Solutions of System of Fractional Partial Differential Equations

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Abstract

In this paper, system of fractional partial differential equation which has numerous applications in many fields of science is considered. Adomian decomposition method, a novel method is used to solve these type of equations. The solutions are derived in convergent series form which shows the effectiveness of the method for solving wide variety of fractional differential equations.

1. Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in many fields of science and engineering. Important phenomena in finance, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science [Barkai et al. (2000), Meerschaert et al. (2002), Mainardi (2010), Tadjeran (2007)] are well described by differential equations of fractional order. Recently Magin et al. (2011) published a review article on the fractional signals and systems where we can find applications to control theory. Various applications of fractional calculus like image processing are found in the edited volume of Machado (2011). These applications in interdisciplinary sciences show the importance and necessity of fractional calculus. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier (1974), Miller and Ross (1993), Podlubny (1999), Kilbas, Srivastava and Trujillo (2006) and others

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[Samko et al. (1993), Caponetto et al. (2010), Diethelm (2010)]. Machado et al. (2010) published a review on article recent history of fractional calculus. Hernández et al. (2010) published a paper on recent developments in the theory of abstract differential equations with fractional derivatives. These works form an introduction to the theory of fractional differential equations and provide a systematic understanding of the fractional calculus such as the existence and the uniqueness of solutions, some analytical methods for solving fractional differential equations like Green's function method, the Mellin transform method, the power series method and etc. In the literature, there exists no method that yields an exact solution for nonlinear fractional differential equations. Only approximate solutions can be derived using linearization or perturbation methods. All this motivates us to construct an efficient numerical method for fractional differential equations.

The Adomian decomposition method (ADM), introduced by Adomian (1980), provides an effective procedure for finding explicit and numerical solutions of a wider and general class of differential systems representing real physical problems. This method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations, and even for stochastic systems as well. Moreover no linearization or perturbation is required in this method. In the last two decades, extensive work has been done using ADM as it provides analytical approximate solutions for nonlinear equations and considerable interest in solving fractional differential equations using ADM has been developed.

Javadi (2007) proposed new ideas for the implementation of the Adomian decomposition method to solve nonlinear Volterra integral equations. The study of delay differential equations using ADM is done by Evans et al. (2005). Fakharian (2010) solves the Hamilton - Jacobi - Bellman equation arising in nonlinear optimal problem using Adomian decomposition method. It is interesting to note that Biazar et al. (2004) solved the system of first order ordinary differential equations and higher order differential equations which can be converted into a system of first order differential equations and consequently this method has been employed to study the system of integro - differential equations by Biazar (2005). Hosseini et al. (2006) introduced a simple method to determine the rate of convergence of ADM. Further they extended their earlier work of second order ordinary differential equations to higher-order ODE's and systems of nonlinear differential equations and obtained solutions by using modified ADM in Hosseini et al. (2009). The method of Adomian decomposition was used successfully to solve a class of coupled systems of two linear second order and two nonlinear first order differential equations by Bougoffa et al. (2006). Gu and Li (2007) introduced a modified ADM to solve a system of nonlinear differential equations and also they proved that the calculating speed of the method is faster than that of the original Adomian method. Abassy (2010) considered Boussinesq equation and obtained the power series solution by using the method of improved Adomian decomposition.

It should be emphasized that, to the best of our knowledge, as for as ADM for fractional differential equations is concerned, there exists a vast number of publications in the literature. Referring to all these items are beyond the scope of this paper. So let us mention briefly. System of differential equations with fractional derivatives of Riemann-Liouville and Caputo types was studied by Junsheng et al. (2007). Also they developed an algorithm to convert the multi-order fractional differential equation into a system of fractional differential equations and found their

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solutions in Gejji et al. (2007). Moreover the multi-term linear and nonlinear diffusion-wave equation of fractional order is solved in Gejji (2008). Recently Saha Ray (2009) solved the spatially fractional order diffusion equation by using two-step ADM and the results obtained in his paper have been compared with the solutions obtained from ADM. Wang (2006) developed the method to obtain the solutions for the nonlinear fractional KdV-Burgers equation with time and space fractional derivatives. Hu et al. (2008) found the analytical solution of the linear fractional differential equation and shown that the solution by Adomian decomposition method is the same as the solution by the Green's function method. Li (2009) provided a new algorithm based on ADM for fractional differential equations and compared the obtained numerical results with those obtained in the fractional Adams method. El-Sayed (2010) gave the solution to the model that describes the intermediate process between advection and dispersion via fractional derivative in the Caputo sense.

In this work, we consider the system of fractional partial differential equations which is studied by Jafari et al. (2009) using homotopy analysis method. Mamchuev (2008) proved the unique solvability of a boundary value problem for a system of fractional partial differential equations in a rectangular domain and constructed the solution in closed form. Relatively Adomian (1996) presented a method of solving system of coupled nonlinear partial differential equations. In this paper, we find the solution of system of fractional partial differential equations using ADM. The graphs of their solutions are provided at the end.

2. Preliminaries

In this section, we provide some basic definitions of fractional integral and differential operators [Luchko et al. (1999)] and also we briefly describe the Adomian Decomposition Method (ADM).

Fractional Calculus

Definition 1.

A real function $f(t)$, $t > 0$, is said to be in the space C_a, $a \in R$, if there exists a real number $p(>\alpha)$ such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty)$. Clearl, $C_\alpha \subset C_\beta$, if $\beta \leq \alpha$.

Definition 2.

A function $f(t)$, $t > 0$, is said to be in the space C_a^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_a$.

Definition 3. [Riemann - Liouville Fractional Integral Operator]

The (left sided) Riemann-Liouville fractional integral of order $\mu \geq 0$ of a function $f \in C_a$, $a \geq -1$, is defined as

$$D_t^{-\mu} = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s,x)}{(t-s)^{1-\mu}} \, ds, \quad \mu > 0, \ t > 0.$$

Definition 4. [Caputo Fractional Derivative Operator]

The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in N \cup \{0\}$, is defined as

$$^C D_t^{\mu} = D_t^{-(m-\mu)} \frac{\partial^m}{\partial t^m} f(t,x), \quad m-1 < \mu < m, \ m \in N.$$

Note that

$$D_t^{-\mu} \, ^C D_t^{\mu} f(t,x) = f(t,x) - \sum_{k=0}^{m-1} \frac{\partial^k f}{\partial t^k}(0,x) \frac{t^k}{k!}, \quad m-1 < \mu \leq m, \ m \in N.$$

In this paper, the derivative is considered in Caputo sense, since it provides us the advantage of requiring initial conditions given in terms of integer-order derivatives. Also it allows us to specify inhomogeneous initial conditions for fractional differential equations with the Caputo derivative if it is desired.

Definition 5 [Mittag - Leffler function]

A one-parameter function of the Mittag - Leffler type is defined by the series expansion

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (\alpha > 0).$$

Adomian Decomposition Method

Consider the differential equation

$$Lu + Ru + Nu = g, \tag{2.1}$$

where

    L - Highest order derivative and easily invertible,
    R - Linear differential operator of order less that L,
    Nu - Nonlinear terms, and
    g - Source term.

Solving Lu from (2.1), we have

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$$Lu = g - Ru - Nu.$$

Because L is invertible, the equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.2}$$

If L is a second-order operator, for example, L^{-1} is a two fold integration operator and $L^{-1}Lu = u - u(0) - tu'(0)$, then (2.2) yields,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu.$$

Now the solution u can be presented as a series $u = \sum_{n=0}^{\infty} u_n$ with u_0 identified as $u(0) + tu'(0) + L^{-1}g$ and u_n, $n > 0$, is to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials $Nu = \sum_{n=0}^{\infty} A_n$, where A_n is calculated using the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(\upsilon(\lambda))\right]_{\lambda=0}, \quad n = 0, 1, \cdots,$$

where

$$\upsilon(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n.$$

Now the series solution $u = \sum_{n=0}^{\infty} u_n$ to the differential equation is calculated iteratively as follows:

$$u_0 = u(0) + tu'(0) + L^{-1}g,$$

$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), n \geq 0.$$

The above described method can be easily extended to a system of differential equations and the resulting equations will be of the form

$$u_{i,0} = \Phi_i + L^{-1}g_i,$$

$$u_{i,k+1} = -L^{-1}(Ru_{i,k}) - L^{-1}(A_{i,k}), \quad k \geq 0,$$

where $1 \leq i \leq n$ and Φ_i represent the terms arising from the given initial and boundary conditions for a system containing n equations.

3. System of Fractional Partial Differential Equations

In this section, we apply ADM to derive the solutions of a system of fractional partial differential equations.

Example 1. Consider the following system of linear fractional partial differential equations [Jafari (2009)]:

$$^{C}D_t^{\alpha}u - v_x + u + v = 0;$$
$$^{C}D_t^{\beta}v - u_x + u + v = 0; \qquad (0 < \alpha, \beta < 1) \qquad (3.1)$$

with initial conditions

$u(x,0)$ and $v(x,0)$.

The above system of equations can be rewritten in equivalent form using the above properties as

$$^{C}D_t u = {}^{C}D_t^{1-\alpha}v_x - {}^{C}D_t^{1-\alpha}u - {}^{C}D_t^{1-\alpha}v,$$

$$^{C}D_t v = {}^{C}D_t^{1-\beta}u_x - {}^{C}D_t^{1-\beta}u - {}^{C}D_t^{1-\beta}v.$$

In the above equation let us denote v_x by $L_x(v)$, u_x by $L_x(u)$ and $^{C}D_t$ by L_t which is invertible. Then, L_t^{-1} may be taken as $\int_0^t (.)dt$. Therefore, the last equations can be written as

$u(x,t) \quad =$

$$u(x,0) + L_t^{-1}({}^{C}D_t^{1-\alpha}L_x(v(x,t))) - L_t^{-1}({}^{C}D_t^{1-\alpha}u(x,t)) - L_t^{-1}({}^{C}D_t^{1-\alpha}v(x,t)),$$

$$= v(x,0) + L_t^{-1}({}^{C}D_t^{1-\beta}L_x(u(x,t))) - L_t^{-1}({}^{C}D_t^{1-\beta}u(x,t)) - L_t^{-1}({}^{C}D_t^{1-\beta}v(x,t)).$$

Now if we take the initial conditions as

$u_0 = u(x,0)$

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$$v_0 = v(x,0) \quad ,$$

then, by using ADM, we can write

$$u_1 = L_t^{-1}({}^CD_t^{1-\alpha}L_x(v_0)) - L_t^{-1}({}^CD_t^{1-\alpha}u_0) - L_t^{-1}({}^CD_t^{1-\alpha}v_0),$$

$$v_1 = L_t^{-1}({}^CD_t^{1-\beta}L_x(u_0)) - L_t^{-1}({}^CD_t^{1-\beta}u_0) - L_t^{-1}({}^CD_t^{1-\beta}v_0),$$

and, for $n > 1$, we have

$$u_n = L_t^{-1}({}^CD_t^{1-\alpha}L_x(v_{(n-1)})) - L_t^{-1}({}^CD_t^{1-\alpha}u_{(n-1)}) - L_t^{-1}({}^CD_t^{1-\alpha}v_{(n-1)})$$

$$v_n = L_t^{-1}({}^CD_t^{1-\beta}L_x(u_{(n-1)})) - L_t^{-1}({}^CD_t^{1-\beta}u_{(n-1)}) - L_t^{-1}({}^CD_t^{1-\alpha}v_{(n-1)}).$$

Then, the solution to the system of above equations (3.1) is given by

$$u = \sum_{n=0}^{\infty} u_n \text{ and } v = \sum_{n=0}^{\infty} v_n. \tag{3.2}$$

For example, let us consider the initial conditions as $\sinh x$ and $\cosh x$. Therefore, the iterations become

$$u_0 = \sinh x,$$
$$v_0 = \cosh x,$$

$$u_1 = L_t^{-1}({}^CD_t^{1-\alpha}L_x(v_0)) - L_t^{-1}({}^CD_t^{1-\alpha}u_0) - L_t^{-1}({}^CD_t^{1-\alpha}v_0)$$

$$= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}\cosh x,$$

$$v_1 = L_t^{-1}({}^CD_t^{1-\beta}L_x(u_0)) - L_t^{-1}({}^CD_t^{1-\beta}u_0) - L_t^{-1}({}^CD_t^{1-\beta}v_0)$$

$$= -\frac{t^{\beta}}{\Gamma(\beta+1)}\sinh x,$$

$$u_2 = L_t^{-1}({}^CD_t^{1-\alpha}L_x(v_1)) - L_t^{-1}({}^CD_t^{1-\alpha}u_1) - L_t^{-1}({}^CD_t^{1-\alpha}v_1)$$

$$= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right)\cosh x + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\sinh x,$$

$$v_2 = L_t^{-1}(^C D_t^{1-\beta} L_x(u_1)) - L_t^{-1}(^C D_t^{1-\beta} u_1) - L_t^{-1}(^C D_t^{1-\beta} v_1)$$

$$= \left(\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} \right)\sinh x + \frac{t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)}\cosh x,$$

$$u_3 = L_t^{-1}(^C D_t^{1-\alpha} L_x(v_2)) - L_t^{-1}(^C D_t^{1-\alpha} u_2) - L_t^{-1}(^C D_t^{1-\alpha} v_2)$$

$$= \left(\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \right)\cosh x$$

$$+ \left(\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} \right)\sinh x - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\cosh x,$$

$$v_3 = L_t^{-1}(^C D_t^{1-\beta} L_x(u_2)) - L_t^{-1}(^C D_t^{1-\beta} u_2) - L_t^{-1}(^C D_t^{1-\beta} v_2)$$

$$= \left(\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \right)\cosh x - \left(\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} - \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \right)\sinh x - \frac{t^{3\beta}}{\Gamma(3\beta+1)}\sinh x,$$

and so on.

By collecting the terms and substituting in equations (3.2), we get the solution to (3.1) as

$$u(x,t) = \sinh x \left(1 + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} - \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \cdots \right)$$

$$- \cosh x \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} \right.$$

$$\left. + \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right)$$

$$= \sinh(x) \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} (-1)^{k+1} \frac{t^{(n-k)\alpha+k\beta}}{\Gamma((n-k)\alpha+k\beta+1)} \right)$$

$$- \cosh(x) \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} (-1)^{k+1} \frac{t^{(n-k)\alpha+k\beta}}{\Gamma((n-k)\alpha+k\beta+1)} + (-1)^{n+1} t^{n\alpha} \right) \right),$$

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$$
\begin{aligned}
v(x,t) \quad &= \cosh x \left(1 + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \cdots \right) \\[2mm]
&\quad - \sinh x \left(\frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} \right. \\[2mm]
&\qquad\qquad \left. - \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots \right) \\[2mm]
&= \cosh(x) \left(1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} (-1)^{k+1} \frac{t^{(n-k)\beta+k\alpha}}{\Gamma((n-k)\beta+k\alpha+1)} \right) \\[2mm]
&\quad - \sinh(x) \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} (-1)^{k+1} \frac{t^{(n-k)\beta+k\alpha}}{\Gamma((n-k)\beta+k\alpha+1)} + (-1)^{n+1} t^{n\beta} \right) \right).
\end{aligned}
$$

Table 1. Numerical results showing u(x,t) and v(x,t) for different values of α and β by arbitrarily fixing x=3 and t=0.1 using Adomian Decomposition Method and Homotopy Analysis Method (2009)

α	β	ADM		HAM (2009)	
		$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$
0.2	0.2	5.82663	5.96672	5.82663	5.96672
	0.4	5.83636	6.85362	5.82663	6.84975
	0.6	5.84475	7.74879	5.82663	7.74517
	0.8	5.85115	8.52987	5.82663	8.52726
	1.0	5.85546	9.11625	5.82663	9.11458
0.4	0.2	6.76122	5.95700	6.76509	5.96672
	0.4	6.76509	6.84975	6.76509	6.84975
	0.6	6.76834	7.74680	6.76509	7.74517
	0.8	6.77076	8.52876	6.76509	8.52726
	1.0	6.77235	9.11562	6.76509	9.11458
0.6	0.2	7.67438	5.94860	7.67799	5.96672
	0.4	7.67636	6.84649	7.67799	6.84975
	0.6	7.67799	7.74517	7.67799	7.74517
	0.8	7.67917	8.52788	7.67799	8.52726
	1.0	7.67992	9.11513	7.67799	9.11458
0.8	0.2	8.46545	5.94221	8.46805	5.96672
	0.4	8.46655	6.84408	8.46805	6.84975
	0.6	8.46743	7.7440	8.46805	7.74517
	0.8	8.46805	8.52726	8.46805	8.52726
	1.0	8.46844	9.11479	8.46805	9.11458
1.0	0.2	9.0579	5.9379	9.05956	5.96672
	0.4	9.05853	6.84248	9.05956	6.84975
	0.6	9.05902	7.74324	9.05956	7.74517
	0.8	9.05935	8.52687	9.05956	8.52726
	1.0	9.05956	9.11458	9.05956	9.11458

The solution obtained above in series form is convergent for all values of $0 < \alpha, \beta < 1$. The convergence of the series can be easily examined by using the ratio test.

Table 1 shows the numerical results obtained from Adomian decomposition method and homotopy analysis method [Jafari et al. (2009)]. From the table, one can observe that there is no effect of β on $u(x,t)$ and α on $v(x,t)$ while using HAM obtained in [Jafari et al. (2009)], which is not accurate since the problem considered is coupled. This difficulty is overcome when we use the ADM. The exactness of the solution is clear by its convergence to the exact solution when $\alpha, \beta \to 1$ (see Table 1). Also the implementation of this method to the system of fractional partial differential equation is easy and accurate compared with other methods.

Moreover the solution obtained here is in simple form through which the implementation into the computers can be done easily. Also when we take $\alpha = \beta = 1$, the solution converges to $u(x,t) = \sinh(x - t)$ and $v(x,t) = \cosh(x - t)$ which is an exact solution that was proved in [Wazwaz (2007)] using variational iteration method. The graphs were drawn for $\alpha \neq \beta$ (Figure 3.1) and for $\alpha = \beta$ by fixing an arbitrary value to x, say $x = 3$ (Figure 3.2).
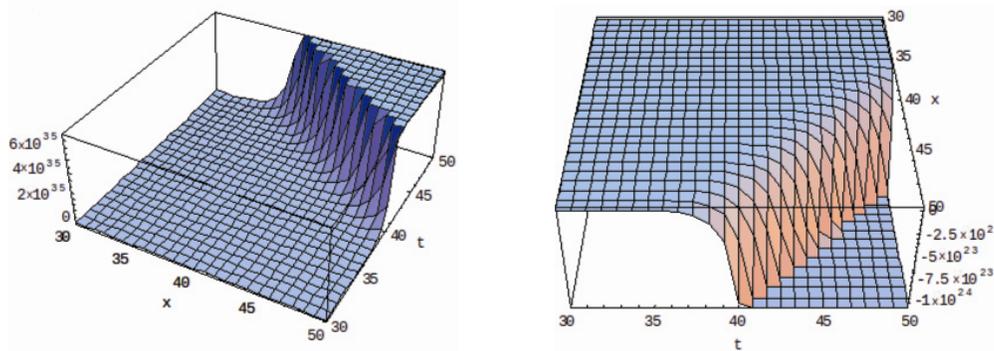

Figure 3.1. Graph showing $u(x,t)$ and $v(x,t)$ when $\alpha = 0.3$ and $\beta = 0.8$ by varying x and t
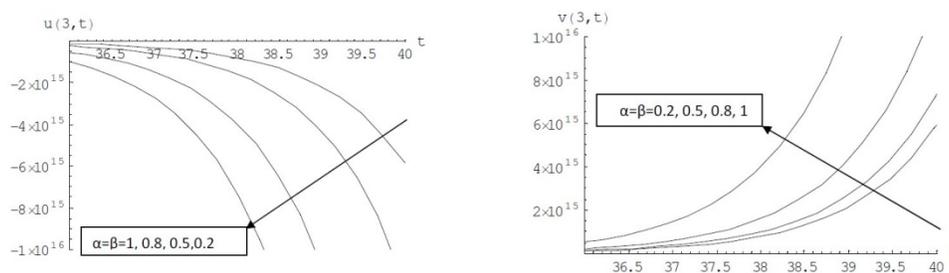

Figure 3.2. Graph showing u(x,t) and v(x,t) when x=3 and for different values of α and β

Example 2. Consider the following nonlinear system of fractional differential equations [**Error! Reference source not found.**affari(2009)]:

$$^{C}D_{t}^{\alpha}u + v_{x}w_{y} - v_{y}w_{x} = -u; \quad ^{C}D_{t}^{\beta}v + u_{x}w_{y} + u_{y}w_{x} = v; \quad ^{C}D_{t}^{\gamma}w + u_{x}v_{y} + u_{y}v_{x} = w; \quad (0 < \alpha, \beta, \gamma < 1).$$

with initial conditions

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$u(x,y,0)=u_0, v(x,y,0)=v_0$ and $w(x,y,0)=w_0$.

By using the properties of Caupto derivative, we can rewrite the given system as

$$^CD_t u=^CD_t^{1-\alpha}(-v_xw_y+v_yw_x)-^CD_t^{1-\alpha}u,$$
$$^CD_t v=^CD_t^{1-\beta}(-u_xw_y-u_yw_x)+^CD_t^{1-\beta}v,$$
$$^CD_t w=^CD_t^{1-\gamma}(-u_xv_y-u_yv_x)+^CD_t^{1-\gamma}w.$$

As we did in the first example let us take $^CD_t(\cdot)$ as $L_t(\cdot)$. So $L_t^{-1}(\cdot)$ may be taken as $\int_0^t (\cdot)dt$.

Then, the above equation becomes

$$u = u(x,y,0)+L_t^{-1}(^CD_t^{1-\alpha}(-v_xw_y+v_yw_x))-L_t^{-1}(^CD_t^{1-\alpha}u),$$
$$v = v(x,y,0)+L_t^{-1}(^CD_t^{1-\alpha}(-u_xw_y-u_yw_x))-L_t^{-1}(^CD_t^{1-\alpha}v),$$
$$w = w(x,y,0)+L_t^{-1}(^CD_t^{1-\gamma}(-u_xv_y-u_yv_x))+L_t^{-1}(^CD_t^{1-\gamma}w).$$

If we take the initial conditions as

$u_0=u(x,y,0),$
$v_0=v(x,y,0),$
$w_0=w(x,y,0),$

then, by using the ADM, we get

$$u_1=L_t^{-1}(^CD_t^{1-\alpha}(A_0))-L_t^{-1}(^CD_t^{1-\alpha}u),$$
$$v_1=L_t^{-1}(^CD_t^{1-\beta}(B_0))+L_t^{-1}(^CD_t^{1-\beta}v),$$
$$w_1=L_t^{-1}(^CD_t^{1-\gamma}(C_0))+L_t^{-1}(^CD_t^{1-\gamma}w),$$

and, for $n>1$, we have

$$u_n= L_t^{-1}(^CD_t^{1-\alpha}(A_{n-1}))-L_t^{-1}(^CD_t^{1-\alpha}u),$$
$$v_n=L_t^{-1}(^CD_t^{1-\beta}(B_{n-1}))+L_t^{-1}(^CD_t^{1-\beta}v),$$
$$w_n=L_t^{-1}(^CD_t^{1-\gamma}(C_{n-1}))+L_t^{-1}(^CD_t^{1-\gamma}w),$$

where A_n, B_n, and C_n are Adomian polynomials corresponding to the non-linear terms $(-v_x w_y + v_y w_x)$, $(-u_x w_y - u_y w_x)$, and $(-u_x v_y - u_y v_x)$ respectively. Then the solution is given by

$$u = \sum_{n=0}^{\infty} u_n, v = \sum_{n=0}^{\infty} v_n \text{ and } w = \sum_{n=0}^{\infty} w_n.$$

For example, if we take the initial conditions as e^{x+y}, e^{x-y}, e^{-x+y}, then the iterations become

$u_0 = e^{x+y}$,

$v_0 = e^{x-y}$,

$w_0 = e^{-x+y}$,

$u_1 = L_t^{-1}(^C D_t^{1-\alpha}(A_0)) - L_t^{-1}(^C D_t^{1-\alpha} u_0) = -\dfrac{e^{x+y} t^{\alpha}}{\Gamma(1+\alpha)}$,

$v_1 = L_t^{-1}(^C D_t^{1-\beta}(B_0)) + L_t^{-1}(^C D_t^{1-\beta} v_0) = \dfrac{e^{x-y} t^{\beta}}{\Gamma(1+\beta)}$,

$w_1 = L_t^{-1}(^C D_t^{1-\gamma}(C_0)) + L_t^{-1}(^C D_t^{1-\gamma} w_0) = \dfrac{e^{-x+y} t^{\gamma}}{\Gamma(1+\gamma)}$,

$u_2 = L_t^{-1}(^C D_t^{1-\alpha}(A_1)) - L_t^{-1}(^C D_t^{1-\alpha} u_1) = \dfrac{e^{x+y} t^{2\alpha}}{\Gamma(1+2\alpha)}$,

$v_2 = L_t^{-1}(^C D_t^{1-\beta}(B_1)) + L_t^{-1}(^C D_t^{1-\beta} v_1) = \dfrac{e^{x-y} t^{2\beta}}{\Gamma(1+2\beta)}$,

$w_2 = L_t^{-1}(^C D_t^{1-\gamma}(C_1)) + L_t^{-1}(^C D_t^{1-\gamma} w_1) = \dfrac{e^{x-y} t^{2\gamma}}{\Gamma(1+2\gamma)}$,

and so on.

By collecting the terms, we get the solution as

$$u(x,y,t) \qquad = e^{x+y}\left(1 + \sum_{m=1}^{\infty} \frac{(-t^{\alpha})^m}{\Gamma(m\alpha+1)}\right) = e^{x+y} E_{\alpha}(-t^{\alpha}),$$

$$v(x,y,t) \qquad = e^{x-y}\left(1 + \sum_{m=1}^{\infty} \frac{(t^{\beta})^m}{\Gamma(m\beta+1)}\right) = e^{x-y} E_{\beta}(t^{\beta}),$$

$$w(x,y,t) \qquad = e^{-x+y}\left(1 + \sum_{m=1}^{\infty} \frac{(t^{\gamma})^m}{\Gamma(m\gamma+1)}\right) = e^{-x+y} E_{\gamma}(t^{\gamma}).$$

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The series solution converges to the Mittag-Leffler function of the first kind which is the exact solution of the problem that can be verified by directly substituting into the equation. Also, when we substitute $\alpha = \beta = \gamma = 1$, the solution converges to

$$u(x,y,t)=e^{x+y-t}, \ v(x,y,t)=e^{x-y+t}, \ w(x,y,t)=e^{-x+y+t},$$

which was proved by Wazwaz(2007) for integer type problem using variational iteration method. Also the same results are obtained in [Jafari et al. (2009)] using homotopy analysis method. The graphs (Figures 3.3 - 3.5) are plotted for different values of α, β and γ by fixing arbitrary values of x and y, say x=0.5 and y=0.5.
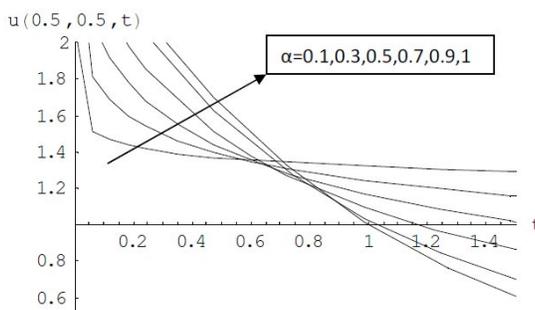
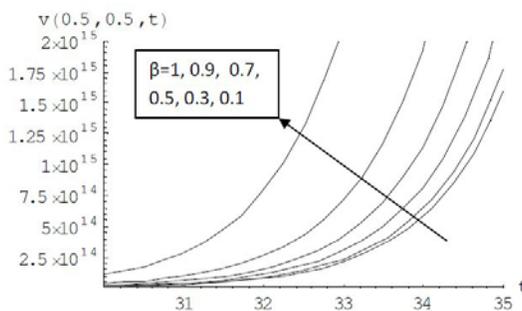**Figure 3.3.** Graph showing $u(x,y,t)$, when x=y=0.5 and for different values of α, β and γ

Figure 3.4. Graph showing $v(x,y,t)$ when x=y=0.5 and for different values of α, β and γ
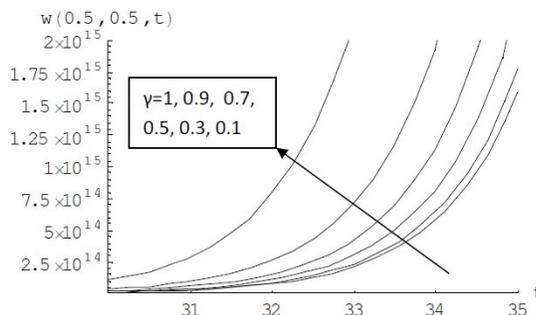
Figure 3.5. Graph showing $w(x,y,t)$ when x=y=0.5 and for different of α, β and γ

4. Conclusion

In this paper, Adomain decomposition method is efficiently used to obtain the solution of the system of fractional partial differential equations. Since its convergence to the exact solution, when the system is of integer order, proves that this method is very much effective to solve the system of partial differential equations. However, the necessity of discretization or perturbation is not needed, the method can be easily implemented using any symbolic mathematical softwares. It is observed that the results obtained in this article can be extended to any system of n fractional partial differential equations. Moreover, it provides more realistic series solutions that converge very rapidly in real physical problems.

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