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On The Geometrıc Interpretatıons of The Kleın-Gordon Equatıon And Solution of The Equation by Homotopy Perturbation Method

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Abstract

This paper is organized in the following ways: In the first part, we obtained the Klein Gordon Equation (KGE) in the Galilean space. In the second part, we applied Homotopy Perturbation Method (HPM) to this differential equation. In the third part, we gave two examples for the Klein Gordon equation. Finally, We compared the numerical results of this differential equation with their exact results. We also showed that approach used is easy and highly accurate.

Keywords: The Galilean space, The Homotopy perturbation method, The Klein-Gordon Equation, Linear and nonlinear partial differential equations

MSC 2010 No.: 35A20, 35A25, 81Q05

1. Introduction

1.1. The Klein-Gordon, Sine-Gordon and Sinh-Gordon Equations

The relativistic wave equation of the motion of a free particle with zero spin found by physicists O. Klein and V. Gordon is called Klein-Gordon equation (KGE). If the particle is characterized by one space coordinate x , and then the equation has the form as

$$u_{tt} - u_{xx} = m^2 u. \tag{1.1}$$

where t is time, $u = u(t,x)$ is wave function, and m is the mass of the particle. Note that Equation (1.1) for $m = 0$ is a classical one-dimensional wave equation (the equation of a vibrating string) whose solution $u(t,x)$ has the form $f(x+t) + f(x-t)$, where $f(t)$ is an arbitrary function. By analogy with KGE the equations

$$u_{tt} - u_{xx} = \sin u \tag{1.2}$$

$$u_{tt} - u_{xx} = \sinh u \tag{1.3}$$

are called Sine-Gordon equations (SGE) and Sinh-Gordon equation (SHGE), respectively. Equation (1.2) has also an important physical meaning: since the left-hand side of this equation coincides with the left-hand side of the equation of a vibrating string of the wave [Inc and Ugurlu (2007)] Equation (1.1), it is also a wave equation, but, unlike Equation (1.1), it is nonlinear (Liao, He (2003, 2004, 2006, 2007, Bektas et al. (2004)) and describes physical processes related to the nonlinear [Adomian (1986)] waves, in particular solitary waves (solitons) [Drazin(1989)] which preserve their shape under interaction. This theory is very important for the theory of plasm.

The n-dimensional Galilean and pseudo-Galilean spaces Γ^n and Γ_1^n can be defined as the affine space E^n whose hyperplane at infinity is endowed by the geometry of the Euclidean space Rosenfeld (1969) R^{n-1} or the pseudo – Euclidean space R_1^{n-1} [see, Rosenfeld (1969) pp. 295-297]. If a system of affine coordinates in the space E^n is chosen such that the basis vectors $e_2, e_3, \cdots, e_n$ are directed to the hyperplane at infinity of R^{n-1} or R_1^{n-1}, the distance d between two points $X = (x^1, x^2, \cdots, x^n)$ and $Y = (y^1, y^2, \cdots, y^n)$ is equal to $\left|y^1 - x^1\right|$. If $x^1 = y^1$, when $d = 0$, then these points have another distance d^1 equal to the distance between the points $X^1 = (x^2, x^3, \cdots, x^n)$ and $Y^1 = (y^2, y^3, \cdots, y^n)$ in R^{n-1} or R_1^{n-1}, respectively. The motions in Γ^n and Γ_1^n have the form

$$^1x^1 = x^1 + a^1, \quad {}^1x^i = A_I^i x^1 + A_j^i x^i + a^i \quad (i, j = 2, 3, \cdots, n),$$

where (A_j^i) is an orthogonal or pseudo–orthogonal $(n-1)\times(n-1)$– matrix, respectively (here the Einstein rule for summation is used). These formulas coincide with the formulas of transformation of orthogonal coordinates in the n-dimensional isotropic or pseudo isotropic spaces I^n or I_I^n, respectively, I^n is the n-dimensional affine space E^n whose hyperplane at infinity is endowed with the geometry of the co-Euclidean space $(R^{n-I})^*$ or the copseudo-Euclidean space $(R_I^{n-I})^*$ corresponding to R^{n-I} and R_I^{n-I} in the duality principle of the projective space P^{n-I}. The motions in I^n and I_I^n have the from

$$^{1}x^{1} = x^{1} + A_{i}^{1}x^{i} + a^{1}, \quad {}^{i}x^{i} = A_{j}^{i}x^{j} + a^{i}, \ (i, j = 2,3,\cdots,n).$$

The hyperplane at infinity of R^{n-1} and R_{I}^{n-1}, that is, the $(n-2)$ plane $x^{1} = 0$ in the hyperplane $x^{0} = 0$, and the absolute imaginary or real hyperquadric in this $(n-2)$-plane, which is the intersection of all hyperspheres in R^{n-1} or R_{I}^{n-1}, form the absolutes of Γ^{n} and Γ_{1}^{n}. For Γ^{3} and Γ_{1}^{3} the absolutes consist of the plane at infinity, line $x^{1} = 0$, and two imaginary conjugate or real points on this line. Depending on a position relative to the absolute of For Γ^{3} and Γ_{1}^{3} the lines and planes in these spaces are divided into two classes: lines of general position which do not meet the line $x^{1} = 0$, and special lines which meet the line: planes of general position which do not contain the line $x^{1} = 0$ (the planes Γ^{2} or Γ_{1}^{2}),and special planes which contain this line (the planes R^{2} or R_{1}^{2}).

At each point X in Γ^{3} or Γ_{1}^{3} we determine orthonormal frames which consist of vectors e_{1}, e_{2}, e_{3} of length 1 or i such that the line Xe_{1} is a line of general position, and the lines Xe_{2}, Xe_{3} are special lines which divide harmonically the lines joining X with two imaginary conjugate or real points of the absolute whose equations will be written as

$$g_{22}\left(x^{2}\right)^{2} + g_{33}\left(x^{3}\right)^{3} = 0,$$

where $g_{22} = g_{33} = 1$ for Γ^{3} and $g_{22} = -g_{33} = \pm 1$ for Γ_{1}^{3}.

If a point X is characterized by a position vector **x**, then the derivation formulas for these frames are

$$dx = \omega^{i}e_{i}, \quad de_{1} = \omega_{1}{}^{u}e_{u}, \quad de_{2} = \omega_{2}{}^{3}e_{3}, \quad de_{3} = \omega_{3}{}^{2}e_{2}, \quad \omega_{3}{}^{2} = -\delta\omega_{2}{}^{3},$$

where $i, j = 1,2,3, \quad u = 2,3, \quad \delta = 1$ for Γ^{3} and $\delta = -1$ for Γ_{1}^{3}. Exterior differentiation of these equations gives

$$d\omega^{I} = 0, d\omega^{u} = \omega^{i} \wedge \omega_{i}^{u}, \quad d\omega_{1}^{u} = \omega_{1}^{v} \wedge \omega_{v}^{u}, \quad d\omega_{2}^{3} = 0. \ , \tag{1.4}$$

where $v = 2,3$. Formulas (1.4) show that the linear forms ω^{1} and ω_{2}^{3} are locally exact differentials, therefore

$$\omega^{1} = du, \qquad \omega_{2}^{3} = dv. \tag{1.5}$$

Consider a curve C of general position in Γ^{3} or Γ_{1}^{3}. If X is a point on this curve, e_1 is tangent vector to this curve at X, e_{2} is a special vector of the oscillating plane of this curve at X, and e_{3} is the third vector of an orthonormal frame. The derivation equations of this curve are

$$\frac{dx}{dt} = e_1, \quad \frac{de_1}{dt} = ke_2, \quad \frac{de_2}{dt} = \kappa e_3, \quad \frac{de_3}{dt} = -\delta\kappa e_2, \tag{1.6}$$

where t is the natural parameter He (2001), k and κ are the curvature and the torsion of the curve.

Consider a surface S of general position in Γ^3 or Γ_1^3. Let us suppose that the intersections of this surfaces with Euclidean or pseudo-Euclidean planes x^1 = const. are not straight lines, that is, this surface has no special rectilinear generators. We determine at a point X of this surface the orthonormal frame, whose vectors e_1 and e_2 are tangent vectors to S at X and e_3 is the normal vector to S at X, that is, this vector is orthogonal to e_2 (the vectors e_2 and e_3 of this are in a plane $x^1 = 0$). The differential equation of Pfaff of the surface S is

$$\omega^3 = 0. \tag{1.7}$$

The exterior differentiation of the equation gives

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0, \tag{1.8}$$

hence, by means of the Cartan lemma, we obtain

$$\omega^3{}_1 = a\omega^1 + b\omega^2, \quad \omega_2^3 = b\omega^1 + c\omega^2 \;. \tag{1.9}$$

The first fundamental forms of the surface S for general curves are

$$I = ds^2 = \left(\omega^1\right)^2, \tag{1.10}$$

and for special curves are

$$I_1 = \left(ds_I\right)^2 = g_{22}\left(\omega^2\right)^2 \;, \tag{1.11}$$

that is, intersections of S, with planes $x = 1$.

The second fundamental form of the surface S is

$$II = \left(d^2a, e_3\right) = g_{33}\left(\omega^1\omega_1^3 + \omega^2\omega_2^3\right) = g_{22}\left[a\left(\omega^1\right)^2 + 2b\omega^1\omega^2 + c\left(\omega^2\right)^2\right]. \tag{1.12}$$

We call a surface in Γ_1^3 spacelike if $g_{22} = +1$ and timelike if $g_{22} = -1$. The line of the absolute determines on the surface S, the Koenigs net consisting of special curves which are intersections of S with the planes $x^I = 0$, and of curves of tangency of cones with apices on the line of the absolute tangent to S. The curves of this net are curvature lines of S, since normal lines to S along curves of this net forms developable surface.

At the points of the curvature lines of S of general position, vectors e_2 of moving frames have constant directions, since they are directed to the apices of cones, therefore for curvature curves of general position, $\omega_2^3 = 0$. It follows from (1.5), that the equations of curvature curves of S are $u = \text{const.}\ v = \text{const}$. The coordinates u and v are called canonical coordinates on the surface S.

The principal curvatures of S, that is normal curvatures $k_n = II/I$ for curves $\omega^1 = 0$ and $\omega_2^3 = 0$ are, respectively,

$$k_1 = \delta c, \quad k_2 = g_{33}\frac{ac-b^2}{c}. \tag{1.13}$$

Hence, the Gaussian curvature $K = K_e = k_1 k_2$ of the surface S is

$$K = k_1 k_2 = \delta g_{33}\left(ac - b^2\right). \tag{1.14}$$

Therefore, the Gaussian curvature of a surface S in Γ^3 and of a timelike surface in Γ_1^3 is equal to $ac - b^2$ and of a spacelike surface in Γ_1^3 is equal to $b^2 - ac$. Note that the condition for the surfaces S in Γ^3 and Γ_1^3 which have no special rectilinear generators is the equality $c = 0$.
The curves on a surface S which are determined by the equation II = 0 are asymptotic curves. Since the form II is expressed by the formula (1.12), the condition for finding asymptotic directions $\varphi = \omega^2/\omega^1$ of general position is

$$a + 2b\varphi + c\varphi^2 = 0. \tag{1.15}$$

In the case when vector e_1 of the moving frame is fixed at every point A on S, the moving frame is canonical and all other forms are principal, that is

$$\omega_1^2 = a\omega_1 + \beta\omega^2. \tag{1.16}$$

The exterior differentiation of the forms (1.9) and (1.16) and the substitution of expression (1.7), (1.16), (1.14) and (1.16) into (1.4) give

$$-\alpha_2 + \beta_1 + \beta = g_{33}K, \tag{1.17}$$

$$-a_2 + 2b\beta + b_1 - ca = 0, \tag{1.18}$$

$$-b_2 + c_1 + c\beta = 0, \tag{1.19}$$

where the indices p_1 and p_2 mean Pfaffian derivatives determined by the formula

$$dp = p_I \omega^I + p_2 \omega^2 .$$

Let the vector e_1 e1 be tangent to *a* curvature curve of general position, that is, the coordinate system on the surface S is canonical system *u*, *v*, and let us find the corresponding differential forms of the moving frame. Since the curvature of a special curvature curve is $k_1 = dv/ds_1$, where *v* is on angle between tangent lines and s_1 is the length of special curve then formulas (1.11) and (1.13) imply that $dv = cw^2$. Let us denote the radius of curvature of this curve by

$$\gamma = k_1^{-1} = \delta c^{-1} \tag{1.20}$$

then we have

$$\omega^2 = \delta\gamma dv . \tag{1.21}$$

By the substitution (1.21) into the second formula (1.16) and by formulas (1.5) and $a = 0$, we obtain

$$b = 0 . \tag{1.22}$$

Therefore, the formulas (1.13) have the from

$$k_1 = \delta c, \quad k_2 = g_{33} a . \tag{1.23}$$

By formulas (1.5) and (1.21) we can express the Pfaffian derivatives p_1 and p_2 through the partial derivates p_u and p_v in the form $p_1 = p_u$, $p_2 = cp_v$. Therefore, by the equations (1.18), (1.19) and (1.22) we obtain

$$\beta = -\gamma c_u - \frac{\gamma_u}{\gamma} , \qquad \alpha = -a_v ,$$

that is,

$$\omega_1^2 = -a_v \omega^1 + \gamma_u \gamma^{-1} \omega^2 = -a_v du + \delta \gamma_u dv.$$

In this case formula (1.17) can be written as

$$a_{vv} + \delta \gamma_{uu} = -\delta a . \tag{1.24}$$

This formula shows that the conditions of integrability of the differential equations of a surface S are reduced to single differential equation for its principal curvatures k_1 and k_2.

Let a surface S have a curvature $K = \text{const} = \varepsilon m^2$ where $\varepsilon = \pm 1$. When a surface S is referred to the canonical coordinates, let us show that the equation (1.24) can be reduced to the form (1.1). In this case formulas (1.23) can be written as

$$k_1 = \delta c = \gamma^{-1}, \quad k_2 = K k_1^{-1} = \varepsilon \delta m^2 \gamma$$

and also formula (1.24) can be written as

$$\varepsilon g_{33} m^2{}_{vv} + \delta \gamma_{uu} = -\delta \varepsilon g_{33} m^2 \gamma \, . \tag{1.25}$$

Let us set $\varepsilon = -1$ for a surface in Γ^3 and for a spacelike surface in Γ_1^3; then we obtain

$$\gamma_{uu} - m^2 \gamma_{vv} = m^2 \gamma \, ,$$

that is, if we denote u by t, v/m by x, the function γ by u for $m = 1$, we obtain an equation

$$u_{tt} - u_{xx} = u \, .$$

2. An Analysis of the HPM

A lot of methods have been used to obtain solutions of partial differential equations in the literature [He and Elagan (2011)]. We consider HPM which is one of the most used methods. The first of all, we must obtain form HMP [Abbasbandy (2006), He (1999)] for HPM. To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (2.1) can therefore be rewritten as follows;

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{2.3}$$

Homotopy perturbation structure is shown as following;

$$\mathrm{H}(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[A(v)-f(r)\right]=0, \tag{2.4}$$

where

$$v(r,p):\Omega\times[0,1]\to\Re. \tag{2.5}$$

In Equation (2.4), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Equation (2.4) can be written as a power series in p, as following;

$$v=v_0+pv_1+p^2v_2+p^3v_3+\cdots, \tag{2.6}$$

and the best approximation for solution is

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+v_3+\cdots, \tag{2.7}$$

The convergence of series Equation (2.7) has been proved by He (2000). This technique can have full advantage of the traditional perturbation techniques. The series Equation (2.7) is convergent rate depends on the non-linear operator $A(v)$ (the following opinions are suggested by He (2000):

(1) The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively largee, i.e., $p\to 1$.

(2) The norm of $L^{-1}(\partial N/\partial v)$ must be smaller than one so that the series converges.

3. Applications of The HPM

Example 1.

We consider a linear partial differential equation in order to illustrate the technique discussed above. The problem of the form is

$$u_{tt}-u_{xx}=-u, \tag{3.1}$$

where exact solution of the differential Equation (3.1) is given by;

$$u(x,t)=(x+1)\sin t \tag{3.2}$$

and initial conditions

$$u_t(x,0)=x+1.$$

Structure of HPM He (2004, 2003, 2001, 2009) for Equation (3.1) is

$$(1-p)[\ddot{Y}-\ddot{u}_0]+p\left[\ddot{Y}-Y''+Y\right]=0\,, \tag{3.3}$$

$$\ddot{Y}-\ddot{u}_0-p\ddot{Y}+p\ddot{u}_0+p\ddot{Y}-pY''+pY=0\,,$$

$$\ddot{Y}-\ddot{u}_0+p\ddot{u}_0-pY''+pY=0\,, \tag{3.4}$$

where $\ddot{Y}=\dfrac{\partial^2 y}{\partial t^2}$, $Y''=\dfrac{\partial^2 y}{\partial x^2}$ and $p\in[0,1]$. We suppose that the solution of Equation (3.1) has the form as following;

$$Y=Y_0+pY_1+p^2Y_2+p^3Y_3+\cdots=\sum_{n=0}^{\infty}p^nY_n(x,t) \tag{3.5}$$

$$\ddot{Y}=\ddot{Y}_0+p\ddot{Y}_1+p^2\ddot{Y}_2+p^3\ddot{Y}_3+\cdots,$$
$$Y''=Y_0''+pY_1''+p^2Y_2''+p^3Y_3''+\cdots.$$

Then, substituting Equation (3.5) into Equation (3.4), and rearranging based on powers of p -terms, we obtain:

$$\left(\ddot{Y}_0+p\ddot{Y}_1+p^2\ddot{Y}_2+p^3\ddot{Y}_3\right)-\ddot{u}_0+p\ddot{u}_0-p\left(Y_0''+pY_1''+p^2Y_2''+p^3Y_3''\right)+p\left(Y_0+pY_1+p^2Y_2+p^3Y_3\right)=0\,,$$

$$\ddot{Y}_0+p\ddot{Y}_1+p^2\ddot{Y}_2+p^3\ddot{Y}_3-\ddot{u}_0+p\ddot{u}_0-pY_0''-p^2Y_1''-p^3Y_2''+pY_0+p^2Y_1+p^3Y_2=0\,,$$

$$p^0:\ddot{Y}_0-\ddot{u}_0=0 \tag{3.6}$$

$$p^1:\ddot{Y}_1+\ddot{u}_0-Y_0''+Y_0=0 \tag{3.7}$$

$$p^2:\ddot{Y}_2-Y_1''+Y_1=0 \tag{3.8}$$

$$p^3:\ddot{Y}_3-Y_2''+Y_2=0 \tag{3.9}$$

with solving Equation (3.6- 3.9);

$$p^0:\ddot{Y}_0-\ddot{u}_0=0\Rightarrow Y_0=u_0(x,0)=x+1, \tag{3.10}$$

$$p^1 : \ddot{Y}_1 + \ddot{u}_0 - Y_0'' + Y_0 = 0 \Rightarrow \ddot{Y}_1 = -\ddot{u}_0 + Y_0'' - Y_0$$

$$\Rightarrow Y_1 = \int_0^t \int_0^t \left[-\ddot{u}_0 + Y_0'' - Y_0 \right] dt\, dt$$

$$\Rightarrow Y_1 = -\frac{1}{6} t^3 \left(x+1 \right),$$

$$p^2 : \ddot{Y}_2 - Y_1'' + Y_1 = 0 \Rightarrow \ddot{Y}_2 = Y_1'' - Y_1$$

$$\Rightarrow Y_2 = \int_0^t \int_0^t \left[Y_1'' - Y_1 \right] dt\, dt \tag{3.11}$$

$$\Rightarrow Y_2 = \frac{1}{5!} t^5 \left(x+1 \right),$$

$$p^3 : \ddot{Y}_3 - Y_2'' + Y_2 = 0 \Rightarrow \ddot{Y}_3 = Y_2'' - Y_2$$

$$\Rightarrow Y_3 = \int_0^t \int_0^t \left[Y_2'' - Y_2 \right] dt\, dt \tag{3.12}$$

$$\Rightarrow Y_3 = -\frac{1}{7!} t^7 (x+1),$$

$$\vdots$$

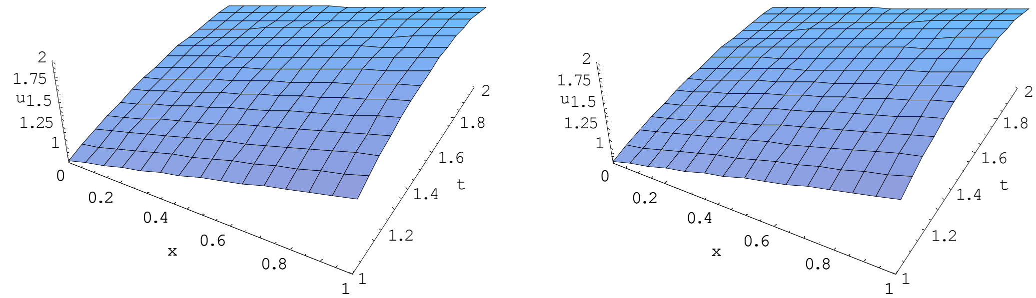
the terms of Equation (3.5) could easily calculated. When we consider the series Equation (3.5) with the terms Equations (3.6-3.9) and suppose $p = 1$, we obtain approximation solution of Equation (3.1) as following;

$$u\left(x,t\right) = Y_0 + Y_1 + Y_2 + Y_3 + \cdots. \tag{3.13}$$

As a result, the components $Y_0, Y_1, Y_2, Y_3, \cdots$ are identified. We obtain analytic solution of Equation (3.1) as following;

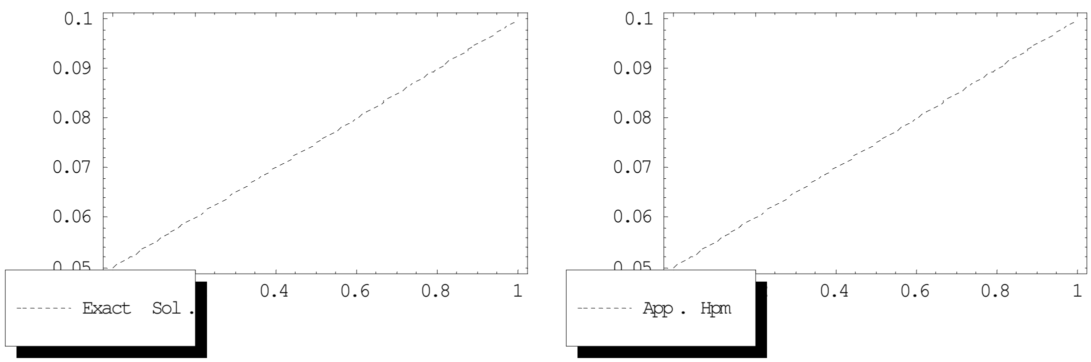
$$u\left(x,t\right) = t\left(x+1\right) - \frac{1}{6} t^3 \left(x+1\right) + \frac{1}{120} t^5 \left(x+1\right) - \frac{1}{5040} t^7 (x+1) + \cdots$$

$$u\left(x,t\right) = \left(x+1\right) \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \right]$$



a. Exact Solution b. Approximation Solution HPM

Figure 1. The 3D surfaces for Y_3 in comparison with the analytic solution $u(x,t)$ when t = 0.05 with initial condition of Equation (3.1) by means of HPM



a. Exact Solution b. Approximation Solution HPM

Figure 2. The 2D plots of the numerical results for Y_3 in comparison with the analytic solution $u(x,t)$ when $t = 0.05$ with initial condition of Equation (3.1) by means of HPM

Example 2. We consider a nonlinear partial differential equation given as

$$u_{tt} - u_{xx} + u^2 - uu_x = 0, \qquad 0 \le x \le 1, \ t > 0. \tag{3.14}$$

The boundary conditions and initial condition are

$$\begin{aligned} &u(x,0) = u_t(x,0) = e^x, \ 0 < x < 1, \\ &u(0,t) = e^t, \ (t \ge 0). \end{aligned} \tag{3.15}$$

Structure of HPM He (2004, 2003, 2001, 2009) for Equation (3.1) is

$$\left(1-p\right)[\ddot{Y}-\ddot{u}_0]+p\left[\ddot{Y}-Y''+Y^2+YY'\right]=0, \tag{3.16}$$

$$\ddot{Y}-\ddot{u}_0-p\ddot{Y}+p\ddot{u}_0+p\ddot{Y}-pY''+pY^2+pYY'=0,$$

$$\ddot{Y}-\ddot{u}_0+p\ddot{u}_0-pY''+pY^2+pYY'=0, \tag{3.17}$$

where $\ddot{Y}=\dfrac{\partial^2 y}{\partial t^2}$, $Y''=\dfrac{\partial^2 y}{\partial x^2}$, $Y'=\dfrac{\partial y}{\partial x}$, and $p\in[0,1]$. We suppose that the solution of Equation (3.14) has the form as following;

$$Y=Y_0+pY_1+p^2Y_2+p^3Y_3+\cdots=\sum_{n=0}^{\infty}p^nY_n\left(x,t\right), \tag{3.18}$$

$$\ddot{Y}=\ddot{Y}_0+p\ddot{Y}_1+p^2\ddot{Y}_2+p^3\ddot{Y}_3+\cdots,$$

$$Y''=Y_0''+pY_1''+p^2Y_2''+p^3Y_3''+\cdots, \tag{3.19}$$

$$Y'=Y_0'+pY_1'+p^2Y_2'+p^3Y_3'+\cdots.$$

Then, substituting Equation (3.19) into Equation (3.17), and rearranging based on powers of p -terms, we obtain;

$$p^0:\ddot{Y}_0-\ddot{u}_0=0, \tag{3.20}$$

$$p^1:\ddot{Y}_1+\ddot{u}_0-Y_0''+Y_0Y_0'+Y_0^2=0, \tag{3.21}$$

$$p^2:\ddot{Y}_2-Y_1''+Y_1'Y_0+Y_0'Y_1+2Y_0\,Y_1=0, \tag{3.22}$$

$$p^3:\ddot{Y}_3-Y_2''+Y_2'Y_0+Y_1'Y_1+Y_1^2+Y_0'Y_2+2Y_0\,Y_2=0, \tag{3.23}$$

$$\vdots$$

with solving Equation (3.20-3.23);

$$p^0:\ddot{Y}_0-\ddot{u}_0=0\Rightarrow Y_0=u_0\Rightarrow Y_0=u_t(x,0)=e^x(t+1), \tag{3.24}$$

$$p^1:\ddot{Y}_1+\ddot{u}_0-Y_0''+Y_0Y_0'+Y_0^2=0\Rightarrow\ddot{Y}_1=-\ddot{u}_0+Y_0''-Y_0Y_0'-Y_0^2,$$

$$\Rightarrow Y_1=\int_0^t\int_0^t\left[-\ddot{u}_0+Y_0''-Y_0Y_0'-Y_0^2\right]dt\,dt,$$

$$\Rightarrow Y_1=\int_0^t\int_0^t\left[Y_0''+Y_0Y_0'-Y_0^2\right]dt\,dt,$$

$$\Rightarrow Y_1=e^x\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right),$$

$$p^2: \ddot{Y}_2 - Y_1'' - Y_1'Y_0 - Y_0'Y_1 + 2Y_0\,Y_1 = 0,$$

$$\Rightarrow \ddot{Y}_2 = Y_1'' + Y_1'Y_0 + Y_0'Y_1 - 2Y_0\,Y_1,$$

$$\Rightarrow Y_2 = \int_0^t\int_0^t \left[Y_1'' + Y_1'Y_0 + Y_0'Y_1 - 2Y_0\,Y_1\right] dt\,dt, \tag{3.25}$$

$$\Rightarrow Y_2 = e^x\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right),$$

$$p^3: \ddot{Y}_3 - Y_2'' - Y_2'Y_0 - Y_1'Y_1 + Y_1^2 - Y_0'Y_2 + 2Y_0\,Y_2 = 0,$$

$$\Rightarrow \ddot{Y}_3 = Y_2'' + Y_2'Y_0 + Y_1'Y_1 - Y_1^2 + Y_0'Y_2 - 2Y_0\,Y_2,$$

$$\Rightarrow Y_3 = \int_0^t\int_0^t \left[Y_2'' + Y_2'Y_0 + Y_1'Y_1 - Y_1^2 + Y_0'Y_2 - 2Y_0\,Y_2\right] dt\,dt, \tag{3.26}$$

$$\Rightarrow Y_3 = e^x\left(\frac{t^6}{6!} + \frac{t^7}{7!}\right),$$

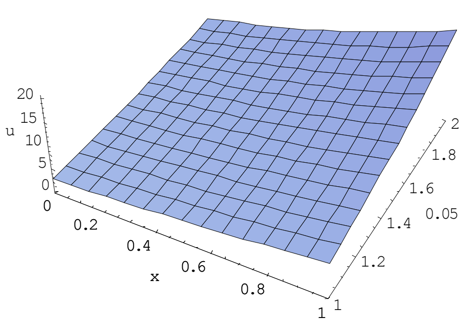
$$\vdots$$

the terms of Equation (3.18) could calculated. When we consider the series Equation (3.18) with the terms Equation (3.24)- Equation (3.26) and suppose $p = 1$, we obtain approximation solution of Equation (3.14) as following;

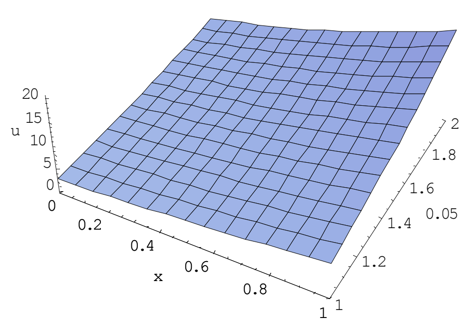
$$u(x,t) = Y_0 + Y_1 + Y_2 + Y_3 + \cdots \tag{3.27}$$

As a result, the components $Y_0, Y_1, Y_2, Y_3, \cdots$ are identified. We obtain analytic solution of Equation (3.14)

$$u(x,t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \cdots\right).$$



a: Exact Solution

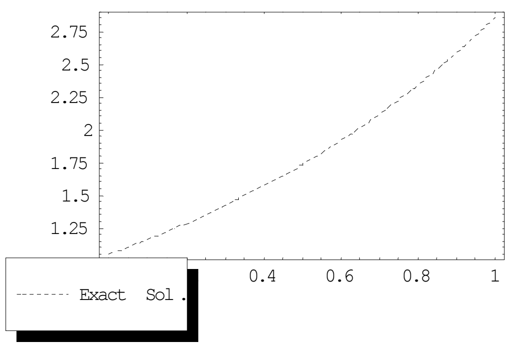


b: Approximation Solution HPM

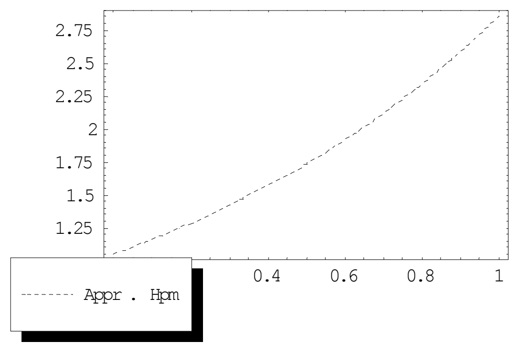
Figure 3. The 3D surfaces for Y_3 in comparison with the analytic solution $u(x,t)$ when t = 0.05 with initial condition of Equation (3.14) by means of HPM

Table. 1 Absolute errors obtain for Example.1.
The numerical results for Y_3 in comparison with the analytic solution $u(x,t)$ when $x=0.1$ with initial condition of Equation (3.1) by means of HPM, ADM, VIM

$x=0.1$			
t	$u_{Exact}-u_{Hpm}$	$u_{Exact}-u_{Vim}$	$u_{Exact}-u_{Adm}$
0.1	1,3877800E–17	-0,329633	1,3877800E–17
0.2	-5,551120E–17	-0,657070	-5,551120E–17
0.3	0	-0,980122	0
0.4	1,2221250E–14	-1,296630	1,2221250E–14
0.5	2,1538300E–14	-1,604450	2,1538300E–14



a. Exact Solution



b. Approximation Solution HPM

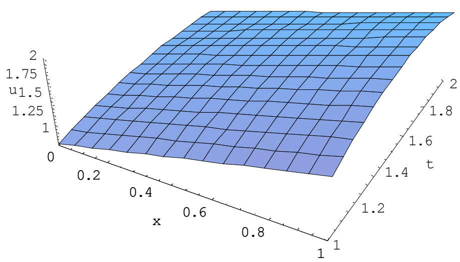
Figure 4. The 2D plots for Y_3 in comparison with the analytic solution $u(x,t)$ when t =0.05 with initial condition of Equation (3.14) by means of HPM

Table 2. Absolute errors obtain for Example. 2.
The numerical results for Y_3 in comparison with the analytic solution $u(x,t)$ when $x=0.1$ with initial condition of Eq.(3.14) by means of HPM, ADM, VIM

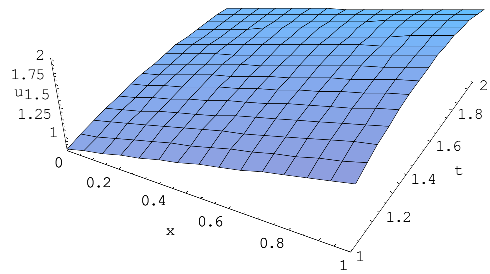
$x=0.1$			
t	$u_{Exact}-u_{Hpm}$	$u_{Exact}-u_{Vim}$	$u_{Exact}-u_{Adm}$
0.1	2,77112E–13	9,36547E–8	2,77112E–13
0.2	7,17608E–11	3,04824E–6	7,17608E–11
0.3	1,86016E–09	2,35485E–5	1,86016E–09
0.4	1,87949E–08	1,00973E–4	1,87949E–08
0.5	1,13330E–07	3,13615E–4	1,13330E–07

4. Numerical Comparison

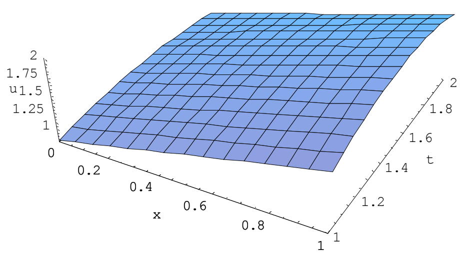
Tables 1 and 2 show the difference of analytical solution and numerical solution of the absolute error. We also demonstrate the numerical solution of Equation (3.1) in Figure 1(a), the corresponding approximate numerical solution in Figure 1(b) and Equation (3.14) in Figure 2(a), the corresponding approximate numerical solution in Figure 2(b). We note that only 3 terms were used in evaluating the approximate solution. We achieved a very good approximation with the actual solution of the equations by using 3 terms only of the Homotopy perturbation method above. It is evident that the overall errors can be made smaller by adding new terms to the perturbation. Numerical approximations shows a high degree of accuracy and in most cases ϕ_n, the n-term approximation, is accurate for quite low values of n. The solution is very rapidly convergent by utilizing the homotopy perturbation method. We obtained that the true of this methodology justify from this numerical results, even in the few terms approximation is accurate.



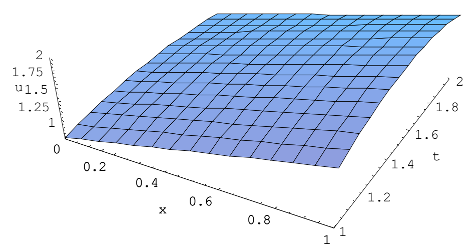
(a) Exact Solution



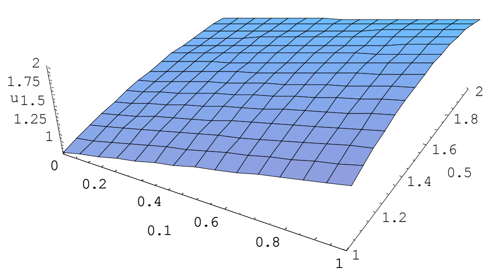
(b) Numerical Solution (HPM)



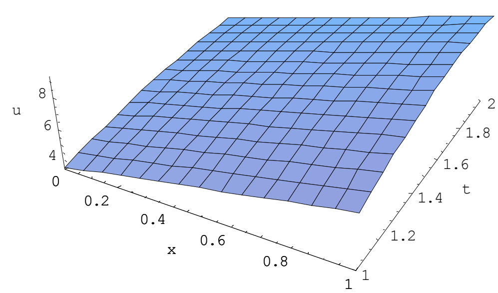
(a) Exact Solution



(b) Numerical Solution (ADM)

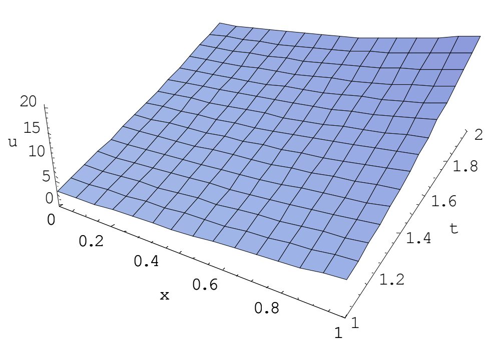


(a) Exact Solution

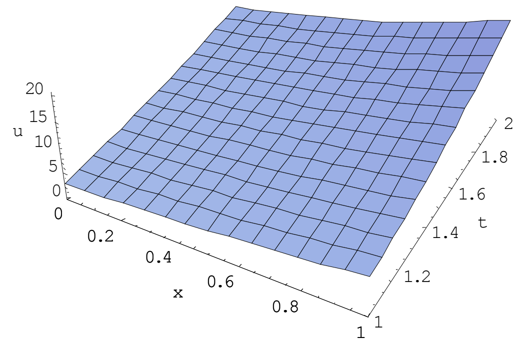


b) Numerical Solution (VIM)

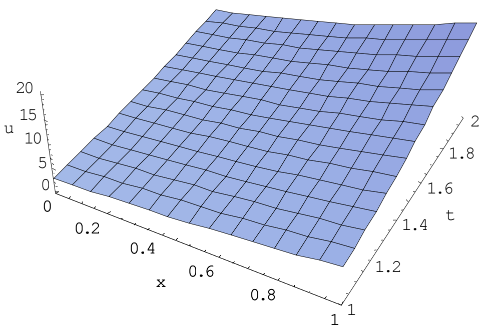
Figures A



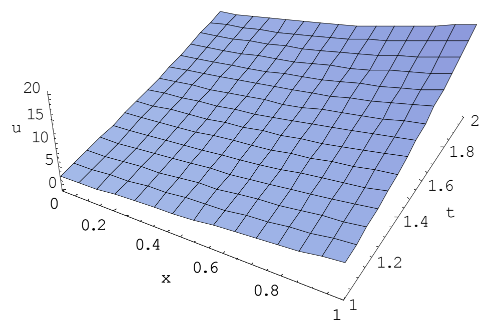
(a) Exact Solution



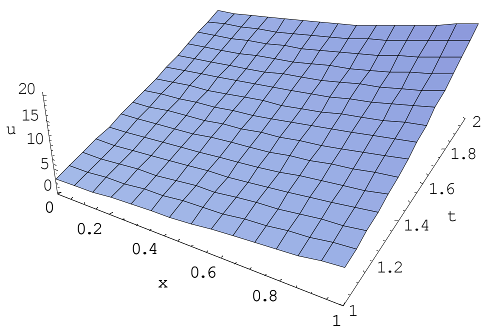
(b) Numerical Solution (HPM)



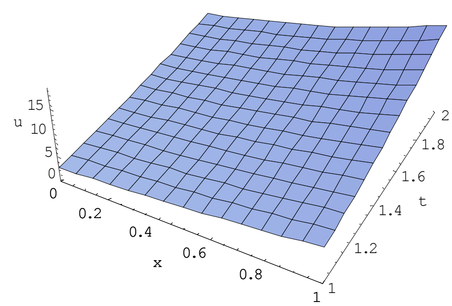
(a) Exact Solution



(b) Numerical Solution (ADM)



(a) Exact Solution



(b) Numerical Solution (VIM)

Figures B

5. Conclusion

The method have been used for solving a lot of differential equations such as ordinary, partial, linear, nonlinear, homogeneous, nonhomogeneous by many researchers. In this research, we used for solving linear and nonlinear Klein-Gordon equations with initial conditions. According to these datas such as 3D, 2D graphics and Tables, one realize that one of the advantages of HPM displays a fast convergence of the solutions.

REFERENCES

Adomian G. (1986). *Nonlinear Stochastic Operator Equations*, Academic Press, San Diego.

Abbasbandy S. (2006). Application of He's homotopy perturbation method for Laplace transform, Chaos Solitons Fractals, 30, 1206–1212.

Bektas, M., Bulut, H. and Ergut, M. (2004). Geometric Interpretations of The Klein-Gordon Equation and Solution of The Equation By The Adomian Decomposition Method", Firat Universitesi Fen ve Mühendislik Bilimler Dergisi, 16(3), 411–423.

Bulut, H. and Baskonus, H.M. (2009). A Comparsion Among Homotopy Perturbation Method And The Decomposition Method With The Variational Iteration Method For Dispersive Equation International Journal of Basic & Applied Sciences IJBAS, 9(10), 32–42.

Drazin, P. G. and Johnson R.S. (1989). Solitons: an Introduction, Cambridge University Press, New York, USA.

He, J.H. (1999). Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering, 178, 257–262.

He, J.H. (2006). Some asymptotic methods for strongly nonlinear equations. International Journal of Modern Physics. B, 20(10), 1141–1199.

He, J.H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation, 151, 287–292.

He, J.H. (2003). Determination of limit cycles for strongly nonlinear oscillators. Physical Review Letter, 90(17), 174-301.

He, J.H. (2001). Bookkeeping parameter in perturbation methods, International Journal Non-Linear Science Numerical Simulation. 2(4), 317–320.

He, J.H.(2000). A coupling method a homotopy technique and a perturbation technique for non-linear problems, International Journal Non-Linear Mechanic, 35, 37–43.

He, J.H. and Elagan S.K. (2011). A first course in functional analysis, Asia Academic Publisher Limited, Hong Kong.

Inc, M. and Ugurlu, Y. (2007). Numerical Simulation of the regularized long wave equation by He's homotopy perturbation method, Phys. Lett. A 369, 173–179.

Liao, S.J. (2004). On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computer, 147, 499–513.

Rosenfeld, B. A. (1969). Non –Euclidean Spaces, Nauka, Moscow, RUSSIA.