5. Approximations by Generalized Laguerre Polynomials

Now in this section, we present a useful theorem which shows the approximations of functions by the generalized Laguerre polynomials. For this purpose, let us define $\Lambda = \{x \mid 0 \le x < \infty\}$ and

$$J_N^{(\alpha)} = span\{L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \ldots, L_n^{(\alpha)}(x)\}.$$

The $L^2_{w(\alpha)}(\Lambda)-$orthogonal projection $\pi_N^{(\alpha)} : L^2(\Lambda) \to J_N^{(\alpha)}$ is a mapping in a way that for any $y(x) \in L^2(\Lambda),$ we have $\langle \pi_N^{(\alpha)}(y) - y, \Phi \rangle = 0, \quad \forall \Phi \in J_N^{(\alpha)}.$

Due to the orthogonally, we can write $\pi_N^{(\alpha)}(y) = \sum_{k=0}^{N-1} c_k L_k^{(\alpha)}(x),$ where $c_i \quad (i = 0,1,\ldots,N-1)$ constants are in the following form

$$c_i = \frac{1}{\gamma_k^{(\alpha)}} < y(x), L_k^{(\alpha)} >_{L^2_{w(\alpha)}} .$$

In the literature of spectral methods, $\pi_N^{(\alpha)}(y)$ is named as the generalized Laguerre expansion of $y(x)$ and approximates $y(x)$ on $(0, +\infty)$. In the spectral methods, by substituting the generalized Laguerre expansion $\pi_N^{(\alpha)}(y)$ in the DDEs and their boundary conditions, we obtain a residual term which is symbolically showed by $Res(x)$ as a function of x, N, and α. Different strategies for minimizing a residual term $Res(x)$, lead to the different versions of spectral methods such as Galerkin, Tau and collocation methods. For instance, in the collocation methods the residual term Res(x) is vanished in particular points named as collocated points. Also estimating the distance between $y(x)$ and its generalized Laguerre expansion as measured in the weighted norm $\|.\|_{w_\alpha}$ is an important problem in numerical analysis. The following theorem provides the basic approximation results for generalized Laguerre expansion.

Theorem 4.

We have

$$\| \frac{d^l}{dx^l}(\pi_N^{(\alpha)}(y) - y) \|_{w(\alpha+l)} \le N^{(l-m)/2} \| \frac{d^m}{dx^m} y(x) \|_{w_{\alpha+m}}, \quad 0 \le l \le m, \quad \forall y \in B^m_{(\alpha)}(\Lambda),$$

where

$$B^m_{(\alpha)}(\Lambda) = \{\forall y \in L^2_{w_\alpha} : \frac{d^l y}{dx^l} \in L^2_{w_{\alpha+l}}(\Lambda),\ 0 \le l \le m\}.$$

Theorem 4 states that if the function $y(x) \in B^{m}_{(\alpha)}(\Lambda)$, or in other words the function $y(x)$, is sufficiently continuous, we recover spectral decay of the expansion coefficients, i.e., $|c_k|$ decays faster than any algebraic order of $1/N$. This result is valid independent of specific boundary conditions on $y(x)$.

Proof:

See Funaro (1992). □

6. The Test Experiments

In this section, some numerical experiments are given to illustrate the properties of the method and all of them were performed on a computer using a program written in Matlab 2013.

Experiment 1.

Consider the following second-order linear differential equation with variable coefficients [Kesan (2003)]:

$$y''(x) + xy'(x) + xy(x) = 1 + x + x^2, \quad -1 \le x \le 1,$$

$$y(0) = 1, \; y'(0) + 2y(1) - y(-1) = -1. \tag{33}$$

Now we approximate the exact solution of equation (33), by

$$y(x) \simeq \sum_{i=0}^{6} a_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T A,$$

where $A = [a_0, a_1, ..., a_6]$. Also we expand the right hand side of equation (33) as

$$1 + x + x^2 = \sum_{i=0}^{6} b_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T B, \text{ where } B = [1, 2, 3/2, 0, 0, 0, 0].$$

First, we reduce equation (33) into the following matrix form $\left(D^2 + GD + G\right)^T A = B$.
Also its boundary conditions as

$$\sum_{i=0}^{6} a_i L_i^{(\alpha)}(0) = \left(L^{(\alpha)}(0)\right)^T A = 1.$$

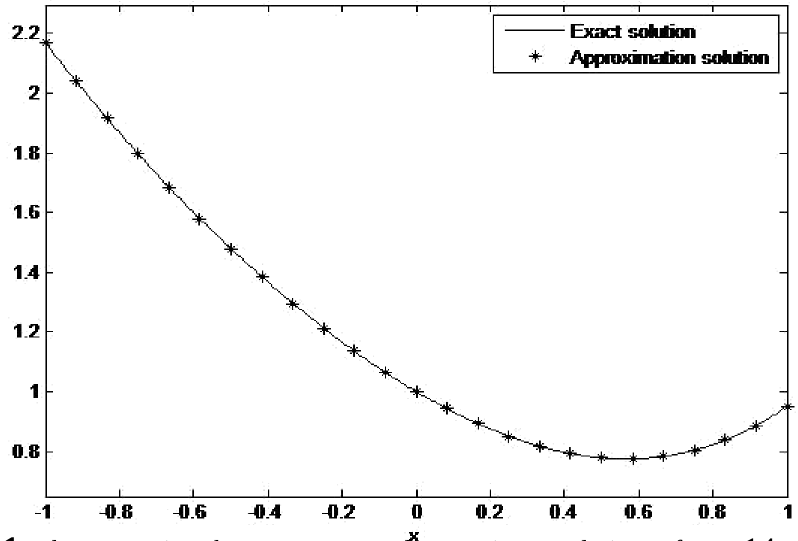

Figure 1. The comparison between exact and approximate solutions of $n = 14$ and $\alpha = 3/2$ of experiment 1

Table 1. The comparison between present method ($n = 6$ and $\alpha = 3$) and the approximate solutions of Taylor method $(n = 4)$ of experiment 1.

x	Present method	Taylor method
-1.0	-1.0000000000	-0.9999999999
-0.8	-0.8000000000	-0.7999999996
-0.6	-0.6000000000	-0.5999999998
-0.4	-0.3879999999	-0.4000000000
-0.2	-0.1889999999	-0.2000000000
0.0	-0.9999999999	-1.0000000000
0.2	0.1999999999	0.2000000000
0.4	0.3899999999	0.4000000000
0.6	0.6000000000	0.5999999999
0.8	0.8000000000	0.7999999997
1.0	1.0000000000	0.9999999988

and

$$\sum_{i=0}^{6} a_i L_i^{(\alpha)}(1) = \left(L^{(\alpha)}(1)\right)^T A \ = 1.$$

By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the numerical solutions. The comparison between our method and Taylor method is shown in table 1. Also the approximate and exact solutions are shown in Figure 1.

Experiment 2.

Consider the second-order linear differential equation:

$$(x^2+1)y''(x)+y'(x)=1, \tag{34}$$

with the boundary conditions y(0)=0, y(1)=1. The exact solution of equation (34) is

$$y(x)=x.$$

Now we approximate the exact solution of equation (34), by

$$y(x)\simeq\sum_{i=0}^{5}a_i L_i^{(\alpha)}(x)=\left(L^{(\alpha)}(x)\right)^T A,$$

where $A=[a_0,a_1,...,a_5]$. Also we expand the right hand side of equation (34) as

$$1\simeq\sum_{i=0}^{5}b_i L_i^{(\alpha)}(x)=\left(L^{(\alpha)}(x)\right)^T B,\ \text{where}\ B=[1,0,0,0,0,0].$$

Now, first we reduce equation (34) into the following matrix form $\left(G^2D^2+D^2+D\right)^T A=B$. Also its boundary conditions as

$$\sum_{i=0}^{5}a_i L_i^{(\alpha)}(0)=\left(L^{(\alpha)}(0)\right)^T A\ =0,$$

and

$$\sum_{i=0}^{5}a_i L_i^{(\alpha)}(1)=\left(L^{(\alpha)}(1)\right)^T A\ =1.$$

By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the solution y(x)=x, which is the exact solution.

Experiment 3.

Consider the third-order linear differential equation:

$$x^2y'''(x)+y''(x)=2,$$

$$y(0) = 0,\ y(1) = 1,\ y(-1)=1. \tag{35}$$

Now we approximate the exact solution of equation (35) by

$$y(x) \simeq \sum_{i=0}^{5} a_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T A.$$

Also we expand the right hand side of equation (35) as

$$2 \simeq \sum_{i=0}^{5} b_i L_i^{(\alpha)}(x) = \left(L^{(\alpha)}(x)\right)^T B,$$

where $B = [2,0,0,0,0,0]$.

Now we must reduce equation (35) into the following matrix form

$$\left(G^2 D^3 + D^2\right)^T A = B,$$

and also its boundary conditions as

$$\sum_{i=0}^{5} a_i L_i^{(\alpha)}(0) = \left(L^{(\alpha)}(0)\right)^T A = 0,$$

$$\sum_{i=0}^{5} a_i L_i^{(\alpha)}(1) = \left(L^{(\alpha)}(1)\right)^T A = 1,$$

and

$$\sum_{i=0}^{5} a_i L_i^{(\alpha)}(-1) = \left(L^{(\alpha)}(-1)\right)^T A = 1.$$

After the augmented matrices of the system and boundary conditions are computed, we obtain the solution $y(x) = x^2$, which is the exact solution.

7. Application of the Method for the High-Order Linear Differential Equation

In this section, we report the numerical results obtained for a high-order linear differential equation by the aforementioned procedure. This shows that it is straightforward to extend the method to the high-order linear differential equations as follows.

Experiment 4.

Let us consider the eighth-order linear differential equation [Kurt and Sezer (2008) and Golbabai and Javidi (2007)]:

$$y^{(8)}(x) - y(x) = -8e^{x}, \; 0 \le x \le 1,$$

with the initial conditions

$$y(0) = 1, \, y'(0) = 0, \, y''(0) = -1,$$

$$y'''(0) = -2, \, y^{(4)}(0) = -3, \, y^{(5)}(0) = -4,$$

$$y^{(6)}(0) = -5, \, y^{(7)}(0) = -6.$$

Table 2. The comparison between the exact and approximate solutions of HPM, MDM and present methods of experiment 4

x	Exact	Present method	HPM method	MDM method
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9946538262	0.9946538261	0.9946538263	0.9946538262
0.2	0.9771222065	0.9771222063	0.9771222065	0.9771222014
0.3	0.9449011653	0.9449011651	0.9449011653	0.9449010769
0.4	0.8950948185	0.8950948183	0.8950948186	0.8950941522
0.5	0.8243606353	0.8243606351	0.8243606356	0.8243574386
0.6	0.7288475201	0.7288475205	0.728847522	0.7288359969
0.7	0.6041258122	0.6041258121	0.6041258211	0.6040917111
0.8	0.4451081856	0.4451081852	0.445108220	0.4450208387
0.9	0.2459603111	0.2459603101	0.2459604249	0.2457599482
1.0	0.0000000000	0.0000000000	3.326×10^{-7}	4.212943×10^{-4}

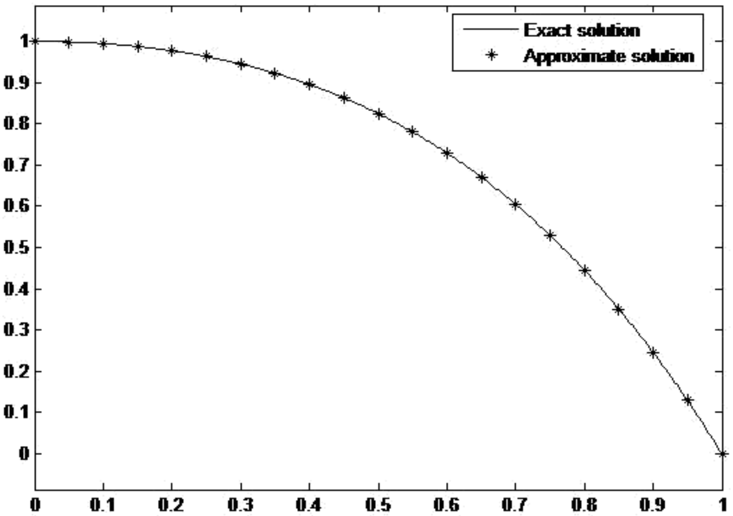
Exact solution
Approximate solution
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 2. The comparison between the exact and approximate solutions of $n = 12$ and $\alpha = 2$ of experiment 4

Table 3. The comparison between the exact and approximate solutions of present and Taylor methods of experiment 4

x	Exact	Present method	Taylor method
0.0	1.0000000000	1.0000000000	1.0000000000
0.1	0.9946538262	0.9946538261	0.9946538266
0.2	0.9771222065	0.9771222063	0.9771222093
0.3	0.9449011653	0.9449011651	0.9449011752
0.4	0.8950948185	0.8950948183	0.8950948487
0.5	0.8243606353	0.8243606351	0.8243607328
0.6	0.7288475201	0.7288475205	0.7288478604
0.7	0.6041258122	0.6041258121	0.6041269662
0.8	0.4451081856	0.4451081852	0.4451117669
0.9	0.2459603111	0.2459603101	0.2459703618
1.0	0.0000000000	0.0000000000	2.57×10^{-5}

The exact solution of this equation is $y(x) = (1-x)e^{x}$. By implementation of our method which is presented in section 4, and also after the augmented matrices of the system and boundary conditions are computed, we obtain the numerical solutions. The comparison between our method and other numerical methods are shown in tables 2 and 3. Also the exact and approximate solutions are shown in figure 2.

We see that our method, HPM and MDM methods obtain better results than the other methods for this experiment. These methods rather than the Taylor polynomial set obtain better results near the corner of interval. In other words, in the interior points between 0 and 1, the Taylor method gives better results. This matter is seen by [Aminataei and Hussaini (2007) and (2010)] also, which is due to the affinity of Taylor series to the origin.

8. Conclusion

In this paper, we have introduced a new and efficient approach for numerical approximation of the linear differential-difference equations. The method is based on the approximation of the exact solution with the generalized Laguerre polynomials approximation with variable coefficients by Taylor series expansion. Implementation of the method reduces the problem to a system of algebraic equations. Some test experiments are presented for showing the accuracy and efficiency of the present method with the other methods such as HPM, MDM and Taylor series. Application of the method for numerical solution of high-order linear differential equations is also considered. In addition, we would like to emphasize that the main importance of the present scheme is considering the general linear DDEs (1) and (2), whereas the other manuscripts only considered the particular cases of our general problem. Further, using the generalized Laguerre polynomials as the basis functions for numerical approximation whereas the classical Laguerre polynomials are particular cases of them is another advantage of the present study.

Acknowledgments

We are grateful to the anonymous reviewers and the editor Professor Dr. Aliakbar Montazer Haghighi for their helpful comments and suggestions which indeed improved the quality of this manuscript.

REFERENCES

Agraval, R.P. and Oregan, D.O. (2009). Ordinary and Partial Differential Equations, *Springer*.

Akkaya, T. and Yalcinbas, S. (2012). Boubaker polynomial approach for solving high-order linear differential-difference equations, A*IP Conference Proceedings of 9th international conference on mathematical problems in engineering,* Vol. 56, pp. 26-33.

Aminataei, A. and Hussaini, S.S. (2007). The comparison of the stability of decomposition method with numerical methods of equation solution, *Appl. Math. Comput.*, Vol. 186, pp. 665-669.

Aminataei, A. and Hussaini, S.S. (2010). The barrier of decomposition method, *Int. J. Contemp. Math. Sci*., Vol. 5, pp. 2487-2494.

Aminataei, A. and Vanani, S.K. (2013). Numerical solution of fractional Fokker-Planck equation using the operational collocation method, *Appl. Comput. Math*., Vol. 12, pp. 33-43.

Askey, R. (1975). Orthogonal Polynomials and Special Functions. *SIAM-CBMS*, Philadelphia.

Ben-yu, G. (1996). The State of Art in Spectral Methods. *Hong Kong University*.

Boyd, J.P. (2000). Chebyshev and Fourier Spectral Methods. Dover *Publications, Inc,* New York.

Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A. (1984). Spectral Method in Fluid Dynamics, *Prentice Hall*, Engelwood Cliffs, NJ.

Canuto, C., Hussaini, M.Y., Quarteroni, A. and Zang, T.A. (2006). Spectral Methods: *Fundamentals in Single Domains*, Springer-Verlag.

Danfu, H. and Xufeng, S. (2007). Numerical solution of integro-differential equations by using

CAS wavelet operational matrix of integration, *Appl. Math. Comput*., Vol. 194, pp. 460-466.

Dunkl, C.F. and Xu, Y. (2001). Orthogonal Polynomials of Several Variables, *Cambridge University Press*.

Erdem, K. and Yalcinbas, S. (2012). Bernoulli polynomial approach to high-order linear differential-difference equations, *AIP Conference Proceedings of Numerical Analysis and Applied Mathematics*, Vol. 73, pp. 360-364.

Eslahchi, M.R. and Dehghan, M. (2011). Application of Taylor series in obtaining the orthogonal operational matrix, *Computers and Mathematics with Applications*, Vol. 61, pp. 2596-2604.

Funaro, D. (1992). Polynomial Approximations of Differential Equations, Springer-Verlag.

Gautschi, W. (2004). Orthogonal Polynomials (Computation and Approximation), *Oxford University Press.*

Golbabai, A. and Javidi, M. (2007). Application of homotopy perturbation method for solving eighth-order boundary value problems. *Appl. Math. Comput*., Vol. 213, pp. 203-214.

Gottlieb, D. and Orszag, S.A. (1977). Numerical Analysis of Spectral Methods: Theory and Applica-tions. *SIAM-CBMS*, Philadelphia.

Gulsu, M. and Sezer, M. (2005). A method for the approximate solution of the high-order linear difference equations in terms of Taylor polynomials, *Int. J. Comput. Math*., Vol. 82, pp. 629-642.

Gulsu, M. and Sezer, M. (2006). A Taylor polynomial approach for solving differential-difference equa-tions, *Comput. Appl. Math*., Vol. 186, pp. 349-364.

Gulsu, M., Sezer, M. and Guney, Z. (2006). Approximate solution of general high-order linear non-homogenous difference equations by means of Taylor collocation method, *Appl. Math. Comput.*, Vol. 173, pp. 683-693.

Hesthaven, J.S., Gottlieb, S. and Gottlieb, D. (2009). Spectral Methods for Time-Dependent Problems. *Cambridge University*.

Hsiao, C. H. (2009). Hybrid function method for solving Fredholm and Volterra integral equations of the second kind, Comput. *Appl. Math*., Vol. 230, pp. 59-68.

Imani, A., Aminataei, A. and Imani, A. (2011). Collocation method via Jacobi polynomials for solving nonlinear ordinary differential equations, *Int. J. Math. Math. Sci*., Article ID 673085, 11P.

Kesan, C. (2003). Taylor polynomial solutions of linear differential equations, *Appl. Math. Comput.*, Vol. 142, pp. 155-165.

Khellat F. and Yousefi, S. A. (2006). The linear Legendre wavelets operational matrix of integration and its application, *J. Frank. Inst*., Vol. 343, pp. 181-190.

King, A.C., Bilingham, J. and Otto, S.R. (2003). Differential Equations (Linear, Nonlinear, Integral, Partial), Cambridge University.

Kurt, N. and Sezer, M. (2008). Polynomial solution of high-order linear Fredholm integro-differential equations with constant coefficients, *J. Frank. Inst.*, Vol. 345, pp. 839-850.

Maleknejad, K. and Mirzaee, F. (2006). Numerical solution of integro-differential equations by using rationalized Haar functions method, Kyber. *Int. J. Syst. Math*., Vol. 35, pp. 1735-1744.

Marcellan, F. and Assche, W.V. (2006). Orthogonal Polynomials and Special Functions (a Computation and Applications), *Springer-Verlag Berlin Heidelberg*.

Ortiz, E. L. (1978). On the numerical solution of nonlinear and functional differential equations with the Tau method, in: Numerical Treatment of Differential Equations in Applications, *In: Lecture Notes in Math*., Vol. 679, pp. 127-139.

Ortiz, E. L. and Samara, L. (1981). An operational approach to the Tau method for the numerical solution of nonlinear differential equations, *Computing*, Vol. 27, pp. 15-25.

Razzaghi, M. and Ordokhani, Y. (2001). Solution of nonlinear Volterra Hammerstein integral equations via rationalized Haar functions, *Math. Prob. Eng*., Vol. 7, pp. 205-219.

Razzaghi, M. and Yousefi, S.A. (2005). Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations, *Math. Comput. Simul*., Vol. 70, pp. 1-8.

Reihani, M. H. and Abadi, Z. (2007). Rationalized Haar functions method for solving Fredholm and Volterra integral equations, *Comput. Appl. Math*, Vol. 200, pp. 12-20.

Sezer, M. and Dascioglu, A.A. (2006). Taylor polynomial solutions of general linear differential-difference equations with variable coefficients. *Appl. Math. Comput*., Vol. 174, pp. 1526-1538.

Sezer, M. and Gulsu, M. (2005). Polynomial solution of the most general linear Fredholm integro-differential-difference equation by means of Taylor matrix method, *Int. J. Complex Variables*, Vol. 50, pp. 367-382.

Shen, J., Tang, T. and Wang, L.L. (2011). Spectral Methods Algorithms, Analysis and Applications, *Springer*.

Trefethen, L.N. (2000). Spectral Methods in Matlab, *SIAM*, Philadelphia, PA.

Vanani, S. K. and Aminataei, A. (2012). A numerical algorithm for the space and time fractional Fokker-Planck equation, Int. J. of Numer. *Meth. for Heat & Fluid Flow*, Vol. 22, pp. 1037-1052.

Vanani, S.K. and Aminataei, A. (2011). Tau approximate solution of fractional partial differential equations, *Comput. Math. Appl.*, Vol. 62, pp. 1075-1083.

Wazwaz, A.M. (2010). The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations, *Appl. Math. Comput*., Vol. 216, pp. 1304-1309.