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Numerical Solution of Fuzzy Arbitrary Order Predator-Prey Equations

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Abstract

This paper seeks to investigate the numerical solution of fuzzy arbitrary order predator-prey equations using the Homotopy Perturbation Method (HPM). Fuzziness in the initial conditions is taken to mean convex normalised fuzzy sets viz. triangular fuzzy number. Comparisons are made between crisp solution given by others and fuzzy solution in special cases. The results obtained are depicted in plots and tables to demonstrate the efficacy and powerfulness of the methodology.

Keywords: Fuzzy, arbitrary order, predator-prey equations, HPM, fuzzy number, triangular fuzzy number

AMS-MSC 2010 No.: 34A07, 34A08, 92D25

1. Introduction

A predator-prey model is the well-known system of differential equations which are used to describe the dynamical relationship between predator and prey. These differential equations are also known as the Lotka Volterra equations. Lotka in 1925 and Volterra in 1926 are the first who studied and applied this model. Many of their contributions can be found in the book "Elements of Physical Biology" Lotka (1925). This model consists of two differential equations, one for the food species prey $x(t)$ and the other for feeding species predator $y(t)$. In this model the parameters and initial conditions are usually considered as crisp (fixed). But in actual practice we may have some vague, insufficient and incomplete information about the variables and parameters due to error in observation, experiment etc., resulting in uncertainties. These uncertainties may be modelled through probabilistic, interval or fuzzy theory.

In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions. Probabilistic methods require sufficient data to get reliable results. In the recent decades, interval analysis and fuzzy theory are becoming powerful tools for the mathematical modelling of real life problems. In these approaches uncertain variables and parameters are represented by interval and fuzzy numbers.

The concept of fuzzy theory was first developed by Zadeh (1965). As discussed above, if the parameters and the initial conditions are described in imprecise forms then fuzzy theory can be applied. Fuzzy Differential Equations (FDEs) are increasingly used for modelling problems in science and engineering. Most of the science and engineering problems require the solution to FDEs. In real life application, it is too complicated to obtain the exact solution of FDEs. Chang and Zadeh (1972) first introduced the concept of a fuzzy derivative followed by Dubois and Prade (1982), who defined and used the extension principle in their approach. Fuzzy differential equation and fuzzy initial value problem (FIVP) were studied by Kaleva (1987, 1990), Seikkala (1987) and others. The numerical methods to solve fuzzy differential equations were introduced in Abbasbandy et al. (2002, 2004, 2008a, 2008b, Friedman et al. (1999), Ma et al. (1999). Allahviranloo et al. (2007, 2009) implemented the predictor corrector method to FDEs.

Shokri (2007) applied the modified Euler's model for first order fuzzy differential equations by an iterative method. Jafari et al. (2012) applied variational iteration method for solving the n-th order fuzzy differential equations to obtain analytical solution. Tapaswini and Chakraverty (2012, 2013a) introduced the Euler and improved Euler type methods to solve fuzzy initial value problems. Also HPM is successfully applied by Tapaswini and Chakraverty (2013b) for the solution of the n^{th} order fuzzy differential equations. Very recently Chakraverty and Tapaswini (2013) also proposed a new concept based on a double parametric form of fuzzy numbers for the numerical solution of the fuzzy beam equation.

In recent years, arbitrary-order (or fractional order) differential equations have been used to model physical and engineering problems. It is too difficult to obtain exact solution of fractional differential equations and so one turns to reliable and efficient numerical techniques to solve these equations. A great deal of important work has been reported regarding fractional calculus in the last few decades. Relating to this field, several books have also been written by different authors representing the scope and various aspects of fractional calculus such as in Samko et al. (1993), Miller and Ross (1993), Oldham et al. (1974), Kiryakova (1993), Podlubny (1993). These books also provide an extensive review on fractional derivatives and fractional differential equations which may help the reader understand the basic concept of fractional calculus. Regarding this concept many authors have developed various methods to solve fractional ordinary and partial differential equations and integral equations of physical systems.

As the governing differential equation of predator-prey model is of arbitrary-order, related literature are also reviewed and cited below for a better understanding of the present investigation. Fractional order predator-prey model (or fractional order Lotka-Volterra equations or fractional order parasite-host equations) is a model of population growth of predator and prey that was excellently explained in Petras (2011) and Xia and Chen (2001). Numerical solutions of the predator-prey model and fractional order rabies model have been done by Ahmed et al. (2007). Shakeri and Dehghan (2007) applied variational iteration method to solve this equation. El-Sayed et al. (2009) discussed the solution of the fractional order biological population model by the adomian method. Recently Das et al. (2009, 2011) applied HPM to solve this equation for various particular cases. Again Liu and Xin (2011) obtained numerical solution of nonlinear fractional partial differential equations arising in predator-prey system using HPM.

Since both fractional and fuzzy equations play an important role in real life physical problems an attempt has been made here to combine both for a more extensive analysis. Some recent useful contributions to the theory of fuzzy fractional differential equations may be seen in Agrawal et al. (2010), Arshad et al. (2011a, 2011b), Mohammed et al. (2011), Wang et al. (2011), Allahviranloo et al. (2012), Salahshour et al. (2012), Salah et al. (2012). Accordingly, the concept of fuzzy fractional differential equation was introduced by Agrawal et al. (2010). Arshad and Lupulescu (2011b) proved some results on the existence and uniqueness of solutions of fuzzy fractional differential equations based on the concept of fuzzy differential equations of fractional order introduced by Agrawal et al. (2010). Arshad and Lupulescu (2011a) investigated the fractional differential equation with fuzzy initial condition. Mohammed et al. (2011) applied the differential transform method for solving fuzzy fractional initial value problems.

Behera and Chakraverty (2013) have investigated uncertain impulse response of an imprecisely defined fractional order mechanical system. Boundary value problem for fuzzy fractional differential equations with finite delay are solved by Wang and Liu (2011). They established the existence of solution by the contraction mapping principle. Also Allahviranloo et al. (2012) studied the explicit solution of fractional differential equations with uncertainty. Salahshour et al. (2012) developed the Riemann-Liouville differentiability by using Hukuhara difference called Riemann-Liouville H-differentiability and solved fuzzy fractional differential equations by Laplace transform. Salah et al. (2012) used homotopy analysis transform method for obtaining a solution of the fuzzy fractional heat equation.

In the present analysis, the homotopy perturbation method (HPM) He (1999, 2000) is used to handle the numerical solution of fuzzy fractional predator-prey system with fuzzy initial conditions. In the following sections the preliminaries are first given. Next, the basic HPM is explained. Then implementations of HPM to fuzzy fractional predator-prey equations with fuzzy initial conditions are discussed. Lastly numerical examples and conclusions are incorporated.

2. Preliminaries

In this section we present some notations, definitions and preliminaries which are used in this paper.

Definition 2.1. Fuzzy Number

A fuzzy number $\tilde{U}$ is a convex normalised fuzzy set $\tilde{U}$ of real line R such that $\{\mu_{\tilde{U}}(x) : R \to [0,1], \forall\, x \in R\}$, where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and is piecewise continuous.

Definition 2.2. Triangular Fuzzy Number

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such that:

i. there exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set, and
ii. $\mu_{\tilde{U}}(x)$ is piecewise continuous.

We denote an arbitrary triangular fuzzy number as $\tilde{U} = (a, b, c)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ is then defined as follows,

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & x \ge c. \end{cases}$$

The triangular fuzzy number $\tilde{U} = (a, b, c)$ can be represented with an ordered pair of functions through the $\gamma-$cut approach viz. $[\underline{u}(\gamma), \overline{u}(\gamma)] = [(b-a)\gamma + a, -(c-b)\gamma + c]$, where $\gamma \in [0,1]$.

For all the above types of fuzzy numbers the left and right bounds of the fuzzy number satisfies the following requirements,

i. $\underline{u}(\gamma)$ is a bounded left continuous non-decreasing function over [0, 1],
ii. $\overline{u}(\gamma)$ is a bounded right continuous non-increasing function over [0, 1],
iii. $\underline{u}(\gamma) \le \overline{u}(\gamma), 0 \le \gamma \le 1.$

Definition 2.3. Fuzzy Arithmetic

For any two arbitrary fuzzy number $\tilde{x} = [\underline{x}(\gamma), \overline{x}(\gamma)]$, $\tilde{y} = [\underline{y}(\gamma), \overline{y}(\gamma)]$ and scalar k, the fuzzy arithmetic's are defined as follows:

i. $\tilde{x} = \tilde{y}$, if and only if $\underline{x}(\gamma) = \underline{y}(\gamma)$ and $\overline{x}(\gamma) = \overline{y}(\gamma)$.
ii. $\tilde{x} + \tilde{y} = [\underline{x}(\gamma) + \underline{y}(\gamma), \overline{x}(r) + \overline{y}(\gamma)]$.
iii. $\tilde{x} \times \tilde{y} = \begin{bmatrix} \min\left(\underline{x}(\gamma) \times \underline{y}(\gamma), \underline{x}(\gamma) \times \overline{y}(\gamma), \overline{x}(\gamma) \times \underline{y}(\gamma), \overline{x}(\gamma) \times \overline{y}(\gamma)\right), \\ \max\left(\underline{x}(\gamma) \times \underline{y}(\gamma), \underline{x}(\gamma) \times \overline{y}(\gamma), \overline{x}(\gamma) \times \underline{y}(\gamma), \overline{x}(\gamma) \times \overline{y}(\gamma)\right) \end{bmatrix}.$
iv. $k\tilde{x} = \begin{cases} [k\overline{x}(\gamma), k\underline{x}(\gamma)], k < 0 \\ [k\underline{x}(\gamma), k\overline{x}(\gamma)], k \ge 0 \end{cases}.$

Lemma 2.1. Bede (2008)

If $\tilde{u}(t) = (x(t), y(t), z(t))$ is a fuzzy triangular number valued function and if $\tilde{u}$ is Hukuhara differentiable, then $\tilde{u}' = (x', y', z')$.

By using this property we intend to solve the fuzzy initial value problem

$$\begin{cases} \tilde{x}' = f(t, \tilde{x}) \\ \tilde{x}(t_0) = \tilde{x}_0 \end{cases},$$

where $\tilde{x}_0 = (\underline{x}_0, x_0^c, \overline{x}_0) \in R, \tilde{x}(t) = (\underline{u}, u^c, \overline{u}) \in R$ and

$$f : [t_0, t_0 + a] \times R \to R, f(t, (\underline{u}, u^c, \overline{u})) = (\underline{f}(t, \underline{u}, u^c, \overline{u}), f^c(t, \underline{u}, u^c, \overline{u}), \overline{f}(t, \underline{u}, u^c, \overline{u})) .$$

We can translate this into the following system of ordinary differential equations as below:

$$\begin{cases} \underline{u} = \underline{f}(t, \underline{u}, u^c, \overline{u}), \\ u^c = f^c(t, \underline{u}, u^c, \overline{u}) \\ \overline{u} = \overline{f}(t, \underline{u}, u^c, \overline{u}) \\ \underline{u}(0) = \underline{x}_0, u^c(0) = x_0^c, \overline{u}(0) = \overline{x}_0. \end{cases}$$

3. Fuzzy Arbitrary Order Predator-Prey

Consider the fuzzy fractional order predator prey equation with triangular fuzzy initial conditions

$$\begin{aligned} D_t^{\alpha} \tilde{x}(t) &= a\tilde{x}(t) - b\tilde{x}(t)\tilde{y}(t) \\ D_t^{\beta} \tilde{y}(t) &= cx(t)\tilde{y}(t) - d\tilde{y}(t) \end{aligned} \tag{1}$$

with

$$\tilde{x}(0) = (a_1, b_1, c_1) \text{ and } \tilde{y}(0) = (a_2, b_2, c_2),$$

where D_t^{α} and D_t^{β} are the Caputo derivative of orders $0 < \alpha, \beta \le 1$. In this study we have considered a, b, c and d as functions of time t. So the related predator-prey system is now defined as,

$$\begin{aligned} D_t^{\alpha} \tilde{x}(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}(t)\tilde{y}(t) \\ D_t^{\beta} \tilde{y}(t) &= c(t)\tilde{x}(t)\tilde{y}(t) - d(t)\tilde{y}(t). \end{aligned} \tag{2}$$

4. Homotopy Perturbation Method [He (1999, 2000)]

To illustrate the basic idea of this method we consider the following nonlinear differential equation of the form

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \quad , \tag{4}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (3) can therefore be written as follows,

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{5}$$

By the homotopy technique, we construct a homotopy $U(r, p) : \Omega \times [0,1] \to R$, which satisfies:

$$H(U, p) = (1-p)[L(U) - L(v_0)] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{6}$$

or

$$H(U, p) = L(U) - L(v_0) + pL(v_0) + p[N(U) - f(r)] = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, v_0 is an initial approximation of Equation (3). Hence, it is obvious that

$$H(U,0) = L(U) - L(v_0) = 0, \tag{8}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{9}$$

and the changing process of p from 0 to 1, is just that of $U(r, p)$ from $v_0(r)$ to $u(r)$. In topology this is called deformation and $L(U) - L(v_0)$, $A(U) - f(r)$ are called homotopic. Applying the perturbation technique He (1999, 2000), due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of Equation (6) or (7) can as a power series in p as follows,

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots. \tag{10}$$

When $p \to 1$, Equation (6) or (7) corresponds to Equation (5). Then, Equation (10) becomes the approximate solution of Equation (5), i.e.,

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + \cdots. \tag{11}$$

The convergence of the series (11) has been proved in He (1999, 2000). In the following section, homotopy perturbation method is applied to solve fuzzy fractional order predator-prey Equations.

5. HPM Solution for Fuzzy Arbitrary Order Predator-Prey Equations

Let us consider the fuzzy arbitrary order predator-prey equations

$$D_t^{\alpha}\tilde{x}(t) = a(t)\tilde{x}(t) - b(t)\tilde{x}(t)\tilde{y}(t),$$
$$D_t^{\beta}\tilde{y}(t) = c(t)\tilde{x}(t)\tilde{y}(t) - d(t)\tilde{y}(t), \tag{12}$$

with initial conditions in the term of triangular fuzzy number viz.

$$\tilde{x}(0) = (a_1, b_1, c_1) \text{ and } \tilde{y}(0) = (a_2, b_2, c_2).$$

Using γ-cut, the triangular fuzzy initial conditions become

$$\tilde{x}(0;\gamma) = [\gamma(b_1 - a_1) + a_1, c_1 - \gamma(c_1 - b_1)] \text{ and } \tilde{y}(0;\gamma) = [\gamma(b_2 - a_2) + a_2, c_2 - \gamma(c_2 - b_2)], \quad 0 \le \gamma \le 1.$$

Here,

$$\tilde{x}(t;\gamma) = [\underline{x}(t;\gamma), \overline{x}(t;\gamma)] \text{ and } \tilde{y}(t;\gamma) = [\underline{y}(t;\gamma), \overline{y}(t;\gamma)]$$

are fuzzy functions of t. Using the Hukahara differentiability, fuzzy arbitrary order predator-prey equation i.e., Equation (12) may be reduced to set of ordinary differential equations. We can readily construct homotopy for Equations (12) as

$$(1-p)D_t^{\alpha}\tilde{x}(t;\gamma) + p[D_t^{\alpha}\tilde{x}(t;\gamma) - a(t)\tilde{x}(t;\gamma) + b(t)\tilde{x}(t;\gamma)\tilde{y}(t;\gamma)] = 0, \tag{13}$$

$$(1-p)D_t^{\beta}\tilde{y}(t;\gamma) + p[D_t^{\beta}\tilde{y}(t;\gamma) - c(t)\tilde{x}(t;\gamma)\tilde{y}(t;\gamma) - d(t)\tilde{y}(t;\gamma)] = 0. \tag{14}$$

One may try to obtain solutions of Equations (12) in the form

$$\tilde{x}(t;\gamma) = \tilde{x}_0(t;\gamma) + p\tilde{x}_1(t;\gamma) + p^2\tilde{x}_2(t;\gamma) + \cdots, \tag{15}$$

$$\tilde{y}(t;\gamma) = \tilde{y}_0(t;\gamma) + p\tilde{y}_1(t;\gamma) + p^2\tilde{y}_2(t;\gamma) + \cdots, \tag{16}$$

where

$$\tilde{x}_i(t;\gamma), i = 0, 1, 2, \cdots \text{ and } \tilde{y}_i(t;\gamma), i = 0, 1, 2, \cdots$$

are functions yet to be determined. Substituting Equations (15) and (16) into Equations (13) and (14) respectively and equating the terms with identical powers of p we have

$$p^0 : \begin{cases} D_t^{\alpha}\tilde{x}_0(t;\gamma) = 0, \\ D_t^{\beta}\tilde{y}_0(t;\gamma) = 0, \end{cases}$$

$$p^1 : \begin{cases} D_t^{\alpha}\tilde{x}_1(t;\gamma) = a(t)\tilde{x}_0(t;\gamma) - b(t)\tilde{x}_0(t;\gamma)\tilde{y}_0(t;\gamma), \\ D_t^{\beta}\tilde{y}_1(t;\gamma) = c(t)\tilde{x}_0(t;\gamma)\tilde{y}_0(t;\gamma) - d(t)\tilde{y}_0(t;\gamma), \end{cases} \tag{17}$$

$$p^2:\begin{cases} D_t^{\alpha}\tilde{x}_2(t;\gamma)=a(t)\tilde{x}_1(t;\gamma)-b(t)\tilde{x}_1(t;\gamma)\tilde{y}_0(t;\gamma)-b(t)\tilde{x}_0(t;\gamma)\tilde{y}_1(t;\gamma), \\ D_t^{\beta}\tilde{y}_2(t;\gamma)=c(t)\tilde{x}_0(t;\gamma)\tilde{y}_1(t;\gamma)+c(t)\tilde{x}_1(t;\gamma)\tilde{y}_0(t;\gamma)-d(t)\tilde{y}_1(t;\gamma), \end{cases}$$

$$p^3:\begin{cases} D_t^{\alpha}\tilde{x}_3(t;\gamma)=a(t)\tilde{x}_2(t;\gamma)-b(t)\tilde{x}_2(t;\gamma)\tilde{y}_0(t)-b(t)\tilde{x}_1(t)\tilde{y}_1(t;\gamma)-b(t)\tilde{x}_0(t;\gamma)\tilde{y}_2(t;\gamma), \\ D_t^{\beta}\tilde{y}_3(t;\gamma)=c(t)\tilde{x}_2(t;\gamma)\tilde{y}_0(t;\gamma)+c(t)\tilde{x}_1(t;\gamma)\tilde{y}_1(t;\gamma)+c(t)\tilde{x}_0(t;\gamma)\tilde{y}_2(t;\gamma)-d(t)\tilde{y}_2(t;\gamma), \end{cases}$$

$$\vdots$$

and so on.

The method is based on applying the operators J_t^{α} and J_t^{β} (the inverse operator of Caputo derivatives D_t^{α} and D_t^{β} respectively) on both sides of each Equations (17) and then we may write the approximate solution bounds as

$$\tilde{x}(t;\gamma)=\left[\sum_{n=0}^{\infty}\underline{x}_n(t;\gamma),\sum_{n=0}^{\infty}\underline{x}_n(t;\gamma)\right] \text{ and } \tilde{y}(t;\gamma)=\left[\sum_{n=0}^{\infty}\underline{y}_n(t;\gamma),\sum_{n=0}^{\infty}\overline{y}_n(t;\gamma)\right].$$

An approximation to the solutions would be achieved by computing few terms say k as

$$\begin{aligned} &\underline{x}(t;\gamma)=\lim_{k\to\infty}\underline{\xi}_k(t;\gamma) \text{ and } \underline{y}(t;\gamma)=\lim_{k\to\infty}\underline{\varphi}_k(t;\gamma), \\ &\overline{x}(t;\gamma)=\lim_{k\to\infty}\overline{\xi}_k(t;\gamma) \text{ and } \overline{y}(t;\gamma)=\lim_{k\to\infty}\overline{\varphi}_k(t;\gamma), \end{aligned} \tag{18}$$

where

$$\underline{\xi}_k(t;\gamma)=\sum_{n=0}^{k}\underline{x}_n(t;\gamma),\ \underline{\varphi}_k(t;\gamma)=\sum_{n=0}^{k}\underline{y}_n(t;\gamma),$$

$$\overline{\xi}_k(t;\gamma)=\sum_{n=0}^{k}\overline{x}_n(t;\gamma) \text{ and } \overline{\varphi}_k(t;\gamma)=\sum_{n=0}^{k}\overline{y}_n(t;\gamma).$$

6. Particular Cases

Case 1. Let us consider $a(t)=t, b(t)=1, c(t)=1, d(t)=t$ and initial conditions in term of triangular fuzzy numbers viz. $\tilde{x}(0)=(1.2,1.3,1.4)$ and $\tilde{y}(0)=(0.5,0.6,0.7)$.

Through the γ-cut approach initial conditions
become $\tilde{x}(0;\gamma)=[0.1\gamma+1.2,\ 1.4-0.1\gamma]=[\delta_1,\delta_2]$ and
$\tilde{y}(0;\gamma)=[0.1\gamma+0.5,\ 0.7-0.1\gamma]=[\eta_1,\eta_2]$. Solving Equation (17) we have

$$\underline{x}_0(t;\gamma)=\delta_1,$$

$$\overline{x}_0(t;\gamma)=\delta_2,$$

$$\underline{y}_0(t;\gamma) = \eta_1,$$

$$\bar{y}_0(t;\gamma) = \eta_2,$$

$$\underline{x}_1(t;\gamma) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\delta_1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\delta_2\eta_2,$$

$$\bar{x}_1(t;\gamma) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\delta_2 - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\delta_1\eta_1,$$

$$\underline{y}_1(t;\gamma) = \frac{t^{\beta}}{\Gamma(\beta+1)}\delta_1\eta_1 - \frac{t^{\beta+1}}{\Gamma(\beta+2)}\eta_2,$$

$$\bar{y}_1(t;\gamma) = \frac{t^{\beta}}{\Gamma(\beta+1)}\delta_2\eta_2 - \frac{t^{\beta+1}}{\Gamma(\beta+2)}\eta_1,$$

$$\underline{x}_2(t;\gamma) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\delta_1\eta_1\eta_2 - \frac{(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)}\delta_2\eta_2 + \frac{(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\delta_1 - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\delta_2{}^2\eta_2 + \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\eta_1\delta_2,$$

$$\bar{x}_2(t;\gamma) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2\eta_2\eta_1 - \frac{(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)}\delta_1\eta_1 + \frac{(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\delta_2 - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\delta_1{}^2\eta_1 + \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\eta_2\delta_1,$$

$$\underline{y}_2(t;\gamma) = \frac{t^{2\beta}}{\Gamma(\beta+1)}\delta_1{}^2\eta_1 - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\eta_2\{\delta_1 + (\beta+1)\delta_2\} + \frac{(\beta+2)t^{2\beta+2}}{\Gamma(2\beta+3)}\eta_1 - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\delta_2\eta_2\eta_1 + \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\delta_1\eta_1,$$

$$\bar{y}_2(t;\gamma) = \frac{t^{2\beta}}{\Gamma(\beta+1)}\delta_2{}^2\eta_2 - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\eta_1\{\delta_2 + (\beta+1)\delta_1\} + \frac{(\beta+2)t^{2\beta+2}}{\Gamma(2\beta+3)}\eta_2 - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\delta_1\eta_1\eta_2 + \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\delta_2\eta_1,$$

$$\underline{x}_3(t;\gamma) = -\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\eta_2{}^2\delta_2\eta_1 + \frac{3(\gamma+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)}\eta_1\delta_1\eta_2 - \frac{(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)}\delta_2\eta_2 + \frac{(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\delta_1 - \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\delta_2{}^3\eta_2 + \frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\delta_2\eta_1\left\{\delta_2 + (\beta+1)\delta_1\right\}$$

$$-\frac{(\beta+2)t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\delta_2\eta_2+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\eta_2\eta_1\delta_1\left\{\delta_1+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_2+\delta_2\right\}$$
$$-\frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\xi_1+\frac{t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\eta_1\delta_2\psi_1,$$

where

$$\xi_1=(\alpha+\beta+2)\delta_2{}^2\eta_2+\eta_2{}^2\delta_1+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+1)}(\delta_2{}^2\eta_2+\delta_1\eta_1{}^2)$$

and

$$\psi_1=(\alpha+\beta+2)+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)}.$$

$$\bar{x}_3(t;\gamma)=-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\eta_1{}^2\delta_1\eta_2+\frac{3(\gamma+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)}\eta_2\delta_2\eta_1-\frac{(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)}\delta_1\eta_1$$
$$+\frac{(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\delta_2\,\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\delta_1{}^3\eta_1+\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\delta_1\eta_2\,\{\delta_1+(\beta+1)\delta_2\}$$

$$-\frac{(\beta+2)t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\delta_1\eta_1+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\eta_1\eta_2\delta_2\left\{\delta_2+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_1+\delta_1\right\}$$
$$-\frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\xi_2+\frac{t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\eta_2\delta_1\psi_2,$$

where

$$\xi_2=(\alpha+\beta+2)\delta_1{}^2\eta_1+\eta_1{}^2\delta_2+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+1)}\left(\delta_1{}^2\eta_1+\delta_2\eta_2{}^2\right)$$

and

$$\psi_2=(\alpha+\beta+2)+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)}.$$

$$\underline{y}_3(t;\gamma)=\frac{t^{3\beta}}{\Gamma(3\beta+1)}\delta_1{}^3\eta_1-\frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\left\{(2\beta+1)\delta_2{}^2\eta_2+\delta_1{}^2\eta_2+(\beta+1)\delta_2\eta_2\delta_1\right\}$$
$$+\frac{t^{3\beta+2}}{\Gamma(3\beta+3)}\eta_1\left\{(\beta+2)\delta_1+2(\beta+1)\delta_2+2(\beta+1)^2\delta_1\right\}+\frac{(\beta+2)(2\alpha+3)t^{3\beta+3}}{\Gamma(3\beta+4)}\eta_2$$

$$+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\delta_1\eta_1{}^2\eta_2-\frac{(\alpha+2)t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\delta_2\eta_1\eta_2+\frac{(\alpha+2)t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\delta_1\eta_1$$
$$-\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\eta_1\delta_2\eta_2\left\{\delta_1+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_1+\delta_2\right\}+\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\sigma_1$$

$$-\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\eta_2\rho_1,$$

where

$$\sigma_1=\delta_1{}^2\eta_1\left(1+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+1)}\right)+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+2)}\delta_2\eta_2{}^2+\eta_1{}^2\delta_2+(\alpha+\beta+1)\delta_1\eta_2\eta_1$$

and

$$\rho_1=(\alpha+\beta+2)\delta_2+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)}\delta_1\eta_2.$$

$$\begin{aligned}\overline{y}_3(t;\gamma)=&\frac{t^{3\beta}}{\Gamma(3\beta+1)}\delta_2{}^3\eta_2-\frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\left\{(2\beta+1)\delta_1{}^2\eta_1+\delta_2{}^2\eta_1+(\beta+1)\delta_1\eta_1\delta_2\right\}\\&+\frac{t^{3\beta+2}}{\Gamma(3\beta+3)}\eta_2\left\{(\beta+2)\delta_2+2(\beta+1)\delta_1+2(\beta+1)^2\delta_2\right\}+\frac{(\beta+2)(2\alpha+3)t^{3\beta+3}}{\Gamma(3\beta+4)}\eta_1\\&+\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\delta_2\eta_2{}^2\eta_1-\frac{(\alpha+2)t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\delta_1\eta_1\eta_2+\frac{(\alpha+2)t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\delta_2\eta_2\\&-\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)}\eta_2\delta_1\eta_1\left\{\delta_2+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_2+\delta_1\right\}+\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\sigma_2\\&-\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\eta_1\rho_2.\end{aligned}$$

where

$$\sigma_2=\delta_2{}^2\eta_2\left(1+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta+1)}\right)+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+2)}\delta_1\eta_1{}^2+\eta_2{}^2\delta_1+(\alpha+\beta+1)\delta_1\eta_2\eta_1$$

and

$$\rho_2=(\alpha+\beta+2)\delta_1+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+2)}\delta_2\eta_1.$$

In a similar manner the rest of the components can be obtained. Further we get the approximate solution of $\tilde{x}(t;\gamma)$ and $\tilde{y}(t;\gamma)$ from Equation (18).

Case 2 Next consider $a(t)=1, b(t)=t, c(t)=t, d(t)=1$ with the same initial conditions as consider in Case 1.

Again solving Equation (17) we have

$$\underline{x}_0(t;\gamma)=\delta_1,$$

$$\bar{x}_0(t;\gamma) = \delta_2,$$

$$\underline{y}_0(t;\gamma) = \eta_1,$$

$$\bar{y}_0(t;\gamma) = \eta_2,$$

$$\underline{x}_1(t;\gamma) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\delta_1 - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\delta_2\eta_2$$

$$\bar{x}_1(t;\gamma) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}\delta_2 - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\delta_1\eta_1$$

$$\underline{y}_1(t;\gamma) = \frac{t^{\beta+1}}{\Gamma(\beta+2)}\delta_1\eta_1 - \frac{t^{\beta}}{\Gamma(\beta+1)}\eta_2,$$

$$\bar{y}_1(t;\gamma) = \frac{t^{\beta+1}}{\Gamma(\beta+2)}\delta_2\eta_2 - \frac{t^{\beta}}{\Gamma(\beta+1)}\eta_1,$$

$$\underline{x}_2(t;\gamma) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\delta_1 - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}(\alpha+2)\delta_2\eta_2 + \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}(\alpha+2)\delta_1\eta_1\eta_2$$
$$+\frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\beta+1)\delta_2\eta_1 - \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\beta+2){\delta_2}^2\eta_2,$$

$$\bar{x}_2(t;\gamma) = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\delta_2 - \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}(\alpha+2)\delta_1\eta_1 + \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}(\alpha+2)\delta_2\eta_1\eta_2$$
$$+\frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\beta+1)\delta_1\eta_2 - \frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\beta+2){\delta_1}^2\eta_1),$$

$$\underline{y}_2(t;\gamma) = \frac{t^{2\beta}}{\Gamma(2\beta+1)}\eta_2 - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\delta_1\{(\beta+1)\eta_2+\eta_1\} + \frac{t^{2\beta+2}}{\Gamma(2\beta+3)}(\beta+2){\delta_1}^2\eta_1$$
$$+\frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\alpha+1)\delta_1\eta_1 - \frac{t^{\alpha+\beta+2}}{\Gamma(\alpha+\beta+3)}(\alpha+2)\delta_2\eta_2\eta_1,$$

$$\bar{y}_2(t;\gamma) = \frac{t^{2\beta}}{\Gamma(2\beta+1)}\eta_1 - \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\delta_2\{(\beta+1)\eta_1+\eta_2\} + \frac{t^{2\beta+2}}{\Gamma(2\beta+3)}(\beta+2){\delta_2}^2\eta_2$$
$$+\frac{t^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}(\alpha+1)\delta_2\eta_2 - \frac{t^{\alpha+\beta+2}}{\Gamma(\alpha+\beta+3)}(\alpha+2)\delta_1\eta_2\eta_1,$$

$$\underline{x}_3(t;\gamma)=\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\delta_1-\frac{3t^{3\alpha+1}}{\Gamma(3\alpha+2)}(\alpha+1)\delta_2\eta_2+(\alpha+2)(2\alpha+3)\eta_1\left(\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\delta_1\eta_2-\frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)}\delta_2{\eta_2}^2\right)$$
$$-\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}(2\beta+1)\eta_1\delta_2+\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}{\delta_2}^2\{(\beta+1)\eta_1+\eta_2\}$$
$$-\frac{t^{\alpha+2\beta+3}}{\Gamma(\alpha+2\beta+4)}(2\beta+3)(\beta+2){\delta_2}^3\eta_2+\frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\delta_2\eta_1\left\{(\beta+1)+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}$$
$$-\ \frac{t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\xi_3+\frac{t^{2\alpha+\beta+3}}{\Gamma(2\alpha+\beta+3)}\psi_3,$$

where

$$\xi_3={\delta_2}^2\eta_2\left((\beta+2)+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1)\Gamma(\beta+2)}+(\alpha+1)(\alpha+\beta+2)\right)$$
$$+\delta_1\left((\beta+1)(\alpha+\beta+2){\eta_2}^2+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)}{\eta_1}^2\right)$$

and

$$\psi_3=(\alpha+\beta+3)(\beta+2){\delta_1}^2\eta_1\eta_2+\delta_2\eta_2\delta_1\eta_1\left(\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\alpha+2)\Gamma(\beta+2)}+(\alpha+2)(\alpha+\beta+3)\right).$$

$$\overline{x}_3(t;\gamma)=\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\delta_2-\frac{3t^{3\alpha+1}}{\Gamma(3\alpha+2)}(\alpha+1)\delta_1\eta_1+(\alpha+2)(2\alpha+3)\eta_2\left(\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\delta_2\eta_1-\frac{t^{3\alpha+3}}{\Gamma(3\alpha+4)}\delta_1{\eta_1}^2\right)$$
$$-\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}(2\beta+1)\eta_2\delta_1+\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}{\delta_1}^2\{(\beta+1)\eta_2+\eta_1\}$$
$$-\frac{t^{\alpha+2\beta+3}}{\Gamma(\alpha+2\beta+4)}(2\beta+3)(\beta+2){\delta_1}^3\eta_1+\frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}\delta_1\eta_2\left\{(\beta+1)+\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}$$
$$-\ \frac{t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}\xi_4+\frac{t^{2\alpha+\beta+3}}{\Gamma(2\alpha+\beta+3)}\psi_4,$$

where

$$\xi_4={\delta_1}^2\eta_1\left((\beta+2)+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1)\Gamma(\beta+2)}+(\alpha+1)(\alpha+\beta+2)\right)$$
$$+\delta_2\left((\beta+1)(\alpha+\beta+2){\eta_1}^2+\frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)}{\eta_2}^2\right)$$

and

$$\psi_4=(\alpha+\beta+3)(\beta+2){\delta_2}^2\eta_2\eta_1+\delta_1\eta_1\delta_2\eta_2\left(\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\alpha+2)\Gamma(\beta+2)}+(\alpha+2)(\alpha+\beta+3)\right).$$

$$\underline{y}_3(t;\gamma) = -\frac{t^{3\beta}}{\Gamma(3\beta+1)}\eta_1 + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\{(2\beta+1)\eta_2\delta_1\delta_2((\beta+1)\eta_1+\eta_2)\}$$
$$-\frac{t^{3\beta+2}}{\Gamma(3\beta+3)}\{(2\beta+2)\delta_1{}^2((\beta+1)\eta_2+\eta_1)+(\beta+2)\delta_2{}^2\eta_2\}$$
$$+\frac{t^{3\beta+3}}{\Gamma(3\beta+4)}(2\beta+3)(\beta+2)\delta_1{}^3\eta_1 + \frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}(2\alpha+1)\delta_1\eta_1$$
$$-\frac{2t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}(\alpha+1)(\alpha+2)\delta_2\eta_2\eta_1 + \frac{t^{2\alpha+\beta+3}}{\Gamma(2\alpha+\beta+4)}(2\alpha+3)(\alpha+2)\,\delta_1\eta_1{}^2\eta_2$$
$$-\frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\eta_2\left\{\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_1+(\alpha+1)\delta_2\right\}$$
$$+\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\sigma_3 - \frac{t^{\alpha+2\beta+3}}{\Gamma(\alpha+2\beta+4)}\rho_3,$$

where

$$\sigma_3 = (\alpha+\beta+2)(\beta+1)\delta_2\eta_1{}^2 + \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1)\Gamma(\beta+2)}\delta_1{}^2\eta_1 + \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)}\delta_2\eta_2{}^2 +$$
$$(\alpha+\beta+2)(\alpha+1)\delta_1{}^2\eta_1 + (\alpha+2)\delta_1\eta_1\eta_2,$$

and

$$\rho_3 = (\alpha+\beta+3)(\beta+2)\delta_2{}^2\eta_1\eta_2 + \delta_2\delta_1\eta_1\eta_2\left(\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\alpha+2)\Gamma(\beta+2)}+(\alpha+\beta+3)(\alpha+2)\right).$$

$$\bar{y}_3(t;\gamma) = -\frac{t^{3\beta}}{\Gamma(3\beta+1)}\eta_2 + \frac{t^{3\beta+1}}{\Gamma(3\beta+2)}\{(2\beta+1)\eta_1\delta_1\delta_2((\beta+1)\eta_2+\eta_1)\}$$
$$-\frac{t^{3\beta+2}}{\Gamma(3\beta+3)}\{(2\beta+2)\delta_2{}^2((\beta+1)\eta_1+\eta_2)+(\beta+2)\delta_1{}^2\eta_1\} + \frac{t^{3\beta+3}}{\Gamma(3\beta+4)}(2\beta+3)(\beta+2)\delta_2{}^3\eta_2$$
$$+\frac{t^{2\alpha+\beta+1}}{\Gamma(2\alpha+\beta+2)}(2\alpha+1)\delta_2\eta_2 - \frac{2t^{2\alpha+\beta+2}}{\Gamma(2\alpha+\beta+3)}(\alpha+1)(\alpha+2)\delta_1\eta_1\eta_2$$
$$+\frac{t^{2\alpha+\beta+3}}{\Gamma(2\alpha+\beta+4)}(2\alpha+3)(\alpha+2)\,\delta_2\eta_2{}^2\eta_1 - \frac{t^{\alpha+2\beta+1}}{\Gamma(\alpha+2\beta+2)}\eta_1\left\{\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\delta_2+(\alpha+1)\delta_1\right\}$$
$$+\frac{t^{\alpha+2\beta+2}}{\Gamma(\alpha+2\beta+3)}\sigma_4 - \frac{t^{\alpha+2\beta+3}}{\Gamma(\alpha+2\beta+4)}\rho_4.$$

where

$$\sigma_4 = (\alpha+\beta+2)(\beta+1)\delta_1\eta_2{}^2 + \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+1)\Gamma(\beta+2)}\delta_2{}^2\eta_2 + \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)}\delta_1\eta_1{}^2$$
$$+(\alpha+\beta+2)(\alpha+1)\delta_2{}^2\eta_2 + (\alpha+2)\delta_2\eta_1\eta_2,$$

and

$$\rho_4 = (\alpha+\beta+3)(\beta+2)\delta_1{}^2\eta_1\eta_2 + \delta_2\delta_1\eta_1\eta_2\left(\frac{\Gamma(\alpha+\beta+4)}{\Gamma(\alpha+2)\Gamma(\beta+2)}+(\alpha+\beta+3)(\alpha+2)\right).$$

We may obtain rest of the components as discussed above. Substituting these in Equation (18) we may get the approximate solution of $\tilde{x}(t;\gamma)$ and $\tilde{y}(t;\gamma)$.

In the special case one may see that for $\gamma = 1$, the fuzzy initial conditions convert to crisp initial conditions and the solutions obtained by the applied method exactly agree with the solutions of Das et al. (2011).

7. Numerical Results and Discussions

In this section, we present numerical solution of fuzzy arbitrary order predator-prey equations using HPM. It is a gigantic task to include here all the results with respect to various parameters involved in the corresponding equation. As such, few representative results as per the above cases are reported.

In the first case a fuzzy fractional order predator-prey equation for $a(t) = t, b(t) = 1, c(t) = 1$ and $d(t) = t$ with fuzzy initial condition as discussed earlier is solved by HPM. The solutions are given in Tables 1 to 4 for various values of α, β, γ and t. Numerical results are depicted in Tables 1 to 4 respectively by varying t from 0 to 0.6 and keeping γ constant for different types of fractional order viz. $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$ and for integer order $\alpha = \beta = 1$. Crisp results are also included in these tables.

Next by taking $t = 0.5$ as constant and varying the value of γ from 0 to 1, the obtained fuzzy results are depicted in Figures 1(i) to 1(iv) for various order derivatives viz. $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$ and $\alpha = \beta = 1$ respectively. Varying both t and γ from 0 to 0.6 and 0 to 1 for $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$ and $\alpha = \beta = 1$ results are depicted respectively in Figures 2 to 5.

Now for $\gamma = 0.5$ and 1, the interval solution of case 1 with various order derivatives viz. $\alpha = \beta = 1/3$, $\alpha = \beta = 1/2$, $\alpha = \beta = 2/3$ and $\alpha = \beta = 1$ are computed and those are shown in Figures 6 to 9. Similar types of studies have been done using the same parameters as Case 1 with $a(t) = 1, b(t) = t, c(t) = t$ and $d(t) = 1$ for Case 2. Corresponding results are given in Tables 5 to 8 and depicted in Figures 10 to 18. It is interesting to note that for both the cases lower and upper bounds of the fuzzy solutions are the same for $\gamma = 1$, which approximately matches the crisp solution of Das et al. (2011). Also from the Tables 1 to 8 it can be concluded that the crisp solution lies in between the lower and upper bounds of the fuzzy and interval solution.

From Figures 6 to 9 for Case 1, one can see that the prey population decreases and predator population increases with time t. On the other hand for Case 2 (Figures15 to 18), the first prey population increases and after that it decreases with the increase in t. In this case, the predator increases throughout as seen in Figures 15 to 18. We see that in both the cases it takes more time for meeting predator-prey populations as the fractional time derivative increases and finally takes the maximum time for $\alpha = \beta = 1$. It may also be seen from the figures that Case 1 takes the least time to meet predator-prey populations whereas Case 2 takes the maximum time.

Table 1. Fuzzy solution for $\alpha = \beta = 1/3$ of Case 1

$t \rightarrow$		0	0.2	0.4	0.6
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[0.0865, 1.2200]	[-0.5433, 1.4637]	[-1.2409, 1.8615]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.6289, 1.7624]	[0.2490, 2.2562]	[-0.3048, 2.7977]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[0.4116, 0.9784]	[0.0426, 1.0463]	[-0.3299, 1.2217]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.8738, 1.4406]	[0.6792, 1.6829]	[0.3638, 1.9154]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[0.7089, 0.7089]	[0.5725, 0.5725]	[0.4911, 0.4911]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[1.1443, 1.1443]	[1.1572, 1.1572]	[1.1040, 1.1040]
Das et al. (2011)	$x(t)$	1.3	0.7089	0.5725	0.4911
	$y(t)$	0.6	1.1443	1.1572	1.1040

Table 2. Fuzzy solution for $\alpha = \beta = 1/2$ of Case 1

$t \rightarrow$		0	0.2	0.4	0.6
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[0.5157, 1.2006]	[0.0324, 1.3106]	[-0.5382, 1.5688]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.6926, 1.3776]	[0.5133, 1.7916]	[0.1336, 2.2407]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[0.7057, 1.0482]	[0.3983, 1.0375]	[0.0736, 1.1274]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.8463, 1.1888]	[0.7927, 1.4319]	[0.5917. 1.6454]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[0.8832, 0.8832]	[0.7333, 0.7333]	[0.6289, 0.6289]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[1.0117, 1.0117]	[1.0989, 1.0989]	[1.0957, 1.0957]
Das et al. (2011)	$x(t)$	1.3	0.883256	0.733384	0.628987
	$y(t)$	0.6	1.01172	1.09896	1.09571

Table 3. Fuzzy solution $\alpha = \beta = 2/3$ of Case 1

$t \rightarrow$		0	0.2	0.4	0.6
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2,1.4]	[0.7630, 1.2300]	[0.4063, 1.2525]	[-0.0247, 1.3982]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.6785, 1.1455]	[0.6337, 1.4798]	[0.4174, 1.8404]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[0.8884, 1.122]	[0.6426, 1.0657]	[0.3816, 1.0932]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.7869, 1.0204]	[0.8233, 1.2463]	[0.7313, 1.4429]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[1.0081, 1.0081]	[0.8624, 0.8624]	[0.7544, 0.7544]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.9009, 0.9009]	[1.0275, 1.0275]	[1.0731, 1.0731]
Das et al. (2011)	$x(t)$	1.3	1.00815	0.862477	0.75442
	$y(t)$	0.6	0.900967	1.02754	1.07319

Table 4. Fuzzy solution for $\alpha = \beta = 1$ of Case 1

$t \rightarrow$		0	0.2	0.4	0.6
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[1.0058, 1.3041]	[0.8007, 1.2612]	[0.5579, 1.2818]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.6057, 0.9041]	[0.6634, 1.1239]	[0.6425, 1.3665]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[1.0827, 1.2319]	[0.9233, 1.1536]	[0.7562, 1.1182]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.6780, 0.8272]	[0.7715, 1.0018]	[0.8084, 1.1704]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[1.1581, 1.1581]	[1.0409, 1.0409]	[0.9430, 0.9430]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.7518, 0.7518]	[0.8843, 0.8843]	[0.9844, 0.9844]
Das et al. (2011)	$x(t)$	1.3	1.1581	1.0409	0.9430
	$y(t)$	0.6	0.7518	0.8843	0.9844

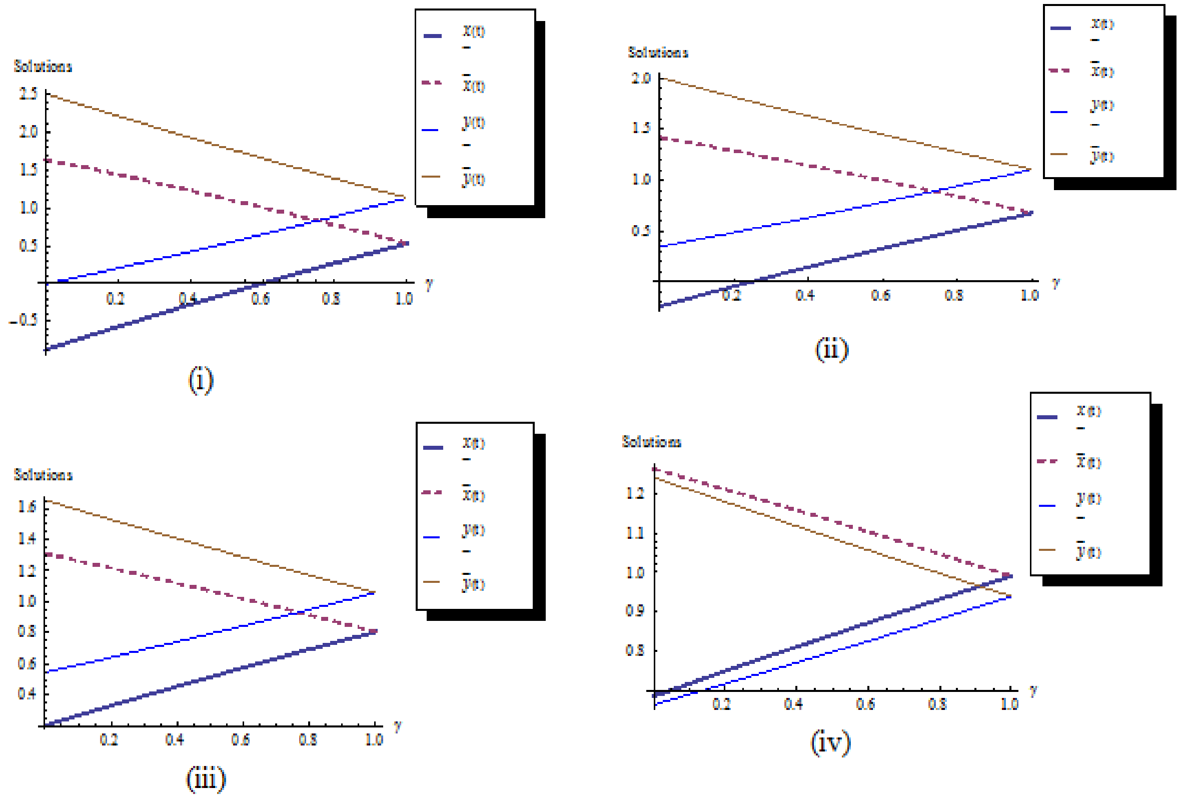

Figure 1. Fuzzy solution of fractional predator-prey equations of Case 1 for $\tilde{x}(t)$ and $\tilde{y}(t)$ at $t = 0.5$ and $\gamma \in [0,1]$ (i) $\alpha = \beta = 1/3$ (ii) $\alpha = \beta = 1/2$ (iii) $\alpha = \beta = 2/3$ (iv) $\alpha = \beta = 1$

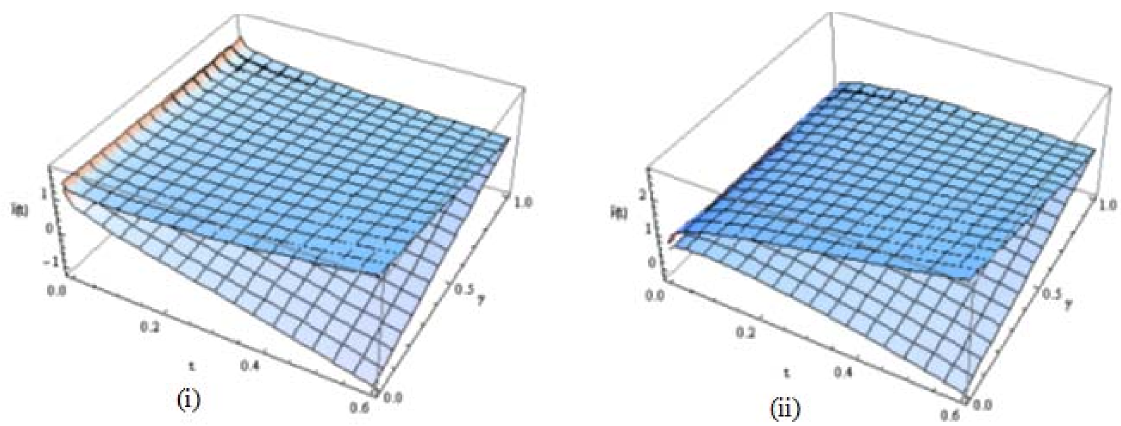

Figure 2. Fuzzy solution of fractional order predator-prey equations of Case 1 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 0.6]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1/3$

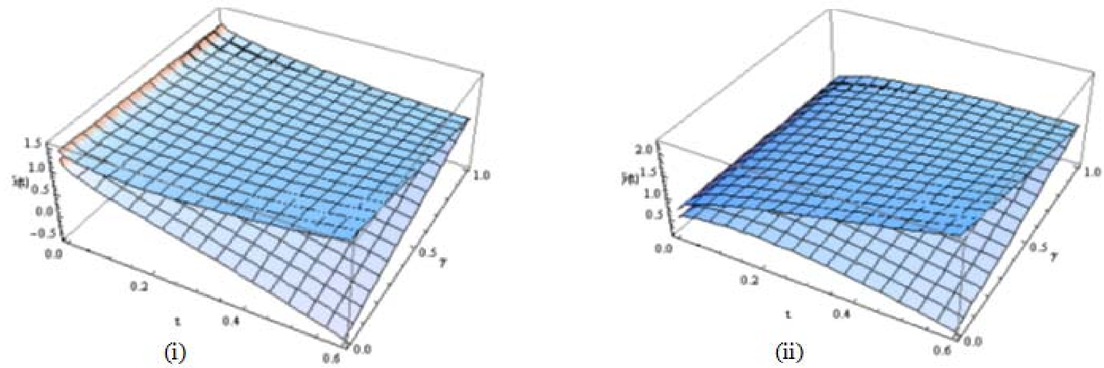

Figure 3. Fuzzy solution of fractional order predator-prey equations of Case 1 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 0.6]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1/2$

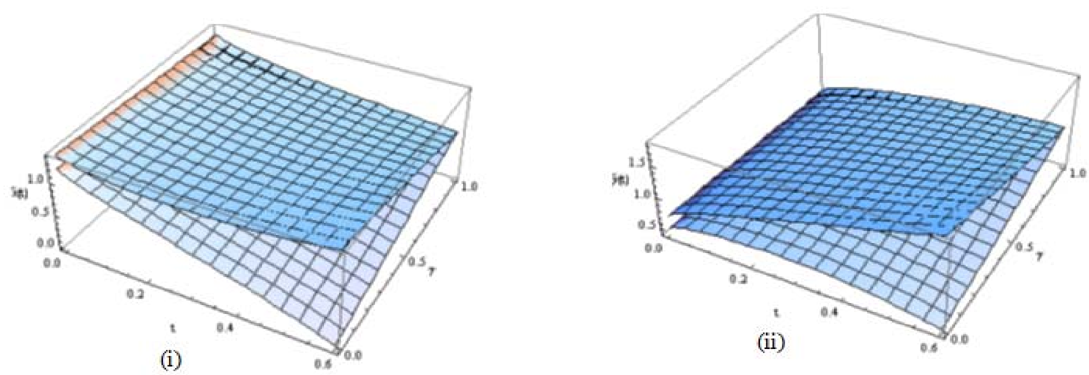

Figure 4. Fuzzy solution of fractional order predator-prey equations of Case 1 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 0.6]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 2/3$

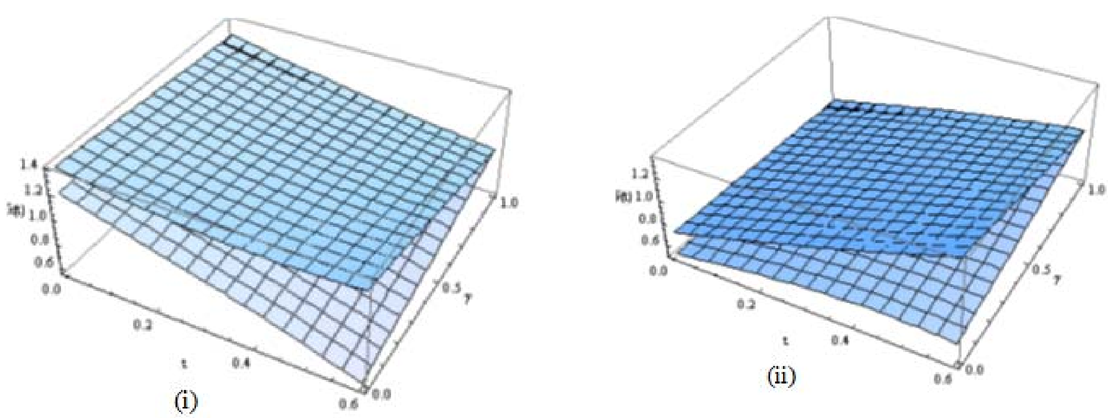

Figure 5. Fuzzy solution of fractional order predator-prey equations of Case 1 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 0.6]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1$

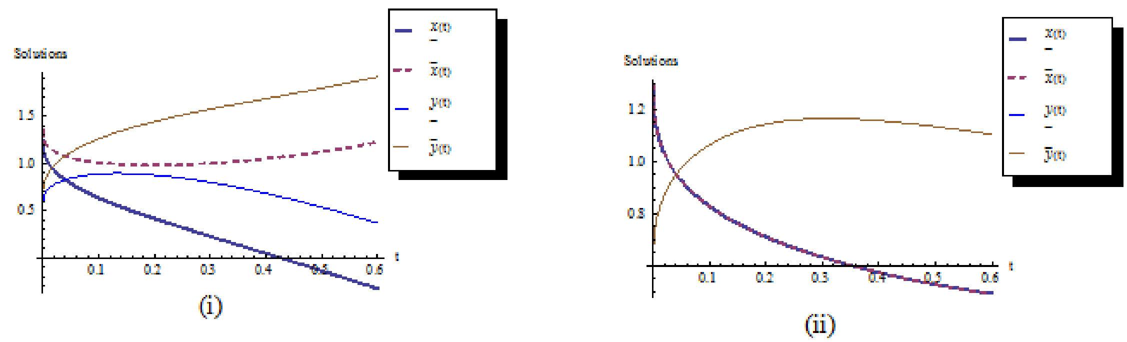

Figure 6. Interval solution of fractional predator-prey equations of Case 1 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1/3$ at (i) $\gamma = 0.5$ and (ii) $\gamma = 1$

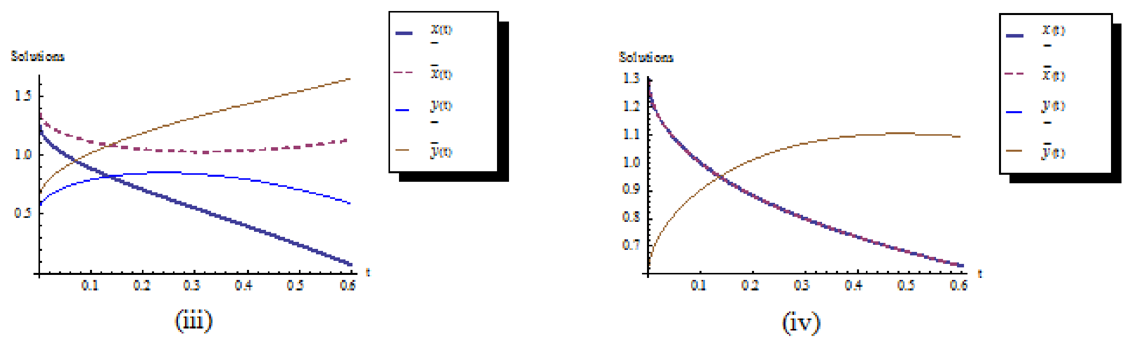

Figure 7. Interval solution of fractional predator-prey equations of Case 1 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1/2$ at (ii) $\gamma = 0.5$ and (iv) $\gamma = 1$

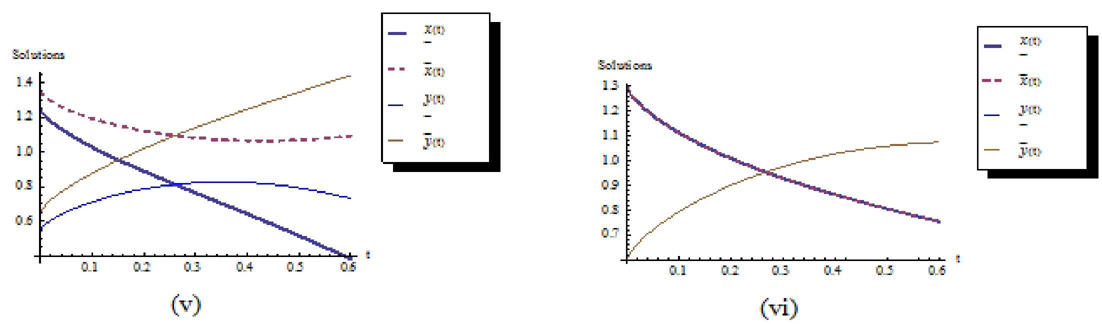

Figure 8. Interval solution of fractional predator-prey equations of Case 1 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 2/3$ at (v) $\gamma = 0.5$ and (vi) $\gamma = 1$

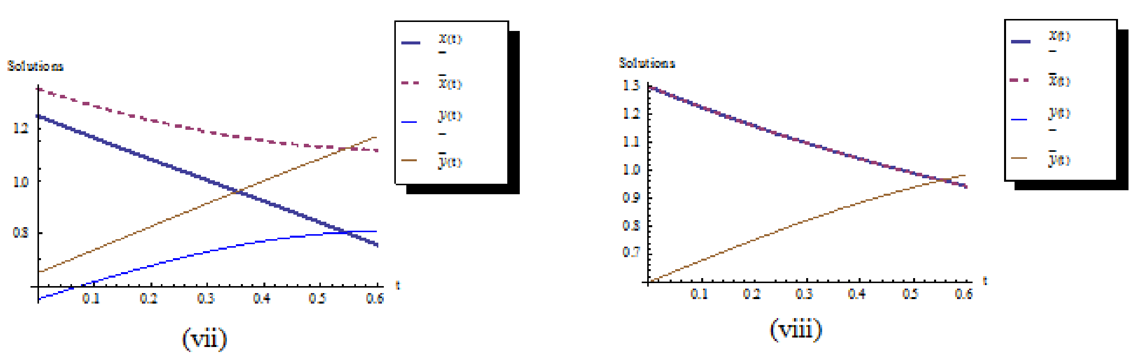

Figure 9. Interval solution of fractional predator-prey equations of Case 1 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1$ at (vii) $\gamma = 0.5$ and (viii) $\gamma = 1$

Table 5. Fuzzy solution for $\alpha = \beta = 1/3$ of Case 2

$t \rightarrow$		0	0.6	1.2	1.8
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[2.1582, 3.9786	[-2.9179, 3.9658	[-16.1175, 3.1179]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7	[0.2725, 1.1484]	[-0.1565, .2201]	[-2.8218, 11.4656]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[2.6551, 3.5654]	[-0.9209, .5222]	[-10.3523, -0.7303]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.4670, 0.9051]	[0.7499, 2.9394]	[0.0406, 7.1882]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[3.1242, 3.1242]	[0.8929, 0.8929]	[-5.2217, -5.2217]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.6779, 0.6779]	[1.7823, 1.7823]	[3.3784, 3.3784]
Das et al. (2011)	$x(t)$	1.3	3.1242	0.8929	-5.2217
	$y(t)$	0.6	0.6779	1.7823	3.3784

Table 6. Fuzzy solution for $\alpha = \beta = 1/2$ of Case 2

$t \rightarrow$		0	0.6	1.2	1.8
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[2.2536, 3.5207]	[-1.1611, 4.1497]	[-13.1353, 3.4797]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.2698, 0.9389]	[0.1060, 3.4349]	[-1.6211, 10.5013]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35	[2.5944, 3.2280]	[0.3644, 3.0207]	[-8.1833, 0.1276]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.4220, 0.7566]	[0.7996, 2.4648]	[0.8067, 6.8710]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[2.9192, 2.9192]	[1.7587, 1.7587]	[-3.7611, -3.7611]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.5842, 0.5842]	[1.5861, 1.5861]	[3.6384, 3.6384]
Das et al. (2011)	$x(t)$	1.3	2.9192	1.7587	-3.7611
	$y(t)$	0.6	0.5842	1.5861	3.6384

Table 7. Fuzzy solution for $\alpha = \beta = 2/3$ of Case 2

$t \rightarrow$		0	0.6	1.2	1.8
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[2.1786, 3.0786]	[0.1815, 4.1388	[-9.7139, 3.9434]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.2604, 0.8003]	[0.2070, 2.6883]	[-0.8021, 9.0370]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[2.4171, 2.867]	[1.3065, 3.2856]	[-5.6666, 1.1647]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.3862, 0.6562]	[0.7292, 1.9703]	[1.1698, 6.0916]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3, 1.3]	[2.6467, 2.6467]	[2.3414, 2.3414]	[-2.0394, -2.0394]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.5181, 0.5181]	[1.3171, 1.3171]	[3.4684, 3.4684]
Das et al. (2011)	$x(t)$	1.3	2.6467	2.3414	-2.0394
	$y(t)$	0.6	0.518172	1.31716	3.46845

Table 8. Fuzzy solution for $\alpha = \beta = 1$ of Case 2

$t \rightarrow$		0	0.6	1.2	1.8
$\gamma = 0$	$[\underline{x}(t), \overline{x}(t)]$	[1.2, 1.4]	[1.8995, 2.4195]	[1.6215, 3.7394]	[-3.6611, 4.6392]
	$[\underline{y}(t), \overline{y}(t)]$	[0.5, 0.7]	[0.2616, 0.6626]	[0.2083, 1.6014]	[-0.0884, 5.8224]
$\gamma = 0.5$	$[\underline{x}(t), \overline{x}(t)]$	[1.25, 1.35]	[2.0339, 2.2939]	[2.2091, 3.2682]	[-1.2336, 2.9178]
	$[\underline{y}(t), \overline{y}(t)]$	[0.55, 0.65]	[0.3584, 0.5589]	[0.5116, 1.2083]	[1.1056, 4.0620]
$\gamma = 1$	$[\underline{x}(t), \overline{x}(t)]$	[1.3,1.3]	[2.1654, 2.1654]	[2.7581, 2.7581]	[0.9597, 0.9597]
	$[\underline{y}(t), \overline{y}(t)]$	[0.6, 0.6]	[0.4575, 0.4575]	[0.8450, 0.8450]	[2.4894, 2.4894]
Das et al. (2011)	$x(t)$	1.3	2.1654	2.7581	0.9597
	$y(t)$	0.6	0.4575	0.8450	2.4894

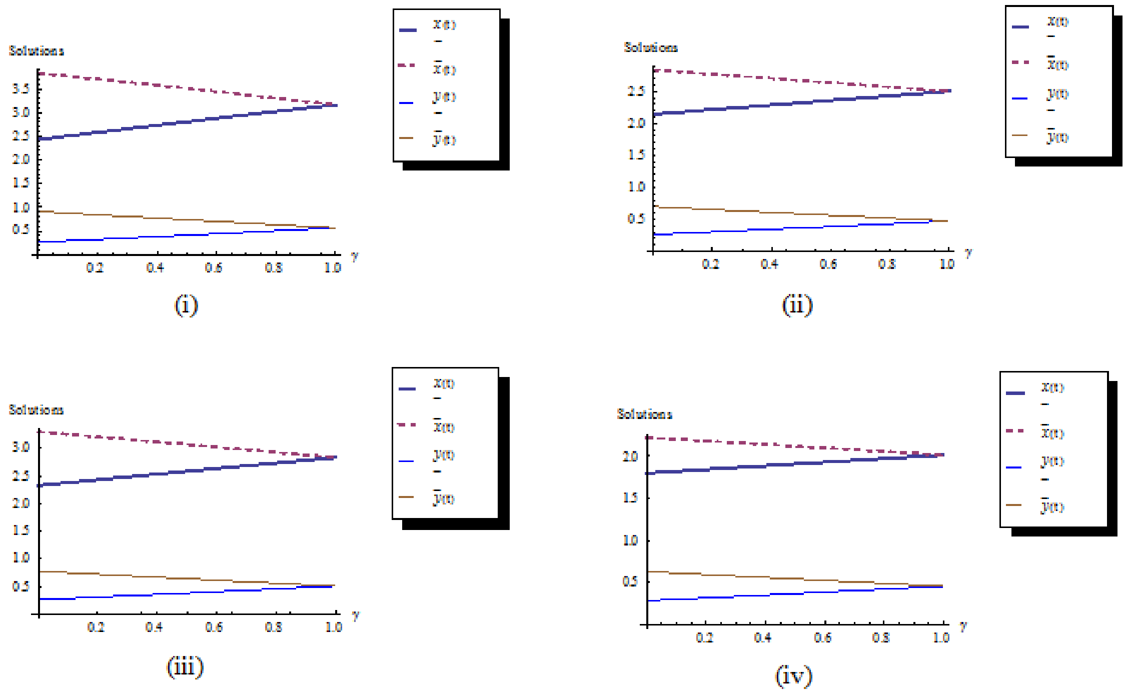

Figure 10. Fuzzy solution of fractional predator-prey equations of Case 2 for $\tilde{x}(t)$ and $\tilde{y}(t)$ at $t = 0.5$ and $\gamma \in [0,1]$ (i) $\alpha = \beta = 1/3$ (ii) $\alpha = \beta = 1/2$ (iii) $\alpha = \beta = 2/3$ (iv) $\alpha = \beta = 1$

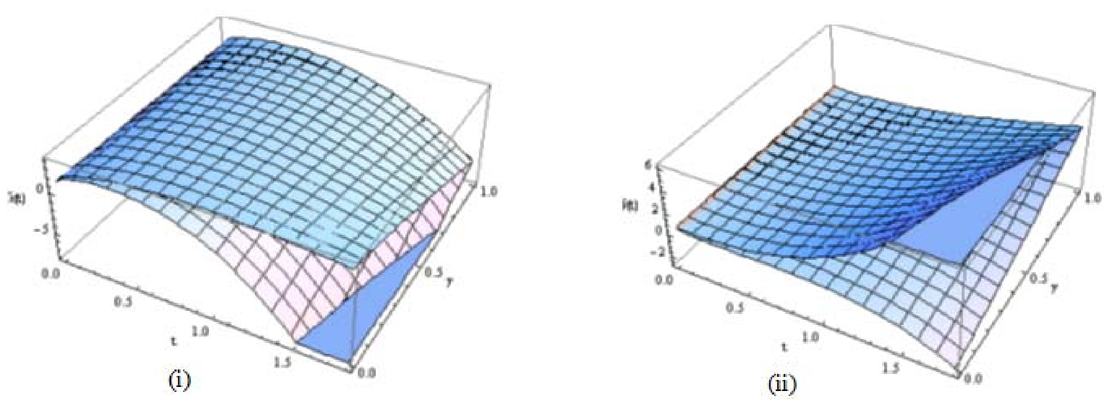

Figure 11. Fuzzy solution of fractional order predator-prey equations of Case 2 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0,1.8]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1/3$

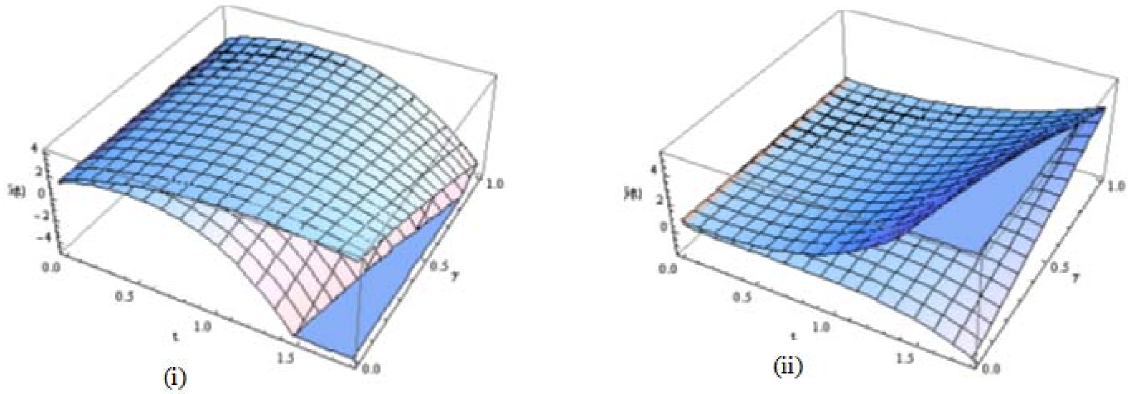

Figure 12. Fuzzy solution of fractional order predator-prey equations of Case 2 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 1.8]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1/2$

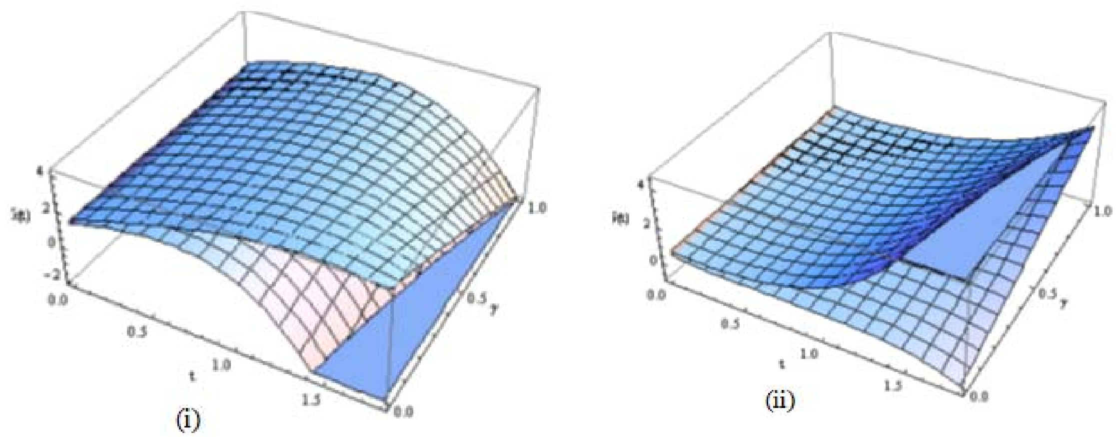

Figure 13. Fuzzy solution of fractional order predator-prey equations of Case 2 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 1.8]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 2/3$

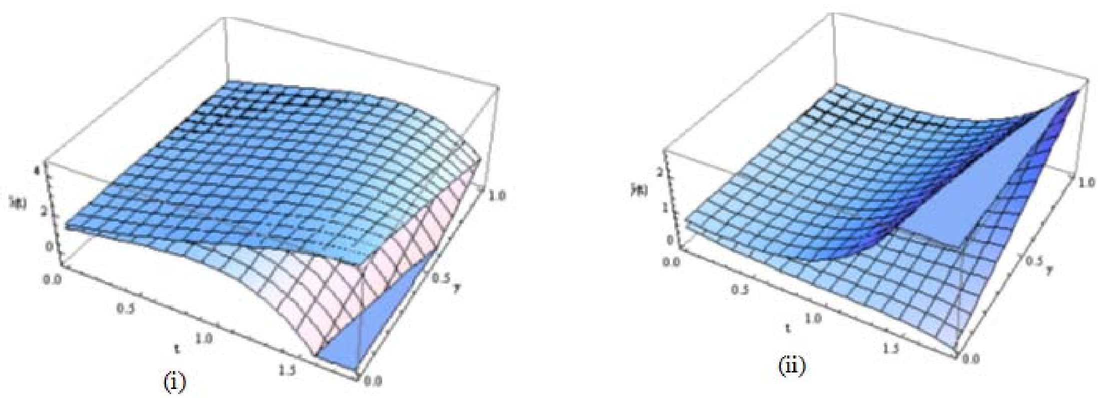

Figure 14. Fuzzy solution of fractional order predator-prey equations of Case 2 for (i) $\tilde{x}(t)$ and (ii) $\tilde{y}(t)$ where $t \in [0, 1.8]$ and $\gamma \in [0,1]$ when $\alpha = \beta = 1$

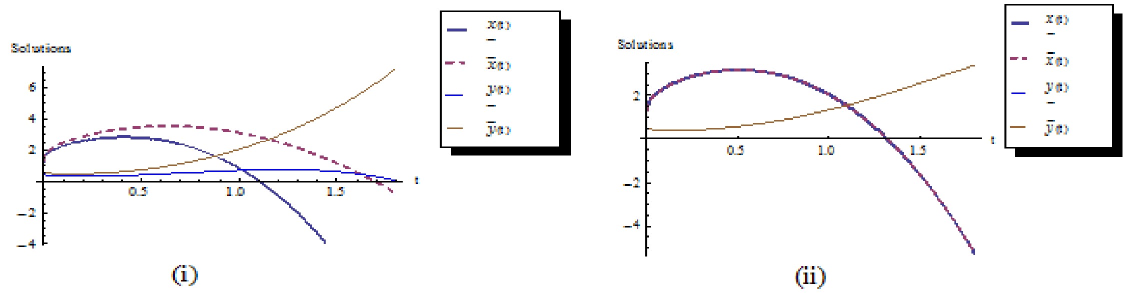

Figure 15. Interval solution of fractional predator-prey equations of Case 2 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1/3$ at (i) $\gamma = 0.5$ and (ii) $\gamma = 1$

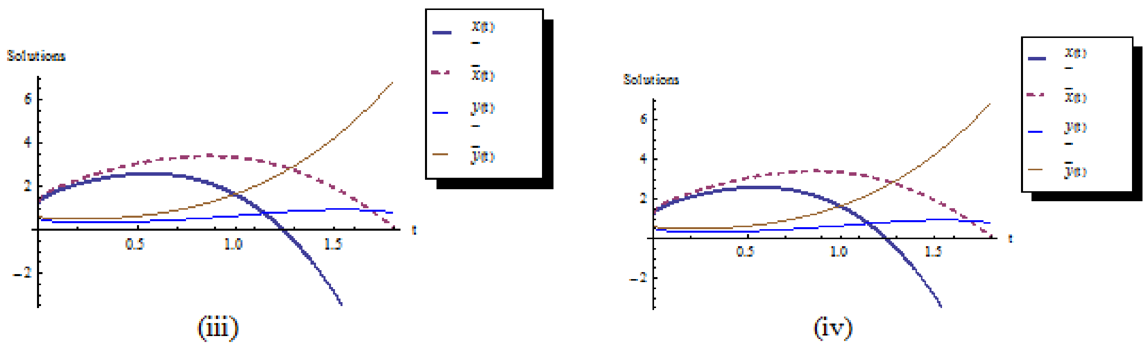

Figure 16. Interval solution of fractional predator-prey equations of Case 2 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1/2$ at (iii) $\gamma = 0.5$ and (iv) $\gamma = 1$

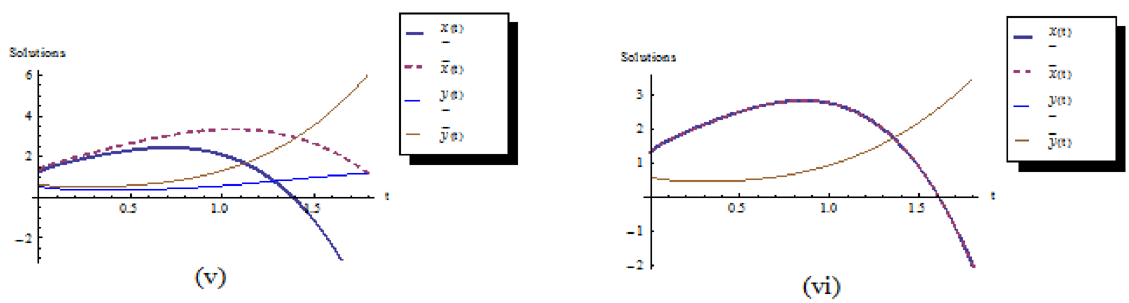

Figure 17. Interval solution of fractional predator-prey equations of Case 2 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 2/3$ at (v) $\gamma = 0.5$ and (vi) $\gamma = 1$

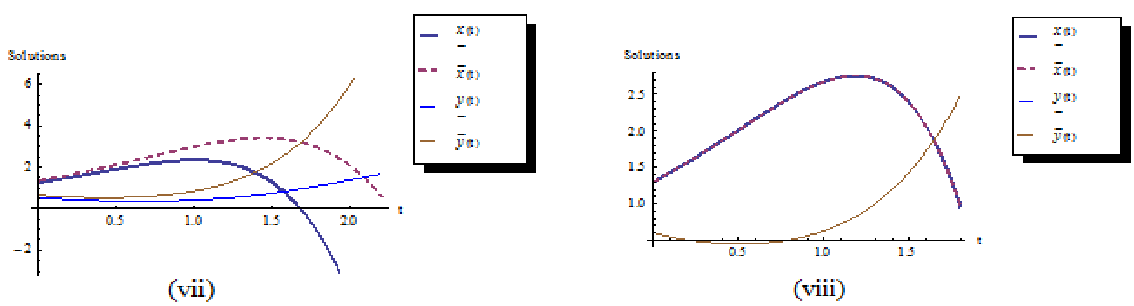

Figure 18. Interval solution of fractional predator-prey equations of Case 2 for $\tilde{x}(t)$ and $\tilde{y}(t)$ with $\alpha = \beta = 1$ at (vii) $\gamma = 0.5$ and (viii) $\gamma = 1$

8. Conclusions

In this paper the HPM has been successfully applied to find the solution of fuzzy fractional order predator-prey equations. The solution obtained by HPM is an infinite series with appropriate initial condition, which may be expressed in a closed form that is the exact solution. The central solution obtained by the fuzzy approach is exactly equal to the crisp solution obtained by Das et al. (2011). It is observed that for Case 1, prey decreases and predator increases with time t and for Case 2 the prey first increases and then decreases with increase in t. One sees that in both the cases it takes more time for meeting predator-prey populations as the fractional time derivative increases and finally takes the maximum time for $\alpha = \beta = 1$. It may also be seen from the figures that Case 1 takes the least time to meet the predator-prey populations whereas Case 2 takes the maximum time. The result shows that the HPM is a powerful tool to fuzzy fractional order predator-prey equations. It seems also to hold great promise for solving other fuzzy nonlinear equations. The solutions obtained are shown in tabular and graphical form.

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