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Solution of Fuzzy System of Linear Equations with Polynomial Parametric Form

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Abstract

This paper proposed two new and simple solution methods to solve a fuzzy system of linear equations having fuzzy coefficients and crisp variables using a polynomial parametric form of fuzzy numbers. Related theorems are stated and proved. The proposed methods are used to solve example problems. The results obtained are also compared with the known solutions and are found to be in good agreement.

Keywords: Fuzzy number, Fuzzy centre, Polynomial parametric fuzzy number, Fuzzy system of linear equations

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1. Introduction

System of linear equations plays a vital role in real life problems such as optimization, economics and in engineering. A standard real system may be written as $AX = B$, where, A and B are crisp real matrix and X is the unknown real vector. In general for the sake of simplicity or for easy computation the variables or parameters are usually taken as crisp numbers. But in the actual case the parameters may be uncertain or a vague estimation about the variables is known as those are found in general by some experiment or experience. So, these variables may be considered as a fuzzy number. In other words, to overcome the uncertainty, one may use fuzzy numbers [Zadeh (1975) and Zimmermann (2001)] in place of crisp numbers. Thus the system of linear equations becomes a Fuzzy System of Linear Equations (FSLE) which is an important area of research in the recent years.

As regards fuzzy real system of linear equations has been investigated by various authors. Different authors used different procedures to solve fuzzy systems. The solution methods depend upon the fuzziness of the variables, coefficient matrix, right hand side vector etc. Moreover due to the complexity of the fuzzy arithmetic, sometimes the methods do give non unique solution. Also the process may be lengthy and not so efficient etc. As such in this paragraph few of the literatures are reviewed for the sake of completeness of the problem. Although there are other papers in the open literature but here only the related papers are discussed. Cong-Xing and Ming (1991) used the concept of embedding approach for fuzzy number space. By using this embedding concept Friedman et al. (1991) proposed a general model for solving a fuzzy system of linear equation. Wang et al. (2001) discussed an iterative method for solving a system of linear equation of the form $X = AX + U$.

Asady et al. (2005) also studied a general fuzzy system and developed different methods using embedding approach. Solution of a system of linear equations with fuzzy numbers is also investigated by Horcik (2008). Vroman et al. (2007) used the parametric form of fuzzy number to solve the fuzzy general linear systems. Recently, Li et al. (2010) presented a new algorithm to solve fuzzy system of linear equations. Sevastjanov and Dymova (2009) proposed a new method for both the interval and fuzzy systems. Garg and Singh (2008) used numerical approach to solve fuzzy system of linear equations with Gaussian fuzzy membership function. Behera and Chakraverty (2012) very recently developed a new solution method which can handle both fuzzy real and complex system of linear equations. Also very recently Amirfakhrian (2012) proposed one solution method for solving system of fuzzy linear equations using fuzzy distance approach. Chakraverty and Behera (2012) developed centre and width based approach for solving fuzzy system of linear equations. Senthilkumara and Rajendran (2011) discussed an algorithmic approach for solving fuzzy linear systems. Authors [Das and Chakraverty (2012); Senthilkumara and Rajendran (2011)] also investigated fully fuzzy system of linear equations.

In most of these studies coefficient matrix is considered as real crisp whereas the unknown variable vector and right hand side vector are considered as fuzzy or whole as fuzzy. In this paper we have considered the fuzzy system of linear equations with fuzzy coefficients and crisp variables using polynomial parametric form of fuzzy numbers. The same type of problem is investigated by Amirfakhrian (2007) in an excellent way. Here fuzzy real system of linear equations is taken as

$$[\tilde{A}]\{X\} = \{\tilde{b}\},$$

where, the coefficient matrix $[\tilde{A}]$ is a real fuzzy matrix, $\{\tilde{b}\}$ is a column vector of fuzzy numbers and $\{X\}$ is the vector of crisp variables.

This paper targets to propose two new methods which can handle the real fuzzy linear systems. Accordingly the second section introduces the preliminaries with fuzzy arithmetic. Fuzzy system of linear equations with the proposed methodologies is explained in section three. In the fourth section numerical examples are discussed. Last section includes the conclusion.

2. Preliminaries

Let $F(R)$ be the set of all normal and convex fuzzy numbers on the real line [Amirfakhrian (2007)].

Definition 1. [Amirfakhrian (2007)]

A generalized LR fuzzy number $\tilde{A}$ with the membership function $\mu_{\tilde{A}}(x), x \epsilon R$ can be defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} l_{\tilde{A}}(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ r_{\tilde{A}}(x), & c \leq x \leq d, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $l_{\tilde{A}}(x)$ is the left membership function that is an increasing function in $[a, b]$ and $r_{\tilde{A}}(x)$ is the right membership function that is a decreasing function in $[c, d]$ such that $l_{\tilde{A}}(a) = r_{\tilde{A}}(d) = 0$ and $l_{\tilde{A}}(b) = r_{\tilde{A}}(c) = 1$. In addition, if $l_{\tilde{A}}(x)$ and $r_{\tilde{A}}(x)$ are linear, then $\tilde{A}$ is a trapezoidal fuzzy number, which is denoted by (a, b, c, d). If $b = c$, then we may write it as (a, c, d) or (a, b, d), which is a triangular fuzzy number.

The parametric form of a fuzzy number is given by $\tilde{v} = \left(\underline{v}(\alpha), \overline{v}(\alpha)\right)$, where functions $\underline{v}(\alpha)$ and $\overline{v}(\alpha)$; $0 \leq \alpha \leq 1$ satisfy the following requirements:

i. $\underline{v}(\alpha)$ is a bounded left continuous non-decreasing function over $[0, 1]$.

ii. $\overline{v}(\alpha)$ is a bounded right continuous non-increasing function over $[0, 1]$.

iii. $\underline{v}(\alpha) \leq \overline{v}(\alpha),\ 0 \leq \alpha \leq 1$.

Definition 2.

Fuzzy centre of an arbitrary fuzzy number $\tilde{v} = \left(\underline{v}(\alpha), \overline{v}(\alpha)\right)$ is defined as $\tilde{v}^c = \frac{\underline{v}(\alpha) + \overline{v}(\alpha)}{2}$, for all $0 \leq \alpha \leq 1$.

Definition 3.

As discussed above, fuzzy numbers may be transformed into an interval through parametric form. So, for any arbitrary fuzzy number $\tilde{x} = (\underline{x}(\alpha), \overline{x}(\alpha))$, $\tilde{y} = (\underline{y}(\alpha), \overline{y}(\alpha))$ and scalar k, we have the interval based fuzzy arithmetic as

i. $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$ and $\overline{x}(\alpha) = \overline{y}(\alpha)$

ii. $\tilde{x} + \tilde{y} = (\underline{x}(\alpha) + \underline{y}(\alpha),\ \overline{x}(\alpha) + \overline{y}(\alpha))$

iii. $\tilde{x} - \tilde{y} = (\underline{x}(\alpha) - \overline{y}(\alpha),\ \overline{x}(\alpha) - \underline{y}(\alpha))$

iv. $\tilde{x} \times \tilde{y} = [\min(\underline{x}(\alpha)\underline{y}(\alpha), \underline{x}(\alpha)\overline{y}(\alpha), \overline{x}(\alpha)\underline{y}(\alpha), \overline{x}(\alpha)\overline{y}(\alpha)),$

$\max(\underline{x}(\alpha)\underline{y}(\alpha), \underline{x}(\alpha)\overline{y}(\alpha), \overline{x}(\alpha)\underline{y}(\alpha), \overline{x}(\alpha)\overline{y}(\alpha))]$

v. $\tilde{x}/\tilde{y} = (\underline{x}(\alpha), \overline{x}(\alpha))/(\underline{y}(\alpha), \overline{y}(\alpha)) = (\underline{x}(\alpha)/\overline{y}(\alpha), \overline{x}(\alpha)/\underline{y}(\alpha))$ provided

$\underline{y}(\alpha) = \overline{y}(\alpha) \neq 0$

vi. $k\tilde{x} = \begin{cases} [k\overline{x}(\alpha), k\underline{x}(\alpha)], & k < 0 \\ [k\underline{x}(\alpha), k\overline{x}(\alpha)], & k \geq 0. \end{cases}$

Now some well-known facts about the fuzzy arithmetic are expressed below for better understanding of the problem as follows

i. $\tilde{x} - \tilde{y}$ can be represented as $\tilde{x} + (-1)\tilde{y}$.

ii. $\tilde{x} + \tilde{0} = \tilde{x}$

iii. $0 \times \tilde{x} = \tilde{0} = 0$.

In the above expressions $\tilde{0}$ is a zero fuzzy number. For triangular and trapezoidal zero fuzzy numbers $\tilde{0}$ may be represented as $(0, 0, 0)$ and $(0, 0, 0,0)$ respectively.

Definition 4: [Amirfakhrian (2007)]

We say a fuzzy number $\tilde{v}$ has m-degree polynomial form if there exist two polynomials $p_m(\alpha)$ and $q_m(\alpha)$, of degree at most m; such that $\tilde{v} = \left(p_m(\alpha),\ q_m(\alpha)\right)$, which is also called polynomial parametric fuzzy number.

3. Fuzzy System of Linear Equations

The $n \times n$ fuzzy system of linear equations with m −degree polynomial parametric form may be written as

$$\begin{aligned} &\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + \cdots + \tilde{a}_{1n}x_n = \tilde{b}_1 \\ &\tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 + \cdots + \tilde{a}_{2n}x_n = \tilde{b}_2 \\ &\vdots \\ &\tilde{a}_{n1}x_1 + \tilde{a}_{n2}x_2 + \cdots + \tilde{a}_{nn}x_n = \tilde{b}_n. \end{aligned} \tag{2}$$

In matrix notation the above system may be written as $[\tilde{A}]\{X\} = \{\tilde{b}\}$, where the coefficient matrix $[\tilde{A}] = (\tilde{a}_{kj}) = \left(\underline{a}_{kj}(\alpha),\ \overline{a}_{kj}(\alpha)\right) = (\underline{A}, \overline{A})$, $1 \le k,\ j \le n$ is a fuzzy $n \times n$ matrix, $\{\tilde{b}\} = \{\tilde{b}_k\} = \left\{\left(\underline{b}_k(\alpha),\ \overline{b}_k(\alpha)\right)\right\} = \{(\underline{b}, \overline{b})\}$, $1 \le k$ is a column vector of fuzzy number and $\{X\} = \{x_j\}$ is the vector of crisp unknown. For positive integer m, all $\tilde{a}_{kj}$ and $\tilde{b}_k$ are fuzzy numbers with m −degree polynomial form.

The above system $[\tilde{A}]\{X\} = \{\tilde{b}\}$, can be written as

$$\sum_{j=1}^{n} \tilde{a}_{kj} x_j = \tilde{b}_k \text{ for } k1,2,\dots,n. \tag{3}$$

As per the parametric form we may write Equation (3) as

$$\sum_{j=1}^{n} \left(\underline{a}_{kj}(\alpha), \overline{a}_{kj}(\alpha)\right) x_j = \left(\underline{b}_k(\alpha), \overline{b}_k(\alpha)\right), \text{ for } k = 1,2,\dots,n\,. \tag{4}$$

Equation (4) can equivalently be written as the following two equations Eqs. (5) and (6)

$$\sum_{x_j \ge 0} \underline{a}_{kj}(\alpha) x_j + \sum_{x_j < 0} \overline{a}_{kj}(\alpha) x_j = \underline{b}_k(\alpha) \tag{5}$$

and

$$\sum_{x_j \ge 0} \overline{a}_{kj}(\alpha) x_j + \sum_{x_j < 0} \underline{a}_{kj}(\alpha) x_j = \overline{b}_k(\alpha). \tag{6}$$

3.1. Proposed Methods for Solving Fuzzy System of Linear Equations

Method 1:

In this section we propose a new method to solve fuzzy system of linear equations. One related theorem is stated and proved related to the present procedure in the following paragraph.

Theorem 1.

If $\{X\}$ is the solution vector of the fuzzy system $[\tilde{A}]\{X\} = \{\tilde{b}\}$, then $\{X\}$ is the solution vector of the crisp system of linear equation $[\underline{A} + \overline{A}]\{X\} = \{\underline{b} + \overline{b}\}$.

Proof:

Let us now first consider the left hand side of the system $[\underline{A} + \overline{A}]\{X\} = \{\underline{b} + \overline{b}\}$.
Hence one may write $[\underline{A} + \overline{A}]\{X\}$ as

$$\sum_{j=1}^{n}\left(\underline{a}_{kj}(\alpha)+\overline{a}_{kj}(\alpha)\right)\{x_j\}, \qquad \text{for } k=1,2,\dots,n.$$

This can be written as

$$\sum_{j=1}^{n}\underline{a}_{kj}(\alpha)x_j+\sum_{j=1}^{n}\overline{a}_{kj}(\alpha)x_j,$$

which is equivalent to

$$\sum_{x_j\geq 0}\underline{a}_{kj}(\alpha)x_j+\sum_{x_j<0}\underline{a}_{kj}(\alpha)x_j+\sum_{x_j\geq 0}\overline{a}_{kj}(\alpha)x_j+\sum_{x_j<0}\overline{a}_{kj}(\alpha)x_j. \tag{7}$$

Using Equations (5) and (6), the above expression can be written as combining first with fourth term and second with third term respectively in Equation (7) we get,

$$\left\{\underline{b}_k(\alpha)+\overline{b}_k(\alpha)\right\}=\left\{\underline{b}+\overline{b}\right\}.$$

Thus, we have $\left[\underline{A}+\overline{A}\right]\{X\}=\left\{\underline{b}+\overline{b}\right\}$. This proves that $\{X\}$ is the solution vector of the system $\left[\underline{A}+\overline{A}\right]\{X\}=\left\{\underline{b}+\overline{b}\right\}$. □

Method 2:

Similarly to find the crisp solution of fuzzy system of linear equations as discussed in Method 1, here one related theorem is stated.

Theorem 2.

If $\{X\}$ is the solution vector of the fuzzy system $\left[\tilde{A}\right]\{X\}=\{\tilde{b}\}$, then $\{X\}$ is the solution vector of the crisp system of linear equation $\left[\tilde{A}^c\right]\{X\}=\{\tilde{b}^c\}$, where $\tilde{A}^c=\left(\underline{a}_{kj}(\alpha)+\overline{a}_{kj}(\alpha)\right)/2$ and $\tilde{b}^c=\left(\underline{b}_k(\alpha)+\overline{b}_k(\alpha)\right)/2$.

Proof:

The proof is straight forward as in Theorem 1.

4. Numerical Examples

Example 1.

Let us consider the 2×2 fuzzy system of linear equations [Amirfakhrian (2007)] where $m = 1$ (by Definition 4) as

$$(-1+2\alpha, 4-2\alpha)x_1 + (-2+3\alpha, 3-2\alpha)x_2 = (-8+13\alpha, 17-10\alpha)$$

$$(1+\alpha, 4-\alpha)x_1 + (2\alpha, 5-2\alpha)x_2 = (2+8\alpha, 23-8\alpha).$$

Using the procedure of Method 1 as discussed in Theorem 1 the above system is now converted to the following system as

$$(-1+2\alpha+4-2\alpha)x_1 + (-2+3\alpha+3-2\alpha)x_2 = (-8+13\alpha+17-10\alpha)$$

$$(1+\alpha+4-\alpha)x_1 + (2\alpha+5-2\alpha)x_2 = (2+8\alpha+23-8\alpha).$$

The above system is now equivalent to the following system as

$$3x_1 + (1+\alpha)x_2 = 9+3\alpha$$

$$5x_1 + 5x_2 = 25.$$

Solving the corresponding system one may have $x_1 = 2$ and $x_2 = 3$. Which is also the solution of the main system. Similarly one may also implement Method 2 to have the same solution.

Example 2.

Now let us consider the 2×2 fuzzy system of linear equations [Amirfakhrian (2007)] as

$$(-1+\alpha, 3-\alpha)x_1 + (1+2\alpha, 4-\alpha)x_2 = (-12+11\alpha, 17-8\alpha)$$

$$(-1+2\alpha, 3-2\alpha)x_1 + (3\alpha, 6-2\alpha)x_2 = (-15+19\alpha, 23-16\alpha),$$

where $m = 1$ (by Definition 4). Similarly as discussed in Example 1, Methods 1 and 2 may be used here to obtain the solution as $x_1 = -5$ and $x_2 = 3$.

Example 3.

Again consider the 2×2 fuzzy system of linear equations [Amirfakhrian (2007)] as

$$\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 = \tilde{b}_1$$

$$\tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 = \tilde{b}_2,$$

where $m = 2$ (by Definition 4) and

$$\tilde{a}_{11} = (3r + \alpha^2, 7 - 3r + 2\alpha^2)$$

$$\tilde{a}_{12} = (2\alpha + \alpha^2, 4 - 2\alpha + 2\alpha^2)$$

$$\tilde{a}_{21} = (1 + 2\alpha + \alpha^2, 8 - 3\alpha + \alpha^2)$$

$$\tilde{a}_{22} = (1 + 2\alpha + \alpha^2, 6 - 3\alpha + 2\alpha^2)$$

$$\tilde{b}_1 = (48.45\alpha + 17.1\alpha^2, 111.15 - 48.45\alpha + 34.2\alpha^2)$$

$$\tilde{b}_2 = (17.1 + 34.2\alpha + 17.1\alpha^2, 131.1 - 51.3\alpha + 19.95\alpha^2).$$

Using Methods 1 and 2 one may obtain $x_1 = 14.25$ and $x_2 = 2.85$.

Example 4. [Amirfakhrian (2007)]

Let us consider a fuzzy linear equation for $m = 3$ (by Definition 4) as $\tilde{a}x = \tilde{b}$, where, $\tilde{a} = \tilde{b}$. We have $\tilde{a} = (\underline{a}(\alpha), \ \overline{a}(\alpha))$ and

$$\underline{a}(\alpha) = 2.01846 + 1.4866\alpha - 0.64225\alpha^2 + 0.13718\alpha^3$$

$$\overline{a}(\alpha) = 3.98154 - 1.4866\alpha + 0.64225\alpha^2 - 0.13718\alpha^3.$$

Using Methods 1 and 2 one may obtain $x = 1$.

Obtained results by the present method are now compared with the solution obtained by Amirfakhrian (2007) and are tabulated in Table 1 for all the examples as discussed in Section 4. The results obtained by the proposed method are exactly same as that of Amirfakhrian (2007) for Examples 1, 2 and 4. But the results of Example 3 are quite different from Amirfakhrian (2007). It is worth mentioning that the results obtained by the present method is exactly satisfies the corresponding system whereas for Amirfakhrian (2007) it is not. The solution by Amirfakhrian (2007) may have some typographical error.

Table 1. Comparison of results between Amirfakhrian (2007) and present method for example problems

Examples	Amirfakhrian (2007)	Present method
1	$x_1 = 2$ and $x_2 = 3$	$x_1 = 2$ and $x_2 = 3$
2	$x_1 = -5$ and $x_2 = 3$	$x_1 = -5$ and $x_2 = 3$
3	$x_1 = 14.85$ and $x_2 = 2.25$	$x_1 = 14.25$ and $x_2 = 2.85$
4	$x = 1$	$x = 1$

The main advantages of the proposed methods are that here in the solution procedure the order of original fuzzy systems does not change. But generally in the other methods, order of the fuzzy system changes just by double of its original order. So the present procedures have less effort to solve therefore it is computationally efficient.

5. Conclusions

In this paper a general fuzzy linear system of equations having fuzzy coefficients and crisp variables using a polynomial parametric form of fuzzy numbers is solved by new and simple proposed procedures. Choosing m depends on the shape of left and right spread functions L and R, and their derivation order. The proposed methods can be applied to any system of equations with LR fuzzy number coefficients.

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REFERENCES

Asady, B., Abbasbandy, S. and Alavi, M. (2005). Fuzzy general linear systems, Applied Mathematics and Computation, Vol. 169, pp. 34-40.

Amirfakhrian, M. (2007). Numerical solution of fuzzy system of linear equations with polynomial parametric form, International Journal of Computer Mathematics, Vol. 84, pp. 1089-1097.

Amirfakhrian, M. (2012). Analyzing the solution of a system of fuzzy linear equations by a fuzzy distance, Soft Computing, Vol. 16, pp. 1035-1041.

Behera, Diptiranjan and Chakraverty, S. (2012). A new method for solving real and complex fuzzy system of linear equations, Computational Mathematics and Modeling, Vol. 23, pp. 507-518.

Chakraverty, S. and Behera, Diptiranjan (2012). Fuzzy system of linear equations with crisp coefficients, Journal of Intelligent and Fuzzy Systems, DOI: 10.3233/IFS-2012-0627 (In Press).

Cong-Xing, W. and Ming, M. (1991). Embedding problems of fuzzy number space: part I, Fuzzy Sets and Systems, Vol. 44, pp. 33-38.

Das, S. and Chakraverty, S. (2012). Numerical solution of interval and fuzzy system of linear equations, Applications and Applied Mathematics: An International Journal (AAM), Vol. 7, pp. 334-356.

Friedman, M., Ming, M. and Kandel, A. (1991). Fuzzy linear systems, Fuzzy Sets and Systems, Vol. 96, pp. 201-209.

Garg, Anjeli and Singh, S. R. (2008). Solving fuzzy system of equations using Gaussian membership function, International Journal of Computational Cognition, Vol. 7, pp. 25-32.

Horcik, R. (2008). Solution of a system of linear equations with fuzzy numbers, Fuzzy Sets and Systems, Vol. 159, pp. 1788-1810.

Kaufmann, A., Gupta M. M. (1985). *Introduction to Fuzzy Arithmetic,* Van Nostrand Reinhold Company, New York.

Li, J., Li, W. and Kong, X. (2010) A new algorithm model for solving fuzzy linear systems, Southest Asian Bulletin of Mathematics, Vol. 34, pp. 121-132.

Senthilkumara, P. and Rajendran, G., (2011). An algorithmic approach to solve fuzzy linear systems, Journal of Information & Computational Science, Vol. 8, pp. 503-510

Senthilkumara, P. and Rajendran, G., (2011). New approach to solve symmetric fully fuzzy linear systems, Sadhana, Vol. 36, pp. 933-940.

Sevastjanov, P. and Dymova, L. (2009). A new method for solving interval and fuzzy equations: linear case, Information Sciences, Vol. 179, pp. 925-937.

Vroman, A., Deschrijver, G. and Kerre, E. E. (2007). Solving systems of linear fuzzy equations by parametric functions, IEEE Transaction on Fuzzy Systems, Vol. 15, pp. 370–384.

Wang, X., Zhong, Z. and Ha, M. (2001). Iteration algorithms for solving a system of fuzzy linear equations, Fuzzy Sets and Systems, Vol. 119, pp. 121–128.

Zadeh, L. A., Fu, K. S. and Shimura, M. (1975). *Fuzzy Sets and their Applications to Cognitive and Decision Processes*, Academic Press, New York/San Francisco/London.

Zimmermann, H. J. (2001). *Fuzzy Set Theory and its Application*, Kluwer academic publishers, Boston/Dordrecht/London.