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Distal Fuzzy Dynamical Systems

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Abstract

In this paper the t-distal notion is extended for fuzzy dynamical systems on fuzzy metric spaces. A method for constructing fuzzy metric spaces is studied. The product of t-distal fuzzy dynamical systems is considered. It is proved that: a product of fuzzy dynamical systems is t-distal if and only if its components are t-distal. The persistence of the t-distal property up to a fuzzy factor map is proved.

Keywords: Fuzzy t-distal; Fuzzy dynamical system; Fuzzy factor map; Cartesian product of fuzzy metric spaces.

MSC 2010: 37C70, 54A40

1. Introduction:

In the theory of fuzzy dynamical systems, the fuzzy continuity [Chang (1968), George et al. (1997), George et al. (1994), Grabiec (1998)] and its extensions [Molaei (2004), Rahmat et al. (2008)] are essential means to describe the system behavior. Distal property is the other mathematical property which can describe the behavior of a system without paying attention to the continuity of a system.

In this paper we are going to consider the dynamics on a fuzzy metric space without paying attention to its fuzzy continuity. We are going to define a property which can be a suitable replacement for fuzzy continuity. By this property we can consider the behavior of fuzzy dynamical systems which may not be fuzzy continuous. In section 2 we recall the definition of fuzzy metric spaces and we present a method for constructing new fuzzy metric spaces. By a fuzzy dynamical system, we mean a bijection on a fuzzy metric space. We introduce the notion of t-distal property [Host et al. (2010)] for fuzzy dynamical systems in addition. Intuitively a t-distal fuzzy dynamical system as a kind of dynamics that does not let the points of a fuzzy metric space converge to each other up to a fuzzy metric in the level t. We prove that two fuzzy dynamical systems on two fuzzy metric spaces are t-distal if and only if their product up to a special t-norm is t-distal. In addition we also introduce the notion of a fuzzy factor map as a kind of conjugate relation and show that the t-distal property preserves under a fuzzy factor map.

2. Preliminaries

A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions:

1) $*$ is an associative and commutative operation;
2) $a*1=a$, for all $a\in[0,1]$;
3) $a*b\le c*d$, whenever $a\le b, c\le d$ where $a,b,c,d\in[0,1]$.

A fuzzy metric space $(X,M,*)$ is a triple $(X,M,*)$ where X , is a nonempty set, $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm and $M: X\times X\times(0,\infty)\to[0,1]$ is a mapping which has the following properties:

For every $x,y,z\in X$ and $t,s>0$,
1) $M(x,y,t)>0$;
2) $M(x,y,t)=1$ if and only if $x=y$;
3) $M(x,y,t)=M(y,x,t)$;
4) $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$;
5) $M(x,y,.):(0,\infty)\to[0,1]$ is a continuous map.

Theorem 2.1. If $(X,M,*)$ is a fuzzy metric space then:
i) For given $a,b\in[0,1]$ $a*b=1$ implies $a=b=1$;
ii) The inequality $t\le s$ implies $M(x,y,t)\le M(x,y,s)$, where $x,y\in X$.

Proof:

One can deduce the first property immediately. To prove (ii) put $s=t+r$, then

$$M(x,y,s)=M(x,y,t+r)\ge M(x,y,t)*M(y,y,r)=M(x,y,t)*1=M(x,y,t).$$

One can prove the following theorem by direct calculations.

Theorem 2.2. If $*_1, *_2$ are two t-norms, then

$$*_m : [0,1]\times[0,1] \to [0,1]$$

$$(a,b) \mapsto \min\{a *_1 b, a *_2 b\}$$

is a t-norm.

Theorem 2.3. Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two fuzzy metric spaces. Assume that the t-norm $*$ has the following additional property:

A: $a * a > 0$ for all $0 < a.$,

then $X_1 \times X_2$ with the mapping

$$M : (X_1\times X_2)\times(X_1\times X_2)\times(0,\infty) \to [0,1]$$

$$((x_1,x_2),(y_1,y_2),t) \mapsto M_1(x_1,y_1,t) *_m M_2(x_2,y_2,t)$$

is a fuzzy metric space.

Proof:

The property A of the t-norm implies M is a positive function. More precisely let $M((x_1,x_2),(y_1,y_2),t) = 0$ for some $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$ and $t \in (0,\infty)$. Without less of generality, we assume that $M_1(x_1,y_1,t) *_1 M_2(x_2,y_2,t) = 0$ and $M_1(x_1,y_1,t) \le M_2(x_2,y_2,t)$. So

$$M_1(x_1,y_1,t) *_1 M_1(x_1,y_1,t) = 0.$$

Thus, $M_1(x_1,y_1,t) = 0,$ which is a contradiction.

$M((x_1,x_2),(y_1,y_2),t) = 1$ if and only if $M_1(x_1,y_1,t) = 1$ and $M_2(x_2,y_2,t) = 1$, which is equivalent to $x_1 = y_1, x_2 = y_2$.

Now we want to prove the triangle inequality.

$$M((x_1,x_2),(z_1,z_2),t+s)$$
$$= \min\{M_1(x_1,z_1,t+s) *_1 M_2(x_2,z_2,t+s), M_1(x_1,z_1,t+s) *_1 M_2(x_2,z_2,t+s)\}.$$

Since

$M_i(x_i, z_i, t+s) \geq M_i(x_i, z_i, t)$ and $M_i(x_i, z_i, t+s) \geq M_i(x_i, z_i, s)$, for $i = 1, 2$,

there

$$
\begin{aligned}
&M((x_1, x_2), (z_1, z_2), t+s) \\
&\qquad = \min\{M_1(x_1, z_1, t+s) *_1 M_2(x_2, z_2, t+s), M_1(x_1, z_1, t+s) *_1 M_2(x_2, z_2, t+s)\} \\
&\qquad \geq \min\{\min\{M_1(x_1, y_1, t) *_1 M_2(x_2, y_2, t), M_1(x_1, y_1, t) *_2 M_2(x_2, y_2, t)\},
\end{aligned}
$$

$$
\begin{aligned}
&\min\{M_1(x_1, y_1, s) *_1 M_2(x_2, y_2, s), M_1(x_1, y_1, s) *_2 M_2(x_2, y_2, s)\}\} \\
&\qquad \geq (M_1(x_1, y_1, t) *_m M_2(x_2, y_2, t)) *_m (M_1(x_1, y_1, s) *_m M_2(x_2, y_2, s)) \\
&\qquad = M((x_1, x_2), (y_1, y_2), t) *_m M((x_1, x_2), (y_1, y_2), s).
\end{aligned}
$$

$(X_1 \times X_2, M, *_m)$, presented in Theorem 2.3 is called the Cartesian product of fuzzy metric spaces X_1, X_2.

3. The Distal Property for Fuzzy Dynamics

Now let us to introduce the notion of distality for fuzzy dynamical systems. For this purpose we ssume thata: $(X, T, M, *)$ is a fuzzy dynamical system, i.e., $(X, M, *)$ is a fuzzy metric space and $T : X \to X$ is a bijection.

Remark: If $(X_1, T_1, M_1, *_1)$, and $(X_2, T_2, M_2, *_2)$ are two fuzzy dynamical systems, then $(X_1 \times X_2, M, *_m)$ is a fuzzy dynamical system.

Definition 3.1. Let $(X, T, M, *)$ be a fuzzy dynamical system. Then a pair $(x, y) \in X \times X$ is called a fuzzy t-distal pair if

$$\sup_{n \in z} \{M(T^n(x), T^n(y), t)\} < 1.$$

$(X, T, M, *)$ is called t-distal if for a given pair of distinct points $x, y \in X$, and $t \in (0, \infty)$, (x, y) is a fuzzy t-distal pair.

Theorem 3.1. If $(X_1, M_1, T_1, *_1)$ and $(X_2, M_2, T_2, *_2)$ are two fuzzy dynamical systems, then $(X_1, M_1, T_1, *_1)$ and $(X_2, M_2, T_2, *_2)$ are t-distal fuzzy dynamical systems if and only if $(X_1 \times X_2, T, M, *_m)$ is a t-distal fuzzy dynamical system, where $T = T_1 \times T_2$.

Proof:

Let $(X_1, M_1, T_1, *_1)$ and $(X_2, M_2, T_2, *_2)$ are t -distal fuzzy dynamical systems and $(x_1, x_2) \neq (y_1, y_2)$, $(x_1, x_2) \in X_1 \times X_2$, $(y_1, y_2) \in X_1 \times X_2$. Without less of generality let $x_1 \neq x_2$. Then,

$$\sup_n \{M(T^n(x_1, x_2), T^n(y_1, y_2), t)\} = \sup_n \{M((T_1^n(x_1), T_2^n(x_2)), (T_1^n(y_1), T_2^n(y_2)), t)\}$$
$$= \sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t) *_m M_2(T_2^n(x_2), T_2^n(y_2), t)\}$$
$$= \sup_n \{\min\{M_1((T_1^n(x_1), T_1^n(y_1), t) *_1 M_2(T_2^n(x_2), T_2^n(y_2), t),$$

$$M_1((T_1^n(x_1), T_1^n(y_1), t) *_2 M_2(T_2^n(x_2), T_2^n(y_2), t)\}\}$$
$$\leq \min\{\sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t) *_1 M_2(T_2^n(x_2), T_2^n(y_2), t)\},$$

$$\sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t) *_2 M_2(T_2^n(x_2), T_2^n(y_2), t)\}\} < 1.$$

The last inequality holds because

$$M_1((T_1^n(x_1), T_1^n(y_1), t) \leq \sup_{n \in Z} \{M_1((T_1^n(x_1), T_1^n(y_1), t)\}$$

and

$$M_2((T_2^n(x_2), T_2^n(y_2), t) \leq \sup_{n \in Z} \{M_2((T_2^n(x_2), T_2^n(y_2), t)\}.$$

Thus,

$$M_1((T_1^n(x_1), T_1^n(y_1), t) *_1 M_2(T_2^n(x_2), T_2^n(y_2), t)$$
$$\leq \sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t)\} *_1 \sup_n \{M_2(T_2^n(x_2), T_2^n(y_2), t)\}.$$

So

$$\sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t)\} *_1 M_2(T_2^n(x_2), T_2^n(y_2), t)\}$$
$$\leq \sup_n \{M_1((T_1^n(x_1), T_1^n(y_1), t)\} *_1 \sup_n \{M_2(T_2^n(x_2), T_2^n(y_2), t)\} < 1$$

Thus, $(X_1 \times X_2, T, M, *_m)$ is t-distal.

Conversely, let $(X_1 \times X_2, T, M, *_m)$ be a t-distal fuzzy dynamical system, $x_1, y_1 \in X_1$, and $x_1 \neq y_1$. Let x_2 be a point in X_2 . Then $(x_1, x_2) \neq (y_1, x_2)$, where $(x_1, x_2), (y_1, x_2) \in X_1 \times X_2$. So $\sup_n \{M((T_1^n(x_1), T_2^n(x_2)), (T_1^n(y_1), T_2^n(x_2)), t)\} < 1$.

Because

$$M((T_1^n(x_1),T_2^n(x_2)),(T_1^n(y_1),T_2^n(x_2)),t) = M_1((T_1^n(x_1),T_1^n(y_1),t) *_m M_2(T_2^n(x_2),T_2^n(x_2),t)$$
$$= M_1((T_1^n(x_1),T_1^n(y_1),t) *_m 1 = M_1((T_1^n(x_1),T_1^n(y_1),t).$$

Hence,

$$\sup\{M_1((T_1^n(x_1),T_1^n(y_1),t)\} < 1.$$

So $(X_1, M_1, T_1, *_1)$ is a t-distal fuzzy dynamical system.

Similarly $(X_2, M_2, T_2, *_2)$ is a t-distal fuzzy dynamical system.

Corollary 3.1. If $\{X_i, i = 1,...,k\}$ is a family of fuzzy dynamical systems, then

$$(X_1 \times X_2 \times ... \times X_k, T_1 \times T_2 \times ... \times T_k, M, *_m)$$

is t-distal if and only if $(X_i, T_i, M_i, *_i)$ is t-distal for each $1 \le i \le k$.

In the above corollary $X_1 \times X_2 \times ... \times X_k$ means $(X_1 \times X_2 \times ... \times X_{k-1}) \times X_k$.

Corollary 3.2. If $(X, T, M, *)$ is a fuzzy dynamical system, then the following properties are equivalent:

i) $(X, T, M, *)$ is a t-distal fuzzy dynamical system,
ii) $(X^m, T^m, M, *)$ is a t-distal fuzzy dynamical system for all integer m.
iii) $(X^m, T^m, M, *)$ is a t-distal fuzzy dynamical system for some integer m.

Definition 3.2. Let $(X_1, T_1, M_1, *_1)$ and $(X_2, T_2, M_2, *_2)$ be two fuzzy dynamical systems, an isometry between $(X_1, T_1, M_1, *_1)$ and $(X_2, T_2, M_2, *_2)$ is a mapping $P: X_1 \to X_2$ such that $M_1(x, y, t) = M_2(P(x), P(y), t)$ for all $x, y \in X_1$ and $t \in (0, \infty)$.

Definition 3.3. A fuzzy factor of a fuzzy dynamical system $(X_1, T_1, M_1, *_1)$ is another fuzzy dynamical system $(X_2, T_2, M_2, *_2)$ such that: there is an onto isometry P from X_1 onto X_2, so that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ P\downarrow & & \downarrow P \\ X_2 & \xrightarrow[T_2]{} & X_2 \end{array}$$

In this case, P is called a fuzzy factor map.

Lemma 3.1. If $P: X_1 \to X_2$ is a fuzzy factor map, then $P: X_1 \to X_2$ is a one to one map.

Proof:

If $P(x) = P(y)$, then $M_2(P(x), P(y), t) = 1$. So $M_1(x, y, t) = 1$. Thus, $x = y$.

Now let us to present the main theorem of this section, which prepare a method of congruence for distal fuzzy dynamical systems.

Theorem 3.2. If a fuzzy dynamical system $(X_1, T_1, M_1, *_1)$ has a fuzzy factor $(X_2, T_2, M_2, *_2)$, then $(X_1, T_1, M_1, *_1)$ is a t-distal fuzzy dynamical system if and only if $(X_2, T_2, M_2, *_2)$ is a t-distal fuzzy dynamical system.

Proof:

Let X_1 be a t-distal fuzzy dynamical system and $P: X_1 \to X_2$ be a factor map. Then for given $y_1, y_2 \in X_2$, if $\sup_n\{M_2(T_2^n(y_1), T_2^n(y_2), t)\} = 1$, then there exists $x_1, x_2 \in X_1$ such that $y_1 = P(x_1), y_2 = P(x_2)$, and $\sup_n\{M_2(T_2^n(P(x_1)), T_2^n(P(x_2)), t)\} = 1$. Hence, $\sup_n\{M_2(p(T_1^n(x_1)), P(T_1^n(x_2)), t)\} = 1$. Thus, $\sup_n\{M_1(T_1^n(x_1), T_1^n(x_2), t)\} = 1$. So, $x_1 = x_2$, Thus, $y_1 = y_2$. Hence $(X_2, T_2, M_2, *_2)$ is a t-distal fuzzy dynamical system.

If $(X_2, T_2, M_2, *_2)$ is a t-distal fuzzy dynamical system, then by replacing P with P^{-1} we can show that $(X_1, T_1, M_1, *_1)$ is a t-distal fuzzy dynamical system.

4. Conclusion

In this paper we have successfully deduced a method for constructing fuzzy metric spaces. We considered the product of t-distal fuzzy dynamical systems. We have also studied the distal property for the product of fuzzy dynamical systems. We proved the persistence of t-distal property up to a fuzzy factor map. The fuzzy factor map is a kind of conjugate relation which leads us to the notion of stability for fuzzy dynamical system. The consideration of fuzzy stability using fuzzy factor map can be a topic for further research.

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