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A Subdivision-Regularization Framework for Preventing Over Fitting of Data by a Model

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Abstract

First, we explore the properties of families of odd-point odd-ary parametric approximating subdivision schemes. Then we fine-tune the parameters involved in the family of schemes to maximize the smoothness of the limit curve and error bounds for the distance between the limit curve and the kth level control polygon. After that, we present the subdivision-regularization framework for preventing over fitting of data by model. Demonstration shows that the proposed unified frame work can work well for both noise removal and overfitting prevention in subdivision as well as regularization.

Keywords: Subdivision scheme; model; over fitting; regularization; continuity; error bounds

AMS 2010 No.: 65D17, 65D10, 47A52, 65J22

1. Introduction

Subdivision schemes are mostly used for generating curves and surfaces. It maps initial sequence of control points to a new sequence, by applying subdivision rules. Repetition of this process several times produces a very good approximation to the curve or surface defined by the original set of control points.

In this paper, we first explore some properties of family of odd-point odd-ary schemes presented by Aslam and Abeysinghe (2012) and then study the properties of parameters involved in the family. After that, we propose a combination of the framework of odd-point odd-ary subdivision schemes and a regularization approach as a modeling tool and observe its significant improvement in noise removal and over fitting.

The idea of introducing families of subdivision scheme is not new. It was first done by Dyn et al. (1987). Dubuc and Deslauriers in (1989) study a family of even-point, even-ary (that is, the number of new points inserted corresponding to each old edge at each subdivision step) subdivision schemes. Mustafa and Najma (2010) presented a general formulae for the mask of $(2b + 4)$-point n-ary approximating as well as interpolating subdivision schemes for any integers $b \geq 0$ and $n \geq 2$. These formulae corresponding to the mask not only generalize and unify several well-known schemes but also provide the mask of higher ary schemes. The idea of families of odd-point subdivision scheme of higher ary is relatively new. It was introduced by Lian (2008) and Aslam et al. (2011). There are also some other families of odd-point schemes in the literature Hassan and Dodgson (2003), Siddiqui and Ahmad (2009) and Siddiqui and Rehan (2010). Almost all existing odd-point odd-ary schemes are special cases of family of schemes introduced by Aslam and Abeysinghe (2012). However, they did not explore the properties of their schemes. This motivates us to explore the properties of their generalized family of schemes as well as to use this family in a combined frame work of subdivision-regularization.

Subdivision schemes mostly used for data modelling, by treating the input data as the control points and the subdivision curve as the model. In this context, depending on the nature of the data and requirements of the modelling problem subdivision schemes produce smoother model. If the noise shoots up in the initial data then subdivision technique cannot produce smoother model and in most of the cases over fitting occur (see Lee et al. (2006) for over fitting view). Therefore, we need more sophisticated technique in addition to the subdivision technique to get rid of noise. For the pleasantness and smoothness of the model we suggest the subdivision-regularization technique that is the subdivision operator followed by the regularization operator. In the following paragraph, we give a brief introduction to the regularization technique.

Suppose we are given a noisy signal F. We want to obtain the decomposition $F = U + V$, where U is assumed to be the true signal and the residual V to be the noise. This is very popular inverse problem in image processing literature and there are several approaches to approximate U. Most successful and commonly used method to solve this problem is regularization method [Vogal (2002)] and is given by

$$U_\lambda = \arg \min_{U \in R^n} \{\|F - U\|_2^2 + \lambda \|U\|_{TV}\},$$

for some parameter $\lambda > 0$, where U_λ is the approximation of U and is called regularized solution to the inverse problem and

$$\|U\|_{TV} = \sum_{i=1}^{n-1} \left|\frac{U_{i+1} - U_i}{\Delta x}\right| \Delta x = \sum_{i=1}^{n-1} |U_{i+1} - U_i|,$$

$\|F-U\|_2^2$ is referred to as a least square term and $||U||_{TV}$ is called the penalty functional and it penalizes highly oscillatory regularized solutions. The value of the parameter λ determines the smoothness of U and there are several ways to find its best value; Vogal (2002). This method is extensively explored and implemented for noise removal, image restoration and other applications [Rudin and Osher (1992), Vogal (2002) and Osher et al. (2005)].

Contributions: The main contributions of the paper are:

- A revisitation of generalized family presented Aslam and Abeysinghe (2012) and explores the special cases of this family.

- The fine-tuning of parameters involved in the family of schemes to maximize the smoothness of the limit curve and error bounds for the distance between the limit curve and the control polygon after kth subdivision steps.

- The introduction of subdivision-regularization framework and demonstration for preventing over fitting of data by a model.

The main benefit of the subdivision-regularization approach is that it does not change the beauty of the subdivision and regularization approaches. It simply operates the subdivision operator followed by the regularization operator on the initial noisy data during model fitting process. One limitation is that the subdivision-regularization function contains scalar parameters inherited in both subdivision and regularization techniques and at the moment we cannot propose an automatic way of selecting these parameter to choose the best model from the class of subdivision-regularization models. The rest of the structure of the paper is as follows:

In Section 2, we present a family of odd-point ternary schemes. In Section 3, we present a family of odd-point odd-ary schemes. In Section 4, parameter fine-tuning for continuity and error bound comparison is given. In Section 5, the power of the subdivision-regularization framework is demonstrated. Finally, the conclusion is presented in Section 6.

2. Family of Odd-Point Ternary Schemes

In this section, we present $(2n-1)$-point parametric ternary approximating sub-division schemes for $n \geq 2$ with free parameters and explore its special cases. Consider the polynomial

$$P_{2n-1,t}^{3}(z) = (1+z+z^2)^t \sum_{i=0}^{3(2n-1)-2t-1} u_i z^i, \quad t = 1,2,3,\ldots,3n-2, \tag{2.1}$$

where

$$u_{3n-t-2-i} = u_{3n-t-2+i}, \quad i = 1,2,3,\ldots,3n-t-2$$

and

$$u_{3n-t-2}=\frac{1}{(3)^{t-1}}-2\sum_{i=0}^{3n-t-3}u_i.$$

From (2.1), we get $3n-2$ different polynomials $P^3_{2n-1,t}(z)$ for $t=1,2,3,\dots,3n-2$. If $\alpha^3_{2n-1,t}$ denote the sets of coefficients of polynomials $P^3_{2n-1,t}(z)$ then these sets of coefficients are called masks of $3n-2$ different odd-point ternary subdivision schemes. The lower script $2n-1$ and superscript 3 of $P^3_{2n-1,t}(z)$ and $\alpha^3_{2n-1,t}$ stands for $(2n-1)$ −point and ternary respectively.

It is obvious that the mask $\alpha^3_{2n-1,t}$ obtained from the coefficients of each polynomial $P^3_{2n-1,t}(z)$ satisfies the necessary condition for the uniform convergence of the subdivision schemes. In following, we shall generate families of 3-point and 5-point ternary schemes from (2.1).

2.1. Family of 3-Point Ternary Schemes

For setting $n=2$ in equation (2.1), we obtain four polynomials $P^3_{3,t}(z), t=1,2,3,4$ with following sets of coefficients:

$$\alpha^3_{3,1}=\left\{\begin{matrix}u_0, & u_0+u_1, u_0+u_1+u_2, 1-2u_0-u_1-u_2, 1-2u_0-2u_1, \\ & 1-2u_0-u_1-u_2, u_0+u_1+u_2, u_0+u_1, u_0\end{matrix}\right\},$$

$$\alpha^3_{3,2}=\left\{\begin{matrix}u_0, & 2u_0+u_1, u_0+\frac{1}{3}, \frac{2}{3}-2u_0, 1-4u_0-2u_1, \frac{2}{3}-2u_0, u_0+\frac{1}{3}, \\ & 2u_0+u_1, u_0\end{matrix}\right\},$$

$$\alpha^3_{3,3}=\left\{u_0, \ u_0+\frac{1}{9}, u_0+\frac{1}{3}, \frac{2}{3}-2u_0, \frac{7}{9}-2u_0, \frac{2}{3}-2u_0, u_0+\frac{1}{3}, u_0+\frac{1}{9}, u_0\right\},$$

$$\alpha^3_{3,4}=\frac{1}{27}\{1,4,10,16,19,16,10,4,1\}. \tag{2.2}$$

The above sets of coefficients are also called masks of four different 3-point ternary approximating schemes. These schemes are interrelated that is schemes with more parameters are generalized form of schemes with less parameter or without parameter.

- By taking $u_0=\frac{1}{27}, u_1=\frac{3}{27}$ and $u_2=\frac{6}{27}$ in $\alpha^3_{3,1}$, we get $\alpha^3_{3,4}$,
- For $u_1=\frac{1}{9}$ and $u_2=\frac{2}{9}$ in $\alpha^3_{3,1}$, we get $\alpha^3_{3,3}$,
- If $u_0=\frac{1}{27}$ in $\alpha^3_{3,3}$, we get $\alpha^3_{3,4}$.

2.2. Family of 5-Point Ternary Schemes

From (2.1) for n = 3, we get seven polynomials $P^3_{5,t}(z), t=1,2,\dots,7$. The sets of coefficients of polynomials corresponding to $t=5,6,7$ are given below:

$$\alpha_{5,5}^{3}=\left\{\begin{matrix} u_0,\ \ 5u_0+u_1, 13u_0+3u_1+\frac{1}{3^4}, 20u_0+6u_1+\frac{5}{3^4}, 16u_0+5u_1+\frac{5}{3^3}, \\ \frac{10}{3^3}-4u_0, \frac{5}{3^2}-30u_0-9u_1, \frac{17}{3^3}-42u_0-12u_1, \frac{5}{3^2}-30u_0-9u_1, \\ \frac{10}{3^3}-4u_0, 16u_0+5u_1+\frac{5}{3^3}, 20u_0+6u_1+\frac{5}{3^4}, 13u_0+3u_1+\frac{1}{3^4}, \\ 5u_0+u_1, u_0 \end{matrix}\right\},$$

$$\alpha_{5,6}^{3}=\left\{\begin{matrix} u_0,\ \ 4u_0+\frac{1}{3^5}, 10u_0+\frac{6}{3^5}, 14u_0+\frac{21}{3^5}, 11u_0+\frac{50}{3^5}, \frac{90}{3^5}-4u_0, \frac{126}{3^5}-21u_0, \\ \frac{141}{3^5}-30u_0, \frac{126}{3^5}-21u_0, \frac{90}{3^5}-4u_0, 11u_0+\frac{50}{3^5}, 14u_0+\frac{21}{3^5}, 10u_0+\frac{6}{3^5}, \\ 4u_0+\frac{1}{3^5}, u_0 \end{matrix}\right\},$$

$$\alpha_{5,7}^{3}=\frac{1}{3^6}\{1,7,28,77,161,266,357,393,357,266,161,77,28,7,1\}. \tag{2.3}$$

The above mask is related to each other:

- By taking $u_0=\frac{1}{3^6}$ in $\alpha_{5,6}^{3}$, we get $\alpha_{5,7}^{3}$,
- For $u_0=\frac{1}{3^6}$ and $u_1=\frac{2}{3^6}$ in $\alpha_{5,5}^{3}$, we get $\alpha_{5,7}^{3}$,
- If $u_1=\frac{1}{3^5}-u_0$ in $\alpha_{5,5}^{3}$, we get $\alpha_{5,6}^{3}$.

2.3. Special Cases

- By letting $\{u_0=\frac{1}{72}+\mu, u_1=\frac{1}{9}, u_2=\frac{16}{72} \text{ in } \alpha_{3,1}^{3}\}$ or $\{u_0=\frac{1}{72}+\mu, u_1=\frac{7}{72}-\mu \text{ in } \alpha_{3,2}^{3}\}$ or $\{u_0=\frac{1}{72}+\mu, \text{ in } \alpha_{3,3}^{3}\}$, we get the mask of Siddiqui and Rehan (2010).

- If $\{u_0=\frac{1}{27}, u_1=\frac{3}{27}, u_2=\frac{6}{27} \text{ in } \alpha_{3,1}^{3}\}$ or $\{u_0=\frac{1}{27}, u_1=\frac{2}{27} \text{ in } \alpha_{3,2}^{3}\}$ or $\{u_0=\frac{1}{27}, \text{ in } \alpha_{3,3}^{3}\}$, we get mask of Hassan and Dodgson (2003).

- For $u_0=u-\frac{1}{3}$, in $\alpha_{3,3}^{3}$, we get the mask of 3-point ternary scheme of Aslam et al. (2009).

- For $u_0=\frac{1}{31104}, u_1=\frac{76}{31104}$, in $\alpha_{5,5}^{5}$, we get the mask of Siddiqi and Ahmad (2009).

- For $u_0=u+\frac{4}{81}, u_1=-(4u+\frac{53}{243})$ in $\alpha_{5,5}^{5}$, we get the mask of 5-point ternary scheme of Aslam et al. (2011).

- In fact for $t=2n-1$ in equation (2.1) and with an appropriate choice of parameters, all subdivision schemes and special cases of Aslam et al. (2011), become special cases of this scheme.

3. Family of Odd-Point Odd-Ary Schemes

In this section, we present a generalized version of the family of schemes discussed in the previous section, which in reality is the family of schemes of Aslam and Abeysinghe (2012). Consider a family of polynomial

$$P^{2b+1}_{2n-1,t}(z) = (1+z+z^2+\cdots+z^{2b})^t \sum_{i=0}^{(2b+1)(2n-1)-2bt-1} u_i z^i, \tag{3.1}$$

where $t = 1,2,3,\ldots,(2n-1)+\left\lceil\frac{n-1}{b}\right\rceil$, $\left\lceil\frac{n-1}{b}\right\rceil$, is the largest interger less than or equal to $\frac{n-1}{b}$

$$u_{(2b+1)n-b(t+1)-1-i} = u_{(2b+1)n-b(t+1)-1+i}, \quad i = 1,2,\ldots,(2b+1)n-b(t+1)-1$$

and

$$u_{(2b+1)n-b(t+1)-1} = \frac{1}{(2b+1)^{t-1}} - 2\sum_{i=0}^{(2b+1)n-b(t+1)-2} u_i.$$

By (3.1), we get $(2n-1)+\left\lceil\frac{n-1}{b}\right\rceil$ different polynomials $P^{2b+1}_{2n-1,t}(z)$ one for each t. If $\alpha^{(2b+1)}_{2n-1,t}$ denote the sets of coefficients of polynomials $P^{2b+1}_{2n-1,t}(z)$ then these sets of coefficients are called masks of $(2n-1)-$point $(2b+1)$-ary approximating schemes. Families of schemes corresponding to $(2n-1)-$point as well as $(2b+1)-$ary can be easily obtained.

It is to be noted that the mask $\alpha^{(2b+1)}_{2n-1,t}$ obtained from the coefficients of each polynomial $P^{2b+1}_{2n-1,t}(z)$ satisfies the necessary condition for the uniform convergence of the subdivision schemes.

Moreover, for $b = 1$ in (3.1), we get all families of odd-point ternary schemes discussed in Sections 2. Furthermore, for $\{b=2, n=2\}$ & $\{b=2, n=3, u_1=\frac{1}{5^4}-u_0\}$ in (3.1), we get the following masks of 3-point and 5-point quinary schemes.

$$\alpha^5_{3,3} = \left\{\begin{matrix} u_0, u_0+\frac{1}{25}, u_0+\frac{3}{25}, u_0+\frac{6}{25}, u_0+\frac{10}{25}, \frac{15}{25}-2u_0, \frac{18}{25}-2u_0, \frac{19}{25}-2u_0, \\ \frac{18}{25}-2u_0, \frac{15}{25}-2u_0, u_0+\frac{10}{25}, u_0+\frac{6}{25}, u_0+\frac{3}{25}, u_0+\frac{1}{25}, u_0 \end{matrix}\right\},$$

$$\alpha^{5\star}_{5,5} = \left\{\begin{matrix} u_0, 4u_0+\frac{1}{625}, 10u_0+\frac{4}{625}, 19u_0+\frac{11}{625}, 31u_0+\frac{1}{25}, 41u_0+\frac{2}{25}, \\ 44u_0+\frac{86}{625}, 35u_0+\frac{134}{625}, 14u_0+\frac{191}{625}, \frac{2}{5}-19u_0, \frac{12}{25}-54u_0, \\ \frac{336}{625}-81u_0, \frac{349}{625}-90u_0\ \frac{336}{625}-81u_0, \frac{12}{25}-54u_0, \frac{2}{5}-19u_0, \\ 14u_0+\frac{191}{625}, 35u_0+\frac{134}{625}, 44u_0+\frac{86}{625}, \frac{191}{625}, \frac{2}{5}-19u_0, 44u_0+\frac{86}{625}, \\ 41u_0+\frac{2}{25}, 31u_0+\frac{1}{25}, 19u_0+\frac{11}{625}, 10u_0+\frac{4}{625}, 4u_0+\frac{1}{625}, u_0 \end{matrix}\right\}.$$

4. Parameter Tuning and Error Bound Comparison

We discuss here, a parameter fine-tune for continuity and error bound for some of the schemes in (2.1) and (3.1). The fine-tuning of parameters in other schemes can be discussed analogously. We have the following finding regarding the fine-tuning of parameters.

- The values of the parameters that maximize the continuity of one scheme leads to at least minimum continuity of the others. Since the scheme corresponding to $\alpha^3_{3,3}$ is C^2 for $0 < u_0 < \frac{1}{9}$ then the schemes corresponding to the masks $\alpha^3_{3,1}$, $\alpha^3_{3,2}$ and $\alpha^3_{3,4}$ will be at least C^2. We may refer to Dyn et al. (1987, 2004) for the determination of continuity.

- Moreover, in some cases the continuity of schemes can be increased at the particular value(s) of the parameter(s) instead of parametric interval(s) that is the scheme corresponding to the mask $\alpha^3_{5,6}$ is C^4-continuous over $0 < u_0 < \frac{1}{243}$ but C^5at $u_0 = \frac{1}{729}$.

- The values of the parameter that maximize the order of continuity are shown in Table 1.

- The effect of the parameters on error bounds (maximum distance between limit curve and control polygon after *k*th subdivision level) is shown in Figure 1-4. For a detailed description of error bound sees Mustafa and Deng (2007). We have observed that error bounds for 3-point and 5-point ternary schemes corresponding to the masks $\alpha^3_{3,3}$ & $\alpha^3_{5,6}$ remain constant and attain minimum value over $0 \le u_0 \le \frac{1}{6}$ and $0 \le u_0 \le \frac{14}{801}$ respectively and then increases gradually as we move out of these intervals. Similarly error bounds for 3-point and 5-point quinary schemes corresponding to the masks $\alpha^5_{3,3}$ & $\alpha^{5*}_{5,5}$ remains constant and attains the minimum value over $0 \le u_0 \le \frac{1}{10}$ and $0 \le u_0 \le \frac{1}{200}$, respectively.

Now we give a comparison of error bounds by increasing the complexity (number of point involved to insert new points) of the schemes and by changing the arity of the schemes with fixed complexity. It is observed that the error bounds increase by increasing the complexity of their schemes while they decrease by increasing the arity of the scheme. A graphical representation of error bounds is shown in Figure 5. In this figure, we use 3-point binary Hassan and Dodgson (2003) 3-point ternary (mask $\alpha^3_{3,3}$) 3-point quaternary (mask 1/64{1,5,13,25,38,46,46,38, 25,13, 5,1}), 3-point quinary (mask$\alpha^5_{3,3}$), 4-point ternary Zheng et al. (2009), 5-point ternary (mask $\alpha^3_{5,6}$), and 6-point ternary Zheng et al. (2009) schemes.

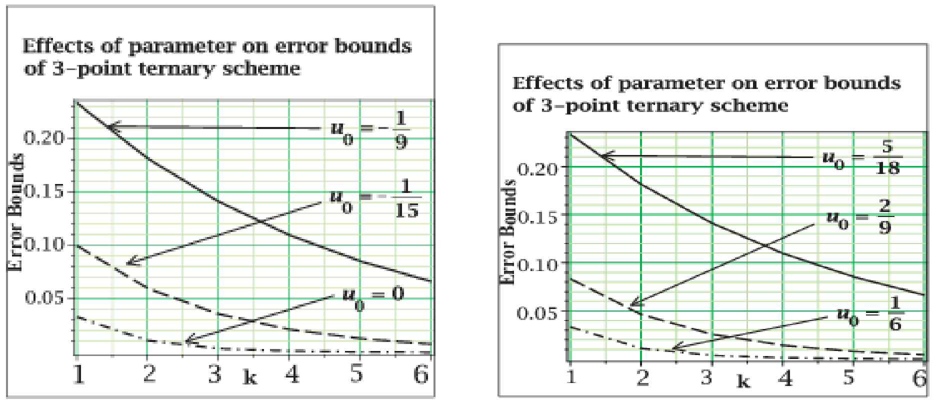

Figure 1. Effects of parametric values of the schemes corresponding to the masks on error bounds

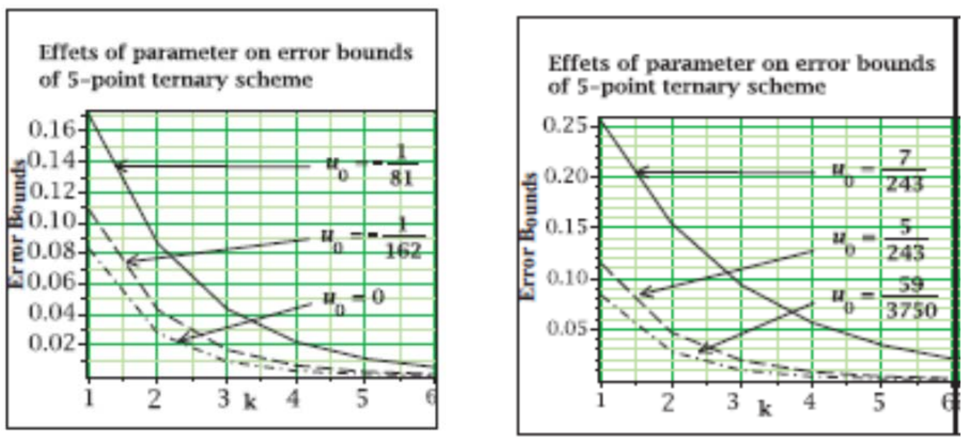

Figure 2. Effects of parametric values of the schemes corresponding to the masks on error bounds

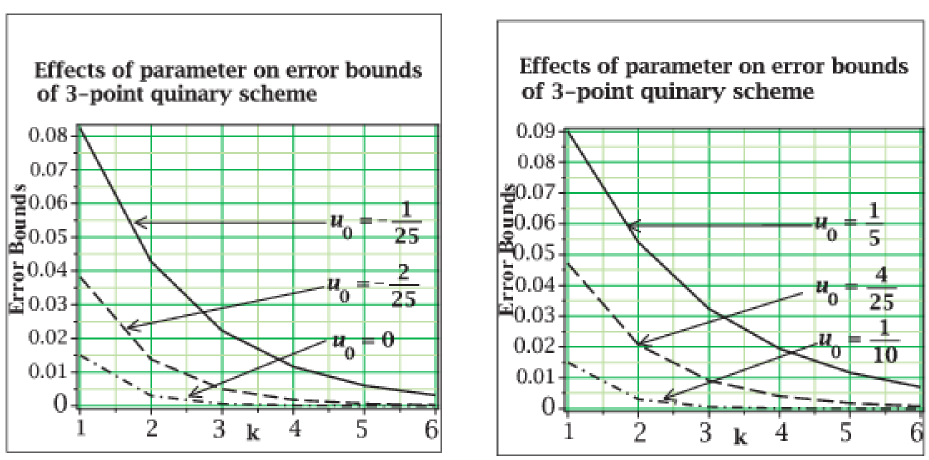

Figure 3. Effects of parametric values of the schemes corresponding to the masks on error bounds.

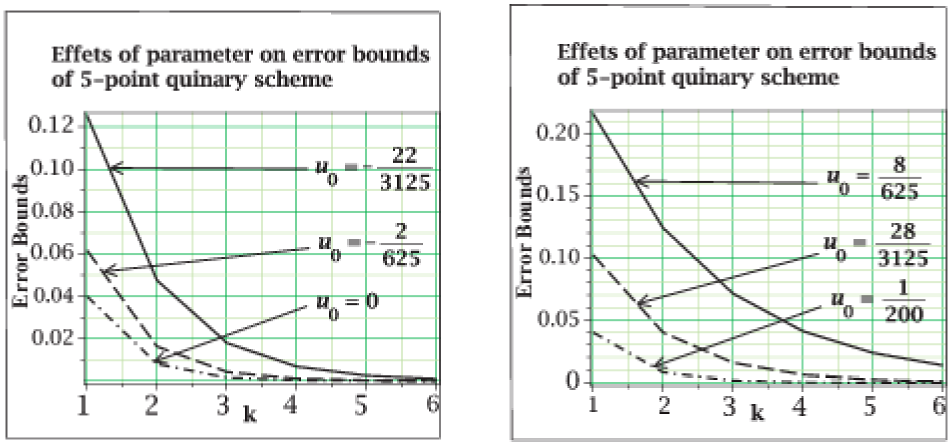

Figure 4: Effects of parametric values of the schemes corresponding to the masks $\alpha_{5,5}^{5*}$ on error bounds

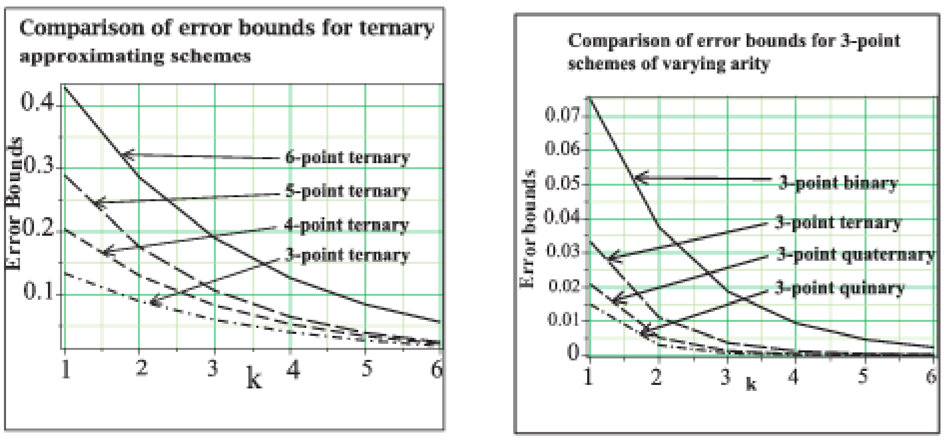

Figure 5. Comparison of error bounds with respect to complexity and arity of the schemes

Table 1: Fine-Tuning of Parameter for Continuity of Schemes

Mask	Parameter	Continuity	Mask	Parameter	Continuity
$\alpha_{3,3}^{3}$	$-\frac{1}{6} < u_0 < \frac{1}{3}$	C^0	$\alpha_{3,3}^{5}$	$-\frac{1}{5} < u_0 < \frac{3}{10}$	C^0
……	$-\frac{1}{9} < u_0 < \frac{2}{9}$	C^1	……	$-\frac{2}{25} < u_0 < \frac{3}{25}$	C^1
……	$0 < u_0 < \frac{1}{9}$	C^2	……	$0 < u_0 < \frac{1}{25}$	C^2
$\alpha_{5,6}^{3}$	$-\frac{28}{729} < u_0 < \frac{11}{243}$	C^0	$\alpha_{5,5}^{5*}$	$-\frac{1}{50} < u_0 < \frac{13}{600}$	C^0
……	$-\frac{35}{1450} < u_0 < \frac{23}{729}$	C^1	……	$-\frac{59}{3750} < u_0 < \frac{11}{625}$	C^1
……	$-\frac{2}{81} < u_0 < \frac{8}{243}$	C^2	……	$-\frac{22}{3125} < u_0 < \frac{28}{3125}$	C^2
……	$-\frac{1}{162} < u_0 < \frac{1}{81}$	C^3	……	$-\frac{2}{625} < u_0 < \frac{3}{625}$	C^3
……	$0 < u_0 < \frac{1}{243}$	C^4	……	$0 < u_0 < \frac{2}{625}$	C^4
……	$u_0 = \frac{1}{729}$	C^5	……		

5. Subdivision-Regularization Framework

In this section, we use the framework of subdivision-regularization that is, use subdivision operator followed by regularization operator, for noise removal and prevention of data over fitting by a model. From experiments we have the following findings:

- When regularization couldn't prevent over fitting (or give unusual fitting) then subdivision-regularization framework can work better.

- When subdivision give over fitted model then you may get the best fitted model by subdivision-regularization.

- Frameworks of "higher arity schemes-regularization" can outperform those of "lower arity schemes-regularization", that is 3-point quinary scheme together with regularization can outperform the 3-point ternary scheme together with regularization.

- "Large support subdivision schemes-regularization" gives loose/ smoother fitted model compared to "small support subdivision schemes-regularization" that is "5-point ternary (quinary) scheme-regularization" may give loose fitted model than 3-point ternary (quinary) scheme-regularization.

We implement our procedure on noisy data of open and closed signals. We use 3- point ternary scheme (mask $\alpha^3_{3,3}$, parameter 1/27), 3-point quinary scheme (mask $\alpha^5_{3,3}$, parameter 1/25), 5-point ternary scheme (mask $\alpha^3_{5,6}$, parameter 1/729), 5- point quinary scheme (mask $\alpha^5_{5,5}$, at $u_0 = u_1 = 1/3125$) and regularization procedure with fixed parametric value 0.01.

In our first test, we consider noisy data of an open signal. Figure 6 (a) displays a noisy signal and model generated by regularization. It shows over fitting of regularization model on the noisy data. Figures 6 $(\mathrm{b}-\mathrm{e})$ shows a noisy signal and models generated by different subdivision schemes. Figures 6 $(\mathrm{f}-\mathrm{i})$ displays subdivision-regularization models together with noisy models. This shows significant improvement over regularization model i.e. Figure 6 (a) and subdivision models i.e., Figures 6 $(\mathrm{f}-\mathrm{i})$. It also shows that models in Figures 6 (h) and (i) are better than models in Figure 6 (f) & (g) respectively. This means that "higher arity schemes-regularization" can outperform "lower arity schemes-regularization". By comparing Figure 6 (f) and (h) with Figure 6 (g) and (i) we may conclude that "Large support subdivision schemes-regularization" gives a loose/ smoother fitted model compared to "small support subdivision schemes-regularization". In our second test, we consider noisy data of a closed signal. Figure 7 also shows that the subdivision-regularization model is better than subdivision as well as regularization models. At the moment we cannot propose an automatic way for selecting the values of the parameter for the best model determination.

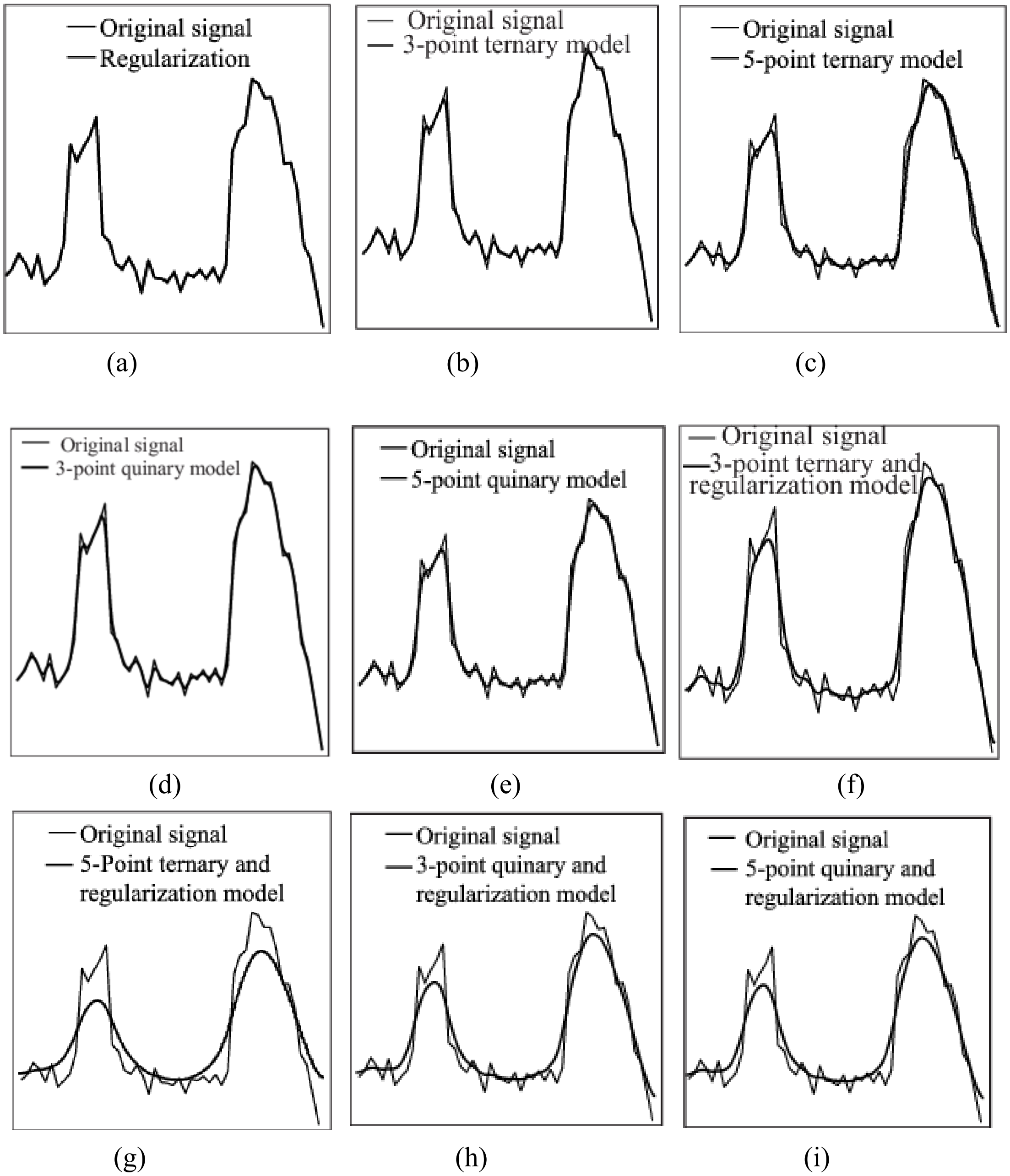

Figure 6. Comparison among regularization, subdivision and combine framework of subdivision-regularization: (a) Original signal and regularization model (b − e) Models generated by 3-, 5-point ternary and quinary schemes. (f − i) Models generated by framework of, 3-, 5-point ternary and quinary schemes and regularization

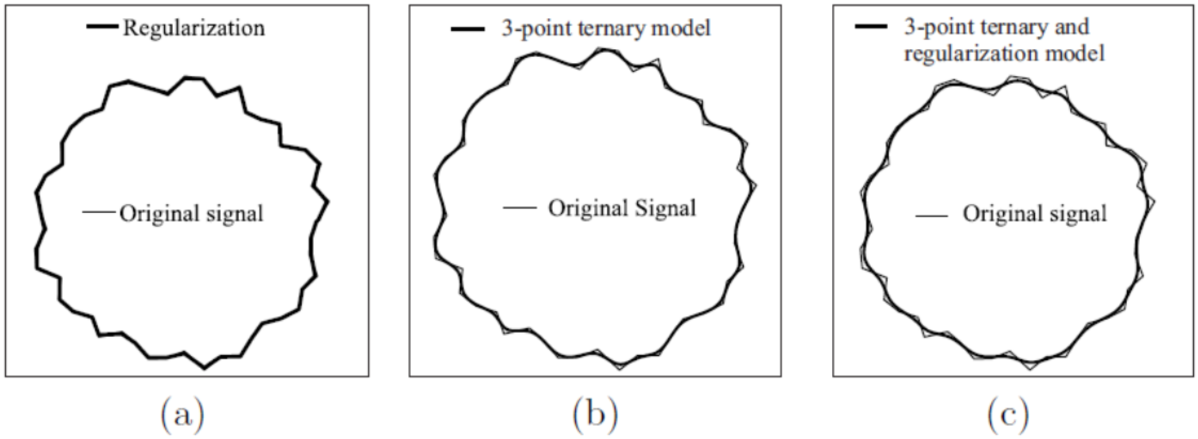

Figure 7. Comparison with original noisy signal: (a) Model generated by regularization (b) Model generated by 3-point ternary scheme (c) Model generated by 3-point ternary scheme and regularization

6. Conclusion

In this article, we have revisited an explicit general formula for the generation of mask of odd-point odd-ary approximating schemes of Aslam and Abeysinghe (2012). We have also studied its continuity and obtained error bounds for it. It is observed that error bound increases by increasing complexity of the schemes while it decreases by increasing the arity of the scheme. Moreover, we have showed that several previously proposed schemes are members of this family.

Next we propose and test the subdivision-regularization framework for preventing over fitting of a data by a model. We conclude that the subdivision regularization model is better than the subdivision as well as regularization models. We also conclude that frameworks of higher arity schemes-regularization" do outperform those of "lower arity schemes-regularization" and "Large support subdivision schemes-regularization" give smoother fitted model compared to "small support subdivision schemes-regularization". In the future, we plan to seek an automatic way for selecting the values of parameters involved in subdivision and regularization techniques for the best model determination.

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