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A New Adjustment of Laplace Transform for Fractional Bloch Equation in NMR Flow

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Abstract

This work purpose suggest a new analytical technique called the fractional homotopy analysis transform method (FHATM) for solving time fractional Bloch NMR (nuclear magnetic resonance) flow equations, which are a set of macroscopic equations that are used for modeling nuclear magnetization as a function of time. The true beauty of this article is the coupling of the homotopy analysis method and the Laplace transform method for systems of fractional differential equations. The solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive.

Keywords: Bloch equations; Laplace transform method; approximate solution; new fractional homotopy analysis transform method

AMS-MSC 2010 No.: 26A33, 34A08, 34A34, 60G22

1. Introduction

In recent years, considerable interest in fractional differential equations has been stimulated by their numerous applications in the areas of physics and engineering [West (2003)]. In past years, differential equations involving derivatives of non-integer order provided adequate models for various physical phenomena [Podlubny (1999)]. In the study of electromagnetics, acoustics, viscoelasticity, electro chemistry and material science. That is so since any realistic modelling of a physical phenomenon having dependence not only on the time instant, but also on the previous time history can only be successfully done using fractional calculus. The book [Oldham and Spanier (1974)] has played a key role in the development of the fractional calculus. Some fundamental results related to solving fractional differential equations may be found in [Miller and Rose (2003); Kilbas and Srivastava (2006); Diethelm and Ford (2002); Diethelm (1997); Samko (1993)].

In physics and chemistry, especially in NMR (nuclear magnetic resonance), MRI (magnetic resonance imaging), or ESR (electron spin resonance) the Bloch equations are a set of macroscopic equations that are used to calculate the nuclear magnetization $M = (M_x, M_y, M_z)$ in the laboratory frame (x, y, z) , T_1 and T_2 are known respectively as the spin lattice and spin-spin relaxation times to measure the interactions of the nuclei with their surrounding molecular environment and those between close nuclei. Magnetic resonance imaging (MRI) is a powerful tool for obtaining spatially localized information from nuclear magnetic resonance (NMR) of atoms within a sample. These equations, introduced by Felix Bloch (1946), have played a central role in elucidating magnetic resonance phenomena ever since Madhu and Kumar (1997). Torrey (1956) modified the Bloch equation by incorporating a diffusion term. The dynamics of an ensemble of spins without mutual couplings are usually well described by the Bloch equations [Jeener, (1999); Rourke et al. (2004)], which can be viewed as mathematical descriptions of the precession of the macroscopic magnetization vector around a (possibly time-dependent) magnetic field. Recently, Petras and Bhalekar et al. (2011) have solved the fractional Bloch equations. The Bloch equations can be expressed in the fractional form as Petras (2011)

$$\begin{aligned} D_t^\alpha M_x(t) &= \omega_0 M_y(t) - \frac{M_x(t)}{T_2}, & 0 < \alpha \leq 1, \\ D_t^\beta M_y(t) &= -\omega_0 M_x(t) - \frac{M_y(t)}{T_2}, & 0 < \beta \leq 1, \\ D_t^\gamma M_z(t) &= \frac{M_0 - M_z(t)}{T_1}, & 0 < \gamma \leq 1, \end{aligned} \quad (1.1)$$

with initial conditions $M_x(0) = 0$, $M_y(0) = 100$ and $M_z(0) = 0$, where $\omega_0 = \gamma B_0$ and $\omega_0 = 2\pi f_0$ (e.g., gyromagnetic ratio $\gamma/2\pi = f_0/B_0 = 42.57 \text{ MHz/T}$ for water protons) and M_0 is the equilibrium magnetization. The complete set of analytic solutions of the system of equations (1.1) is given as

$$\begin{aligned}
 M_x(t) &= e^{-t/T_2} (M_x(0) \cos \omega_0 t + M_y(0) \sin \omega_0 t) \\
 M_y(t) &= e^{-t/T_2} (M_y(0) \cos \omega_0 t - M_x(0) \sin \omega_0 t) \\
 M_z(t) &= M_z(0) e^{-t/T_1} + M_0 (1 - e^{-t/T_1}).
 \end{aligned}
 \tag{1.2}$$

Recently, experts are paying great attention to the construction of the solutions of the Bloch equations by different methods (some exact and approximate solutions) [Murase and Tanki (2011), Yan et al. (1987), Schotland and Leigh (1987), Sivers, (1986)].

The aim of this article is to obtain a new analytical approximate solution of the fractional Bloch equations by using the new fractional homotopy analysis transform method. This new proposed method is a coupling of the homotopy analysis method and the Laplace transform method. Its main advantage is its capability of combining two powerful methods to obtain a convergent series for the fractional partial differential equations. The homotopy analysis method was first introduced and applied by Liao (1992, 1997, 2003, 2004a, 2004b, 2007, 2009). The HAM has been successfully applied by many researchers for solving linear and non-linear partial differential equations [Vishal et al. (2012); Jafari et al. (2010); Zhang (2011); and Ghotbi (2009)]. In recent years, many authors engaged themselves in the study of the solutions of linear and nonlinear partial differential equations by using various methods that incorporate the Laplace transform. Among these are the Laplace decomposition methods [Wazwaz (2010) and Khan et al. (2012)]. Recently, the homotopy perturbation transform method [Kumar et al. (2012) and Khan et al. (2012)] was applied to obtain the solutions of the Blasius flow equation on a semi-infinite domain by coupling the homotopy analysis and the Laplace transform methods. Some authors, such as Wei (2012) and Zhang (2012), have solved the fractional differential equation by using different numerical techniques.

2. Basic Definition of Fractional Calculus and Laplace Transform

Fractional calculus unifies and generalizes the notions of integer-order differentiation and the n -fold integration [Podlubny (1999), Oldham and Spanier (1974), Miller and Ross (2003)]. We give some basic definitions and properties of fractional calculus theory which shall be used in this paper.

Definition 2.1.

A real function $f(t)$, $t > 0$ is said to be in the space C_μ , $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2.2.

The left sided Riemann-Liouville fractional integral operator of order $\mu \geq 0$, of a function $f \in C_\alpha, \alpha \geq -1$ is defined as [Luchko and Gorenflo (1999), Moustafa (2003)]

$$I^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, t > 0, \\ f(t), & \mu = 0, \end{cases} \quad (2.1)$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Definition 2.3.

The left sided Caputo fractional derivative of $f, f \in C_{-1}^m, m \in N \cup \{0\}$ is defined as [Podlubny (1999), Samko (1993)]

$$D_*^\mu f(t) = \frac{d^\mu f(t)}{dt^\mu} = \begin{cases} I^{m-\mu} \left[\frac{d^m f(t)}{dt^m} \right], & m-1 < \mu < m, m \in N. \\ \frac{d^m f(t)}{dt^m}, & \mu = m. \end{cases} \quad (2.2)$$

Note that [Podlubny (1999), Samko (1993)]:

$$(i) \quad I_t^\mu f(x, t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x, s)}{(t-s)^{1-\mu}} ds, \quad \mu > 0, t > 0,$$

$$(ii) \quad D_*^\mu f(x, t) = I_t^{m-\mu} \frac{\partial^m f(x, t)}{\partial t^m} \quad m-1 < \mu \leq m.$$

Definition 2.4.

The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)$ in $[0, \infty)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t) dt, \quad (2.3)$$

where s is real or complex number.

Definition 2.5.

The Laplace transform of $f(t) = t^\alpha$ is defined as Podlubny (1999):

$$L[t^\alpha] = \int_0^\infty e^{-st} t^\alpha dt = \frac{\Gamma(\alpha+1)}{s^{(\alpha+1)}}, \quad R(s) > 0, R(\alpha) > 0. \quad (2.4)$$

Definition 2.6.

The Laplace transform of the Riemann–Liouville fractional integral $I^\alpha f(t)$ is defined as Podlubny (1999)

$$L[I^\alpha f(t)] = s^{-\alpha} F(s). \quad (2.5)$$

Definition 2.7.

The Laplace transform of the Caputo fractional derivative is defined as Podlubny (1999)

$$L[D_t^{n\alpha} f(t)] = s^{n\alpha} L[f(t)] - \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} f^{(k)}(0+), \quad n-1 < n\alpha \leq n. \quad (2.6)$$

Definition 2.8.

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by following series representation, valid in the whole complex plane, Mainardi (1994)

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^\alpha}{\Gamma(n\alpha+1)}, \quad \alpha > 0, \quad z \in C. \quad (2.7)$$

3. Basic Idea of New Fractional Homotopy Analysis Transform Method (FHATM)

To illustrate the basic idea of the HATM, we consider the following fractional partial differential equation:

$$D_t^{n\alpha} u_i(r,t) + R_i[r]u_i(r,t) + N_i[r]u_i(r,t) = g_i(r,t), \quad i=1,2,\dots, t > 0, r \in R^3, \\ n-1 < n\alpha \leq n, \quad (3.1)$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, $R[r]$ is a linear operator in $r \in R^3$, $N_i[r]$ is the general nonlinear operator in $r \in R^3$

and $g_i(r,t)$ are continuous functions. For simplicity we ignore all initial and boundary conditions, which can be treated in a similar way. Now the methodology consists of applying the Laplace transform first on both sides of Equations (3.1); we get

$$L[D_t^{n\alpha} u_i(r,t)] + L[R_i[r]u_i(r,t) + N[r]u_i(r,t)] = L[g_i(r,t)], \quad i = 1, 2, 3, \dots \quad (3.2)$$

Now, using the differentiation property of the Laplace transform, we have

$$L[u_i(r,t)] - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u_i^{(k)}(r,0) + \frac{1}{s^{n\alpha}} L(R_i[r]u_i(r,t) + N_i[r]u_i(r,t) - g_i(r,t)) = 0, \\ i = 1, 2, 3, \dots, t > 0. \quad (3.3)$$

We define the nonlinear operator

$$N[\phi_i(r,t;q)] = L[\phi_i(r,t;q)] - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{(n\alpha-k-1)} u_i^{(k)}(r,0) \\ + \frac{1}{s^{n\alpha}} L(R_i[r]\phi_i(r,t;q) + N_i[r]\phi_i(r,t;q) - g_i(r,t)), \quad i = 1, 2, 3, \dots, t > 0, \quad (3.4)$$

where $q \in [0,1]$ is an embedding parameter and $\phi_i(r,t;q)$ is the real function of r, t and q . By means of generalizing the traditional homotopy methods, Liao (1992, 1997, 2003, 2004a, 2004b, 2007, 2009) constructed the zero order deformation equations as follows

$$(1-q)L[\phi_i(r,t;q) - u_{0_i}(r,t)] = \hbar q H_i(r,t) N[\phi_i(r,t;q)], \quad i = 1, 2, 3, \dots, \quad (3.5)$$

where \hbar is a nonzero auxiliary parameter, $H_i(r,t) \neq 0$ is an auxiliary function, $u_{0_i}(r,t)$ is an initial guess of $u_i(r,t)$ and $\phi_i(r,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary items in HATM. Obviously, when $q=0$ and $q=1$, it holds that

$$\phi_i(r,t;0) = u_{0_i}(r,t), \quad \phi_i(r,t;1) = u_i(r,t). \quad (3.6)$$

Thus, as q increases from 0 to 1, the solution varies from the initial guess $u_{0_i}(r,t)$ to the solution $u_i(r,t)$.

Expanding $\phi_i(r,t;q)$ in Taylor's series with respect to q , we have

$$\phi_i(r,t;q) = u_{0_i}(r,t) + \sum_{m=1}^{\infty} q^m u_{m_i}(r,t), \quad (3.7)$$

where

$$u_{m_i}(r,t) = \frac{1}{m!} \left. \frac{\partial^m \phi_i(r,t;q)}{\partial q^m} \right|_{q=0}. \quad (3.8)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter \hbar , and the auxiliary function are properly chosen, the series (3.7) converges at $q=1$, so that we have

$$u_i(r, t) = u_{0i}(r, t) + \sum_{m=1}^{\infty} u_{mi}(r, t), \quad i = 1, 2, 3, \dots, \quad (3.9)$$

which must be one of the solutions of the original nonlinear equations.

We define the vectors

$$\vec{u}_i = \{u_{0i}(r, t), u_{1i}(r, t), u_{2i}(r, t), \dots, u_{ni}(r, t)\}, \quad i = 1, 2, 3, \dots \quad (3.10)$$

Differentiating equations (3.5) m times with respect to embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the m th order deformation equations

$$L[u_{mi}(r, t) - \chi_m u_{m-1,i}(r, t)] = \hbar q H_i(r, t) R_{mi}(\vec{u}_{m-1}, r, t), \quad i = 1, 2, 3, \dots \quad (3.11)$$

Operating the inverse Laplace transform on both sides, we get

$$u_{mi}(r, t) = \chi_m u_{m-1,i}(r, t) + \hbar q L^{-1}[H_i(r, t) R_{mi}(\vec{u}_{m-1}, r, t)], \quad i = 1, 2, 3, \dots, \quad (3.12)$$

where

$$R_{mi}(\vec{u}_{m-1}, r, t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \phi(r, t; q)}{\partial q^{m-1}} \right|_{q=0}, \quad i = 1, 2, 3, \dots, \quad (3.13)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In this way, it is easy to obtain $u_m(r, t)$ for $m \geq 1$, at M^{th} order, and we have

$$u_i(r, t) = \sum_{m=0}^M u_{mi}(r, t), \quad i = 1, 2, 3, \dots, \quad (3.14)$$

when $M \rightarrow \infty$ we get an accurate approximation of the original equation (3.1).

4. Solution of the Bloch Equation by the New Proposed Method

We consider the following fractional-order Bloch equations in NMR flow as

$$\begin{aligned}
D_t^\alpha M_x(t) &= \omega_0 M_y(t) - \frac{M_x(t)}{T_2}, & 0 < \alpha \leq 1, \\
D_t^\beta M_y(t) &= -\omega_0 M_x(t) - \frac{M_y(t)}{T_2}, & 0 < \beta \leq 1, \\
D_t^\gamma M_z(t) &= \frac{M_0 - M_z(t)}{T_1}, & 0 < \gamma \leq 1,
\end{aligned} \tag{4.1}$$

where α , β and γ are the derivative orders. The total order of the system is (α, β, γ) . Here, all parameters ω_0 , T_1 and T_2 have the units of $(s)^{-q}$ to maintain a consistent set of units for the magnetization.

Operating Laplace transform on both sides in system of equations (4.1) and after using the differentiation property of Laplace transform, we get

$$\begin{aligned}
L[M_x(t)] &= s^{-\alpha} L\left[\omega_0 M_y(t) - \frac{M_x(t)}{T_2}\right], & 0 < \alpha \leq 1. \\
L[M_y(t)] &= \frac{100}{s} - s^{-\beta} L\left[-\omega_0 M_x(t) - \frac{M_y(t)}{T_2}\right], & 0 < \beta \leq 1. \\
L[M_z(t)] &= s^{-\gamma} L\left[\frac{M_0 - M_z(t)}{T_1}\right], & 0 < \gamma \leq 1.
\end{aligned} \tag{4.2}$$

We choose linear operators as

$$\mathfrak{L}[\phi_i(t; q)] = L[\phi_i(t; q)], \quad i = x, y, z, \tag{4.3}$$

with property $\mathfrak{L}[c] = 0$, where c is a constant.

We now define the nonlinear operators as

$$\begin{aligned}
N_1[\phi_x(t; q)] &= L[\phi_x(t; q)] - s^{-\alpha} L\left[\omega_0 \phi_y(t; q) - \frac{\phi_x(t; q)}{T_2}\right], \\
N_2[\phi_y(t; q)] &= L[\phi_y(t; q)] - \frac{100}{s} - s^{-\beta} L\left[-\omega_0 \phi_x(t; q) - \frac{\phi_y(t; q)}{T_2}\right], \\
N_3[\phi_z(t; q)] &= L[\phi_z(t; q)] - s^{-\gamma} L\left[\frac{M_0 - \phi_z(t; q)}{T_1}\right].
\end{aligned} \tag{4.4}$$

Using the above definition, with assumption $H_i(t) = 1$ for $i = x, y, z$, we can construct the zeroth order deformation equations

$$(1-q)\mathfrak{L}[\phi_i(t;q) - M_{i0}(x,t)] = \hbar q N[\phi_i(t;q)], \quad i = x, y, z. \quad (4.5)$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi_i(t;0) = M_{i0}(t), \quad \phi_i(t;1) = M_i(t), \quad i = x, y, z. \quad (4.6)$$

Thus, we obtain the m th order deformation equations

$$\mathfrak{L}[M_{i,m}(t) - \chi_m M_{i,m-1}(t)] = \hbar R_m(\vec{M}_{i,m-1}, t), \quad i = x, y, z. \quad (4.7)$$

Operating the inverse Laplace transform on both sides in Equations (4.7), we get

$$M_{i,m}(t) = \chi_m M_{i,m-1}(t) + \hbar q L^{-1}[R_m(\vec{M}_{i,m-1}, t)], \quad i = x, y, z, \quad m \geq 1, \quad (4.8)$$

where

$$\begin{aligned} R_{x,m}(\vec{M}_{x,m-1}, t) &= L[M_{x,m-1}(t)] - s^{-\alpha} L\left[\omega_0 M_{y,m-1}(t) - \frac{M_{x,m-1}(t)}{T_2}\right], \\ R_{y,m}(\vec{M}_{y,m-1}, t) &= L[M_{y,m-1}(t)] - \frac{100(1-\chi_m)}{s} - s^{-\beta} L\left[-\omega_0 M_{x,m-1}(t) - \frac{M_{y,m-1}(t)}{T_2}\right], \quad m \geq 1, \\ R_{z,m}(\vec{M}_{z,m-1}, t) &= L[M_{z,m-1}(t)] - s^{-\gamma} L\left[\frac{M_0 - M_{z,m-1}(t)}{T_1}\right]. \end{aligned} \quad (4.9)$$

Using

$$R_{i,m}(\vec{M}_{i,m-1}, t) \quad \text{for } i = x, y, z$$

from system of equations (4.9) in (4.8), we get

$$\begin{aligned} M_{x,m}(t) &= (\chi_m + \hbar)M_{x,m-1}(t) - \hbar L^{-1}\left(s^{-\alpha} L\left[\omega_0 M_{y,m-1}(t) - \frac{M_{x,m-1}(t)}{T_2}\right]\right), \\ M_{y,m}(t) &= (\chi_m + \hbar)M_{y,m-1}(t) - 100\hbar(1-\chi_m) \\ &\quad - \hbar L^{-1}\left(s^{-\beta} L\left[-\omega_0 M_{x,m-1}(t) - \frac{M_{y,m-1}(t)}{T_2}\right]\right), \quad m \geq 1, \\ M_{z,m}(t) &= (\chi_m + \hbar)M_{z,m-1}(t) - \frac{\hbar M_0(1-\chi_m)t^\gamma}{T_1 \Gamma(\gamma+1)} + \hbar L^{-1}\left(s^{-\gamma} L\left[\frac{M_{z,m-1}(t)}{T_1}\right]\right). \end{aligned} \quad (4.10)$$

Using the initial approximations $M_{x,1}(0) = M_x(0) = 0$, $M_{y,1}(0) = M_y(0) = 100$ and $M_{z,1}(0) = M_z(0) = 0$, from iterative scheme (4.13), we obtain the various iterates

$$M_{x,1}(t) = -\frac{100\hbar\omega t^\alpha}{\Gamma(\alpha+1)}, \quad M_{y,1}(t) = \frac{100\hbar t^\beta}{T_2 \Gamma(\beta+1)}, \quad M_{z,1}(t) = -\frac{M_0 \hbar t^\gamma}{T_1 \Gamma(\gamma+1)},$$

$$M_{x,2}(t) = \frac{100\hbar\omega(1+\hbar)t^\alpha}{\Gamma(\alpha+1)} - \frac{100\hbar^2\omega t^{2\alpha}}{T_2 \Gamma(2\alpha+1)} - \frac{100\hbar^2\omega t^{\alpha+\beta}}{T_2 \Gamma(\alpha+\beta+1)},$$

$$M_{y,2}(t) = \frac{100\hbar(1+\hbar)t^\beta}{T_2 \Gamma(\beta+1)} - \frac{100\hbar^2\omega^2 t^{2\beta}}{T_2^2 \Gamma(2\beta+1)} - \frac{100\hbar^2\omega^2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

$$M_{z,2}(t) = -\frac{M_0 \hbar(1+\hbar)t^\gamma}{T_1 \Gamma(\gamma+1)} - \frac{M_0 \hbar^2 t^{2\gamma}}{T_1^2 \Gamma(2\gamma+1)}.$$

Proceeding in this manner, the rest of the components $M_{i,n}(x,t)$, $i = x, y, z$ for $n \geq 3$ can be completely obtained and the series solutions are thus entirely determined. Hence, the solution of Equations (4.1) is given as

$$M_i(t) = M_{i,0}(t) + \sum_{m=1}^{\infty} M_{i,m}(t), \quad i = x, y, z. \quad (4.11)$$

If we select $\hbar = -1$,

$$\tilde{M}_x(t) = 100 \omega_0 \left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{T_2 \Gamma(2\alpha+1)} - \frac{t^{\alpha+\beta}}{T_2 \Gamma(\alpha+\beta+1)} + \frac{t^{3\alpha}}{T_2^2 \Gamma(3\alpha+1)} + \frac{t^{2\alpha+\beta}}{T_2^2 \Gamma(2\alpha+\beta+1)} - \frac{\omega_0^2 t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \right. \\ \left. + \frac{t^{\alpha+2\beta}}{T_2^2 \Gamma(\alpha+2\beta+1)} + \dots \right).$$

$$\tilde{M}_y(t) = 100 E_\beta \left(-\frac{t^\beta}{T_2} \right) - \frac{100 \omega_0^2 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{100 \omega_0^2}{T_2} \left(\frac{2t^{\alpha+\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \right) + \dots$$

$$\tilde{M}_z(t) = \frac{M_0}{T_1} (1 - E_\gamma(-t^\gamma)).$$

The series solution converges very rapidly. The rapid convergence means only few terms are required to get the approximate solutions. Clearly, we can conclude that the obtained solution

$$\sum_{m=0}^{\infty} M_{i,m}(x,t), \quad i = x, y, z$$

converges to the exact solution.

5. Numerical Result and Discussion

In this section, approximate solutions are depicted through in Figures 1- 3 for different fractional Brownian motions and standard motions. It is seen from Figures 1 and 3 that the approximate solutions $\tilde{M}_{x,14}(t)$ and $\tilde{M}_{z,14}(t)$ increase with an increase in t for different values of $\alpha = \beta = 0.7, 0.8, 0.9$ and also for standard Bloch equations i.e., for $\alpha = \beta = 1$. It is also seen from Figure 2 that the approximate solution $\tilde{M}_y(t)$ decreases with an increase in t for different value of $\gamma = 0.7, 0.8, 0.9, 1$. It is to be noted that only fourteen terms of the homotopy analysis transform method were used in evaluating the approximate solutions in all figures.

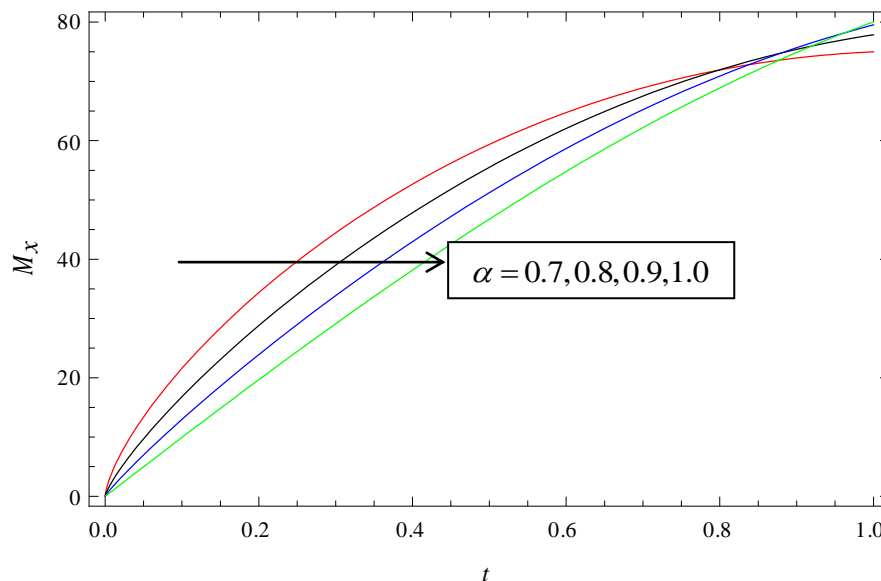


Figure 1. Plot of approximate solution $\tilde{M}_x(t)$ for different values of α at $\omega = 1, \hbar = -1$ and $T_2 = 20 \text{ (ms)}^q$

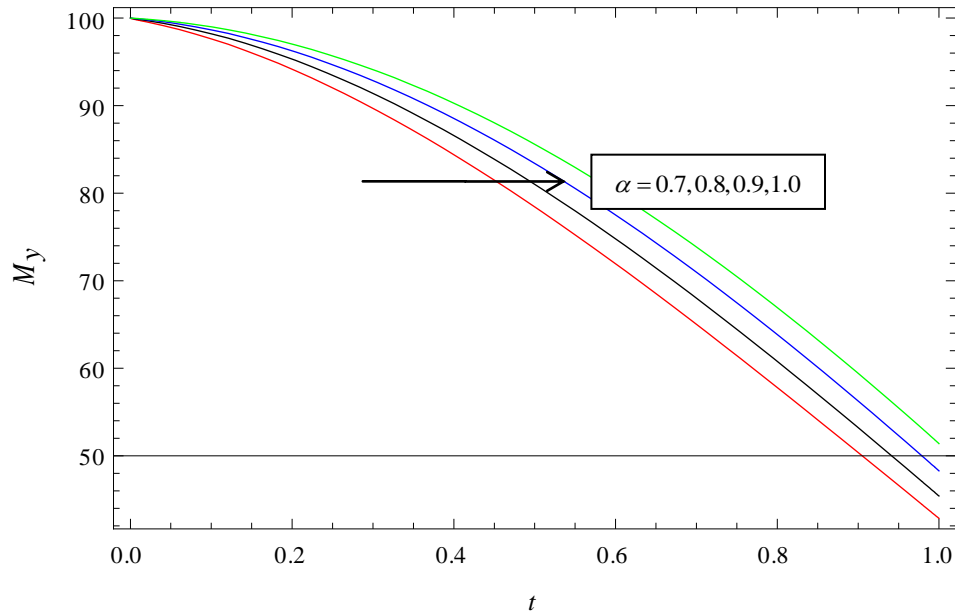


Figure 2. Plot of approximate solution $\tilde{M}_y(t)$ for different values of β at $\omega=1$, $\hbar=-1$ and $T_2=20$ (ms)^q

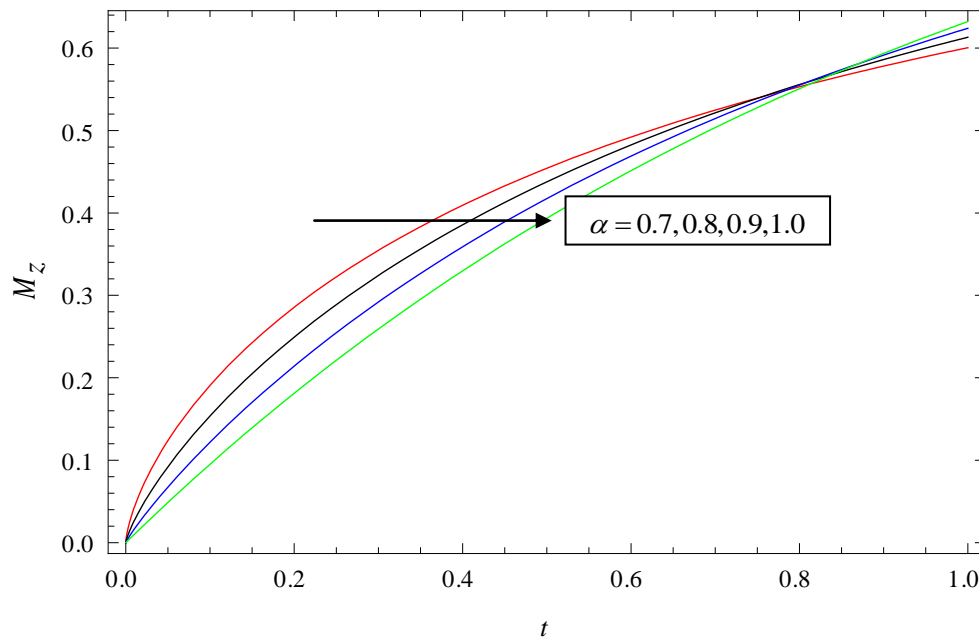


Figure 3. Plot of approximate solution $\tilde{M}_z(t)$ for different values of γ at $\omega=1$, $\hbar=-1$ and $T_1=1$ (ms)^q

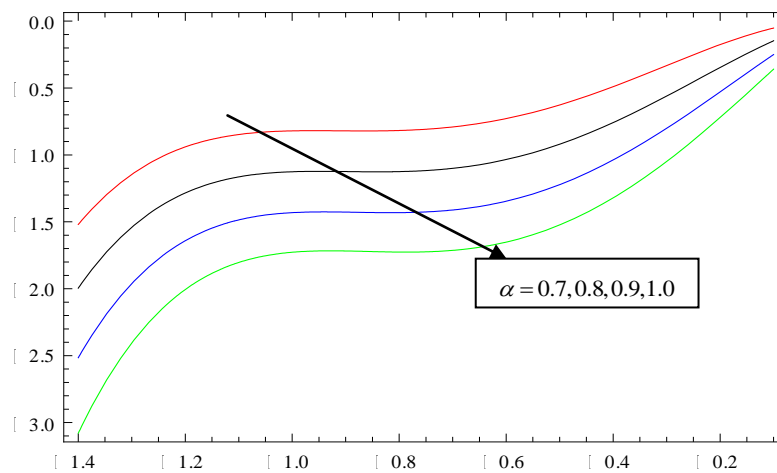
The simplicity and accuracy of the proposed method is illustrated by computing the absolute errors $E_{M_x}(14) = |M_x(t) - \tilde{M}_x(t)|$, $E_{M_y}(14) = |M_y(t) - \tilde{M}_y(t)|$, and $E_{M_z}(14) = |M_z(t) - \tilde{M}_z(t)|$, where $M_x(t)$, $M_y(t)$, $M_z(t)$ are exact solutions and $\tilde{M}_x(t)$, $\tilde{M}_y(t)$, $\tilde{M}_z(t)$ are approximate solutions obtained by truncating the respective solutions series (4.14) at level $N=14$.

Table 1. The values of the approximate solution at different grid points

t	$E_x(14) = M_x(t) - \tilde{M}_{x,14}(t) $	$E_y(14) = M_y(t) - \tilde{M}_{y,14}(t) $	$E_z(14) = M_z(t) - \tilde{M}_{z,14}(t) $
0.2	0	5.79151×10^{-15}	3.15931×10^{-17}
0.4	1.06581×10^{-14}	3.78571×10^{-15}	3.74427×10^{-17}
0.6	4.58665×10^{-12}	4.26774×10^{-12}	3.17064×10^{-16}
0.8	3.42915×10^{-10}	3.11584×10^{-10}	2.56557×10^{-14}
1.0	9.72397×10^{-9}	8.63641×10^{-9}	7.19591×10^{-13}

The absolute errors between exact and approximate solutions with fourteen terms are given in Table for $t \in (0,1)$. From Table 1, it is observed that the values of the approximate solution at different grid points obtained by the present method are close to the values of the exact solution with high accuracy at the level $N=14$. It can also be noted that the accuracy increases as the value of N increases. This shows that the approximate solution is efficient. It is observed that as we move along the domain, we get consistent accuracy.

Figure 4 show the h-curve obtained from the 14th-order HATM approximation solution of fractional-order Bloch equations in NMR flow (4.1). We still have freedom to choose the auxiliary parameter according to \hbar curve. From figure 4, the valid regions of convergence correspond to the line segments nearly parallel to the horizontal axis.

**Figure 4.** Plot of \hbar curve for different values of γ at $\omega=1$, $\hbar=-1$ and $T_1=1(ms)^q$

6. Conclusions

This paper develops an effective and new modification of the homotopy analysis method, which is a coupling of the homotopy analysis and Laplace transform method, and monitors its validity in a wide range of time fractional Bloch equations. The method is applied in a direct way without using linearization, discretization or restrictive assumptions. The method gives more realistic series solutions that converge very rapidly in time fractional Bloch equation equations. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of fractional calculus. It may be concluded that the FHATM methodology is very powerful and efficient in finding approximate solutions as well as analytical solutions of many physical problems.

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