Applications and Applied Mathematics: An International Journal (AAM)

Volume 7 | Issue 1 Article 4

6-2012

A Duhamel Integral Based Approach to Identify an Unknown Radiation Term in a Heat Equation with Non-linear Boundary Condition

R. Pourgholi
Damghan University

M. Abtahi
Damghan University

A. Saeedi
Damghan University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 7, Issue 1 (June 2012), pp. 52 - 70

Applications and Applied Mathematics:
An International Journal
(AAM)

A Duhamel Integral Based Approach to Identify an Unknown Radiation Term in a Heat Equation with Non-linear Boundary Condition

R. Pourgholi, M. Abtahi and A. Saeedi
School of Mathematics and Computer Sciences
Damghan University
P.O.Box 36715-364
Damghan, Iran
pourgholi@du.ac.ir;abtahi@du.ac.ir;saeedi@du.ac.ir

Received: November 21, 2011; Accepted: April 30, 2012

Abstract

In this paper, we consider the determination of an unknown radiation term in the nonlinear boundary condition of a linear heat equation from an overspecified condition. First we study the existence and uniqueness of the solution via an auxiliary problem. Then a numerical method consisting of zeroth-, first-, and second-order Tikhonov regularization method to the matrix form of Duhamel's principle for solving the inverse heat conduction problem (IHCP) using temperature data containing significant noise is presented. The stability and accuracy of the scheme presented is evaluated by comparison with the Singular Value Decomposition (SVD) method. Some numerical experiments confirm the utility of this algorithm as the results are in good agreement with the exact data.

Keywords: IHCP, Radiation term, Existence and Uniqueness, Stability, The Tikhonov regularization Method, SVD Method

MSC 2010: 65M32, 35K05

1. Introduction

Inverse problems are encountered in many branches of engineering and science. In one particular branch, heat transfer, the inverse problem can be used under conditions such as temperature or surface heat flux, or to determine important thermal properties such as the thermal conductivity or heat capacity of solids.

Several functions and parameters can be estimated from the IHCP: static and moving heating sources, material properties, initial conditions, boundary conditions, optimal shape, etc.

Fortunately, many methods have been reported to solve IHCPs (Alifanov, 1994; Beck et al., 1985; Beck and Murio, 1986; Beck et al., 1996; Cabeza et al., 2005; Cannon and Duchateau, 1980; Cannon and Duchateau, 1973; Cannon and Zachmann, 1982; Duchateau, 1981; Dowding and Beck, 1999; Molhem and Pourgholi, 2008; Murio and Paloschi, 1988; Pourgholi and Rostamian, 2010; Pourgholi et al., 2009; Shidfar and Azary, 1996; Shidfar and Azary, 1997; Shidfar and Nikoofar, 1989; Shidfar et al., 2006), and among the most versatile methods the following can be mentioned: Tikhonov regularization (Tikhonov and Arsenin, 1977), iterative regularization (Alifanov, 1994), mollification (Murio, 1993), BFM (Base Function Method) (Pourgholi and Rostamian, 2010), SFDM (Semi Finite Difference Method) (Molhem and Pourgholi, 2008) and the FSM (Function Specification Method) (Beck et al., 1985).

Shidfar (Shidfar et al., 2006) studied the existence and uniqueness of the solution for a one dimensional nonlinear inverse diffusion problem via an auxiliary problem and the Schauder fixed point theorem, furthermore applied a numerical algorithm based on finite differences method and least-squares scheme for solving a nonlinear inverse diffusion problem. Beck and Murio (Beck and Murio, 1986) presented a new method that combines the function specification method of Beck with the regularization technique of Tikhonov. Murio and Paloschi (Murio and Paloschi, 1988) proposed a combined procedure based on a data filtering interpretation of the mollification method and FSM. Beck (Beck et al., 1996) compared the FSM, the Tikhonov regularization method and the iterative regularization method, using experimental data. Another effective technique to solve ill-posed problems is based on the Singular Value Decomposition (SVD) of an ill conditioned matrix (Golub and Van Loan, 1983).

The plan of this paper is as follows: In section 2, we formulate a one-dimensional IHCP for a linear Heat equation with non-linear boundary condition. Existence and uniqueness of the solution via an auxiliary problem will be discussed in section 3. In section 4, a new method consisting of Tikhonov regularization to the matrix form of Duhamel's principle for solving this IHCP will be presented. Finally, some numerical experiment will be given in section 5.

2. Description of the Problem

When the radiation of heat from a solid is considered, the heat flux is often taken to be proportional to the fourth power of difference of the boundary temperature of the solid over the temperature of the surroundings (Cannon, 1984).

In this paper, we consider the problem of determining an unknown function φ , and a function $T(x,t)$ satisfying

$$T_t(x,t) = T_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_M, \tag{1}$$

$$T(x,0) = f(x), \qquad 0 \le x \le 1, \tag{2}$$

$$T_x(0,t) = \phi(T(0,t)) + \zeta(t), \qquad 0 \le t \le t_M, \tag{3}$$

$$T_x(1,t) = p(t), \qquad 0 \le t \le t_M, \tag{4}$$

and the overspecified condition

$$T(a,t) = g(t), \quad 0 \le t \le t_M, \tag{5}$$

where $0 \le a \le 1$ is a fixed point, t_M is a given positive constant, $f(x)$ is the initial temperature of the solid and $\phi(T(0,t)) + \zeta(t)$ represents a general radiation law. In this context we consider that the functions $f(x)$, $p(t)$, $g(t)$ and $\zeta(t)$ are continuous known functions on their domains and the nonlinear terms $\phi(T(0,t))$ an unknown function to be determined with respect to the overspecified condition (5).

3. Existence and Uniqueness

If the function ϕ in (3) is given, then the problem (1)-(4) is a direct problem with unique solution (see Theorem 3.1 below). For an unknown ϕ, we must therefore provide additional information namely (5) to provide a unique solution (T, ϕ) to the inverse problem (1)-(5).

In this section, we give some results on the existence and uniqueness of solution of the IHCP (1)-(5). First of all, let us define, for $-\infty < x < \infty$, $t > 0$,

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} \exp(-\frac{x^2}{4t}), \quad \theta(x,t) = \sum_{m=-\infty}^{\infty} K(x + 2m, t). \tag{6}$$

Theorem 3.1. Suppose that $f(x)$, $\zeta(t)$, $p(t)$ are continuous functions, and that $\phi(s)$ satisfies the following Lipschitz condition,

$$|\phi(s_1) - \phi(s_2)| \le M |s_1 - s_2|, \tag{7}$$

where M is a positive constant, then problem (1)-(4) has a unique solution.

Proof:

Take $F(t,s) = \phi(s) + \zeta(t)$, then by Theorem 7.3.1 in (Cannon, 1984), problem (1)-(4) has a solution of the form

$$T(x,t) = w(x,t) \quad -2\int_0^t \theta(x,t-\tau)F(\tau,\phi(\tau))d\tau + 2\int_0^t \theta(x-1,t-\tau)\,p(\tau)d\tau, \tag{8}$$

where

$$w(x,t) = \int_0^1 \{\theta(x-\xi,t) + \theta(x+\xi,t)\} f(\xi)d\xi, \tag{9}$$

and θ is defined by (6), if and only if φ is a piecewise continuous solution of the following Volterra integral equation of the second kind:

$$\varphi(t) = w(0,t) - 2\int_0^t \theta(0,t-\tau)F(\tau,\varphi(\tau))d\tau + 2\int_0^t \theta(-1,t-\tau)\,p(\tau)d\tau. \tag{10}$$

Define $H(t,\tau,s) = -2\theta(0,t-\tau)\phi(s)$, and

$$G(t) = w(0,t) - 2\int_0^t \theta(0,t-\tau)\zeta(\tau)d\tau + 2\int_0^t \theta(-1,t-\tau)\,p(\tau)d\tau,$$

and write (10) in the following form

$$\varphi(t) = G(t) + \int_0^t H(t,\tau,\varphi(\tau))d\tau. \tag{11}$$

Clearly, $G(t)$ and $H(t,\tau,s)$ are continuous functions and thus, by Theorem 8.2.1 in (Cannon, 1984), the integral equation (11) possesses a unique solution if

$$|\,H(t,\tau,s_1) - H(t,\tau,s_2)\,| \le L(t,\tau)\,|\,s_1 - s_2\,|,$$

where

$$\int_{t_0}^t L(t,\tau)d\tau \le \alpha(t-t_0), \qquad (t > t_0), \tag{12}$$

for some monotone increasing function α, with $\lim_{\eta\downarrow 0}\alpha(\eta) = 0$, and if

$$\int_{t_0}^t |\,H(t,\tau,0)\,|\,d\tau \le \beta(t-t_0), \qquad (t > t_0), \tag{13}$$

for some nonnegative function β, with $\lim_{\eta\downarrow 0}\beta(\eta) = 0$. Since $\phi(s)$ satisfies (7), we have

$$|\,H(t,\tau,s_1) - H(t,\tau,s_2)\,| \le 2M\theta(0,t-\tau)\,|\,s_1 - s_2\,|,$$

moreover,

$$|H(t,\tau,0)| = 2|\phi(0)|\theta(0,t-\tau).$$

Using the inequality $e^{-x} < \frac{1}{x}$, we have

$$\theta(0,t) = \frac{1}{2\sqrt{\pi t}}(1+2\sum_{m=1}^{\infty} e^{-m^2/t}) < \frac{1}{2\sqrt{\pi t}}(1+2t\sum_{m=1}^{\infty}\frac{1}{m^2}) < (\frac{1}{\sqrt{t}}+\sqrt{t}).$$

So that (12) holds if we take $\alpha(\eta) = 4M\sqrt{\eta}(1+\eta/3)$ and (13) holds if we take

$$\beta(\eta) = 4|\phi(0)|\sqrt{\eta}(1+\eta/3).$$

Therefore, the integral equation (10) has a unique solution $\varphi(t)$ and thus problem (1)-(4) has a solution of the form (8) which is, in fact, unique because $F(t,s)$ is Lipschitz continuous; see Theorem 7.3.1 in (Cannon, 1984).

Theorem 3.2. Suppose that $f(x)$, $p(t)$ are continuous functions, that $\zeta(t)$ is a Lipschitz function, that $s = g(t)$ is an invertible function and that $\gamma = g^{-1}$ satisfies the following Lipschitz condition,

$$|\gamma(r) - \gamma(s)| \leq N|r-s|, \tag{14}$$

where N is a positive constant. Then the IHCP (1)-(5), with $a = 0$ in (5), has a unique solution (T,ϕ) and ϕ satisfies (7) for some constant M.

Proof:

Consider the following auxiliary problem

$$T_t(x,t) = T_{xx}(x,t), \qquad a < x < 1, \quad 0 < t < t_M \tag{15}$$

$$T(x,0) = f(x), \qquad a \leq x \leq 1, \tag{16}$$

$$T(0,t) = g(t), \qquad 0 \leq t \leq t_M, \tag{17}$$

$$T_x(1,t) = p(t), \qquad 0 \leq t \leq t_M. \tag{18}$$

By Theorem 7.1.1 in (Cannon, 1984) problem (15)-(18) has a solution of the form

$$T(x,t) = \int_{-\infty}^{\infty} K(x-\xi,t)f(\xi)d\xi$$

$$-2\int_0^t \frac{\partial K}{\partial x}(x,t-\tau)\varphi_1(\tau)d\tau + 2\int_0^t K(x-1,t-\tau)\varphi_2(\tau)d\tau, \tag{19}$$

provided φ_1, φ_2 satisfy the following conditions,

$$\begin{aligned} g(t) &= \varphi_1(t) + \int_{-\infty}^{\infty} K(-\xi,t)f(\xi)d\xi + 2\int_0^t K(-1,t-\tau)\varphi_2(\tau)d\tau, \\ p(t) &= \varphi_2(t) + \int_{-\infty}^{\infty} \frac{\partial K}{\partial x}(1-\xi,t)f(\xi)d\xi - 2\int_0^t \frac{\partial^2 K}{\partial x^2}(1,t-\tau)\varphi_1(\tau)d\tau, \end{aligned} \tag{20}$$

where K is defined by (6). The system of integral equations (20) satisfies all conditions in Corollary 8.2.1 in (Cannon, 1984) and thus there are continuous functions φ_1 and φ_2 satisfying (20). Hence, problem (15)-(18) has a unique solution. Such a solution has the form (19). We impose the condition (5) to this solution and get $\phi(g(t)) + \zeta(t) = T_x(0,t)$ so that

$$\phi(s) = \int_{-\infty}^{\infty} \frac{\partial K}{\partial x}(-\xi,\gamma(s))f(\xi)d\xi - 2\int_0^{\gamma(s)} \frac{\partial^2 K}{\partial x^2}(0,\gamma(s)-\tau)\varphi_1(\tau)d\tau$$

$$+2\int_0^{\gamma(s)} \frac{\partial K}{\partial x}(-1,\gamma(s)-\tau)\varphi_2(\tau)d\tau - \zeta(\gamma(s)). \tag{21}$$

Let $s_1 = g(t_1)$ and $s_2 = g(t_2)$. Then

$$\begin{aligned} |\phi(s_1)-\phi(s_2)| &\le |T_x(0,t_1)-T_x(0,t_2)| + |\zeta(t_1)-\zeta(t_2)| \\ &\le \int_{-\infty}^{\infty} \left|\frac{\partial K}{\partial x}(-\xi,t_1) - \frac{\partial K}{\partial x}(-\xi,t_2)\right| |f(\xi)| d\xi \\ &\quad + 2\int_0^{t_1} \left|\frac{\partial^2 K}{\partial x^2}(0,t_1-\tau) - \frac{\partial^2 K}{\partial x^2}(0,t_2-\tau)\right| |\varphi_1(\tau)| d\tau \\ &\quad + 2\left|\int_{t_1}^{t_2} \frac{\partial^2 K}{\partial x^2}(0,t_2-\tau)\varphi_1(\tau)d\tau\right| \\ &\quad + 2\int_0^{t_1} \left|\frac{\partial K}{\partial x}(-1,t_1-\tau) - \frac{\partial K}{\partial x}(-1,t_2-\tau)\right| |\varphi_2(\tau)| d\tau \\ &\quad + 2\left|\int_{t_1}^{t_2} \frac{\partial K}{\partial x}(-1,t_2-\tau)\varphi_2(\tau)d\tau\right| + |\zeta(t_1)-\zeta(t_2)|. \end{aligned}$$

As it is noted in Section 4.2 of (Cannon, 1984), $\partial^{m+n}K/(\partial t^m \partial x^n)$ are bounded by constants that depend upon x, m and n. Applying the mean value theorem to $\frac{\partial K}{\partial x}(-\xi,t)$, $\frac{\partial^2 K}{\partial x^2}(0,t-\tau)$, and $\frac{\partial K}{\partial x}(-1,t-\tau)$ as functions of $t\in[0,t_M]$ and using the fact that φ_1 and φ_2 are bounded on $[0,t_M]$, and that $\zeta(t)$ is a Lipschitz function, we get a constant R such that

$$|\phi(s_1)-\phi(s_2)|\le R\,|t_1-t_2|=R\,|\gamma(s_1)-\gamma(s_2)|.$$

Finally, using (14), we have

$$|\phi(s_1)-\phi(s_2)|\le M\,|s_1-s_2|,$$

where $M=NR$. The uniqueness of (T,ϕ) follows from the fact that problems (1)-(4) and (15) – (18) have unique solutions.

4. Overview of the Method

To solve the inverse problem (1)-(5), let us consider the following auxiliary inverse problem

$$T_t(x,t)=T_{xx}(x,t), \qquad 0<x<1, \quad 0<t<t_M, \tag{22}$$

$$T(x,0)=f(x), \qquad 0\le x\le 1, \tag{23}$$

$$T_x(0,t)=q(t), \qquad 0\le t\le t_M, \tag{24}$$

$$T_x(1,t)=p(t), \qquad 0\le t\le t_M, \tag{25}$$

and the overspecified condition

$$T(a,t)=g(t), \qquad 0\le t\le t_M, \tag{26}$$

where t_M is a given positive constant and $f(x)$, $g(t)$ and $p(t)$ are continuous known functions on their domains while the heat flux $q(t)$ is unknown which remains to be determined from overspecified condition (26).

The solution of the problem (22)-(26) can be written as follows,

$$T(x,t)=\sum_{i=1}^{3}T_i(x,t), \tag{27}$$

where $T_i(x,t)$, for $i = 1,2,3$, satisfy the following problem:

$$\frac{\partial T_i}{\partial t}(x,t) = \frac{\partial^2 T_i}{\partial x^2}(x,t),\ i = 1,2,3, \qquad 0 \le x \le 1, 0 \le t \le t_M, \tag{28}$$

$$\frac{\partial T_i}{\partial x}(0,t) = \begin{cases} q(t), & i = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad 0 < t < t_M, \tag{29}$$

$$\frac{\partial T_i}{\partial x}(1,t) = \begin{cases} p(t), & i = 2 \\ 0, & \text{otherwise} \end{cases}, \qquad 0 < t < t_M, \tag{30}$$

$$T_i(x,0) = \begin{cases} f(x), & i = 3 \\ 0, & \text{otherwise} \end{cases}, \qquad 0 \le x \le 1. \tag{31}$$

In the linear problem (28)-(31) for $i = 1$, the relation between $q(t)$ and $T_1(x,t)$ can be expressed analytically by the Duhamel's integral as follows (Beck et al., 1985)

$$T_1(x,t) = \int_0^t q(s) \frac{\partial \Phi}{\partial t}(x, t-s) ds + T_1(x,0), \tag{32}$$

where $T_1(x,0)$ is the initial condition for the problem (28)-(31), for $i = 1$, and $\Phi(x,t)$ is the temperature rise at location x for a unit step change in the surface heat flux at $t = 0$ for the same partial differential equation and boundary conditions as the original problem except the differential equation and boundary conditions (other than $x = 0$) are homogeneous.

Equation (32) can be approximated at time t_M, by the following equation

$$(T_1)_M = \sum_{n=1}^{M} q_n \Delta\Phi_{M-n}, \tag{33}$$

where q_n represents the measured heat flux at time t_n, and $\Delta\Phi_i = \Phi_{i+1} - \Phi_i$.

Note that $\Delta\Phi_{i-j} = \dfrac{\partial T_{1i}}{\partial q_j}$, therefore, it represents the sensitivity coefficient measured at time t_i with respect to component q_j. Obviously, the sensitivity coefficients will be zero when $i < j$.

By writing the equation (33), for $M = 1,2,\ldots$ points, we obtain the following matrix equation

$$T_1 = Xq, \tag{34}$$

where

$$T_1 = [(T_1)_1, \ldots, (T_1)_M]^T, \; q = [q_1, \ldots, q_M]^T, \; q_i = q(t_i)$$

and

$$X = \begin{pmatrix} \Delta\Phi_0 & 0 & \ldots & 0 & 0 & 0 \\ \Delta\Phi_1 & \Delta\Phi_0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta\Phi_{M-2} & \Delta\Phi_{M-3} & \ldots & \Delta\Phi_1 & \Delta\Phi_0 & 0 \\ \Delta\Phi_{M-1} & \Delta\Phi_{M-2} & \Delta\Phi_{M-3} & \ldots & \Delta\Phi_1 & \Delta\Phi_0 \end{pmatrix}.$$

By solving the direct problem (28)-(31), for $i = 2,3$, and using the overspecified condition (5) and the equation (27), we have

$$T^*(a,t) = g(t) - T_2(a,t) - T_3(a,t) = T_1(a,t). \quad (35)$$

Considering the Duhamel'theorem, for $M = 1,2,\ldots$, we obtain the following equation

$$T^* = Xq, \tag{36}$$

where

$$T^* = [T_1^*, \ldots, T_M^*]^T, \; T_i^* = T^*(a,t_i).$$

The Matrix X is ill-conditioned. On the other hand, as g is affected by measurement errors, the estimate of q by (36) will be unstable. Therefore, the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution (Tikhonov and Arsenin, 1977; Hansen, 1992;Lawson and Hanson, 1974) to the system of linear algebraic equation (36) is given by

$$\Gamma_\alpha(q) = \| Xq - T^* \|_2^2 + \alpha \| R^{(s)} q \|_2^2.$$

On the case of the zeroth-, first-, and second-order Tikhonov regularization method the matrix $R^{(s)}$, for $s = 0,1,2,$ is given by, see e.g. (Martin et al., 2006):

$$R^{(0)} = I_{M \times M} \in R^{M \times M},$$

$$R^{(1)} = \begin{pmatrix} -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in R^{(M-1)\times M},$$

$$R^{(2)} = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix} \in R^{(M-2)\times M}.$$

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as

$$q_\alpha = [X^T X + \alpha (R^{(s)})^T R^{(s)}]^{-1} X^T T^*.$$

In our computation, we use the GCV scheme to determine a suitable value of α (Elden, 1984; Golub et al. 1979; Wahba, 1990).

For evaluating ϕ, we use

$$T_x(0,t_k) = \phi(T(0,t_k)) + \zeta(t_k), \quad 0 \le t \le T, \tag{37}$$

where $T_x(0,t_k)$ can be obtained from the solution of the inverse problem (22)-(26). Therefore

$$\phi(T(0,t_k)) = q_\alpha(t_k) - \zeta(t_k), \quad 0 \le t \le T. \tag{38}$$

Finally, the MATLAB package is used for interpolating these values and reconstructing the function ϕ.

5. Numerical Results and Discussion

Mathematically, IHCPs belong to the class of ill-posed problems, i.e., small errors in the measured data can lead to large deviations in the estimated quantities. The physical reason for the ill-posedness of the estimation problem is that variations in the surface conditions of the solid body are damped towards the interior because of the diffusive nature of heat conduction. As a consequence, large-amplitude changes at the surface have to be inferred from small-amplitude changes in the measurements data. Errors and noise in the data can therefore be mistaken as significant variations of the surface state by the estimation procedure. Therefore the IHCP has a unique solution, but this solution is unstable. This instability is overcome using the Tikhonov regularization method with the GCV criterion for the choice of the regularization parameter.

In this section the stability and accuracy of the scheme presented in section 4 is evaluated by comparison with the Singular Value Decomposition method. All the computations are performed on the PC (pentium(R) 4 CPU 3.20 GHz). However, to further demonstrate the accuracy and efficiency of this method, the present problem is investigated and some examples are illustrated.

Remark. In an IHCP there are two sources of error in the estimation. The first source is the unavoidable bias deviation (or deterministic error). The second source of error is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors is considered in the mean squared error or total error, (Cabeza et al., 2005).

Therefore, we compare Tikhonov regularization 0th, 1st and 2nd method and SVD method by considering total error S defined by

$$S = [\frac{1}{N-1}\sum_{i=1}^{N}(\hat{q}_i - q_i)^2]^{\frac{1}{2}}, \tag{39}$$

where N is the total number of estimated values.

Example 5.1. In this example, let us consider the following one-dimensional inverse problem, for estimating unknown boundary condition $\phi(T(0,t))$ when $a = 0.1$

$$T_t(x,t) = T_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_M, \tag{40}$$

$$T(x,0) = \sin x, \qquad 0 \le x \le 1, \tag{41}$$

$$T_x(0,t) = \phi(T(0,t)) + e^{-t}, \qquad 0 \le t \le t_M, \tag{42}$$

$$T_x(1,t) = e^{-t}\cos 1, \qquad 0 \le t \le t_M, \tag{43}$$

and the overspecified condition

$$T(0.1,t) = e^{-t}\sin(0.1), \qquad 0 \le t \le t_M. \tag{44}$$

The exact solution of this problem is

$$T(x,t) = e^{-t}\sin x, \quad \phi(T(0,t)) = (T(0,t))^2, \qquad 0 \le x \le 1,\ 0 < t < t_M.$$

Table 1 shows the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization 0th, 1st and 2nd and SVD regularization with noiseless data. Table 2 and figures 1, 2, 3 show this comparison with noisy data (noisy data=input data $+(0.01)\mathrm{rand}(1)$). Finally, we compare two methods with computation total error by (39).

Table 1. The comparison between exact and Tikhonov and SVD solutions for $\phi(T(0,t))$ with noiseless data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

t	Exact	SVD	Tikhonov 0th	Tikhonov 1st	Tikhonov 2nd
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.000000	0.000000	0.000000	0.000000	0.000000
0.3	0.000000	0.000001	0.000001	0.000001	0.000001
0.4	0.000000	0.000001	0.000001	0.000001	0.000001
0.5	0.000000	0.000001	0.000001	0.000001	0.000001
0.6	0.000000	0.000001	0.000001	0.000001	0.000001
0.7	0.000000	0.000001	0.000001	0.000001	0.000001
0.8	0.000000	0.000002	0.000002	0.000002	0.000002
0.9	0.000000	0.000002	0.000002	0.000002	0.000002
1	0.000000	0.000002	0.000002	0.000002	0.000002
S		$1.00e-006$	$1.00e-006$	$1.00e-006$	$1.00e-006$

Table 2. The comparison between exact and Tikhonov and SVD solutions for $\phi(T(0,t))$ with noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

t	Exact	SVD	Tikhonov 0th	Tikhonov 1st	Tikhonov 2nd
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.000000	0.002550	0.005176	0.004625	0.004783
0.2	0.000000	0.002571	0.005496	0.003546	0.003201
0.3	0.000000	0.002889	0.005440	0.001661	0.001710
0.4	0.000000	0.003585	0.004400	0.002558	0.002238
0.5	0.000000	0.003761	0.003274	0.003223	0.003178
0.6	0.000000	0.002845	0.003682	0.003114	0.003364
0.7	0.000000	0.001608	0.004935	0.003414	0.003190
0.8	0.000000	0.001218	0.002681	0.002559	0.002559
0.9	0.000000	0.003167	0.000713	0.002298	0.002530
1	0.000000	0.017198	0.013633	0.003042	0.003130
S		$4.016e-003$	$4.757e-003$	$3.108e-003$	$3.105e-003$

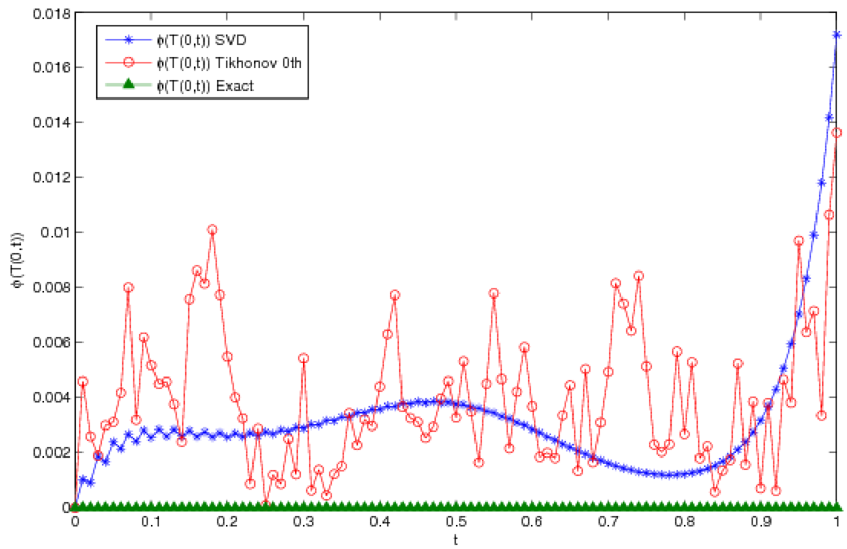

Figure 1. The comparison between the exact results and Tikhonov 0th and SVD of the problem (40)-(44) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

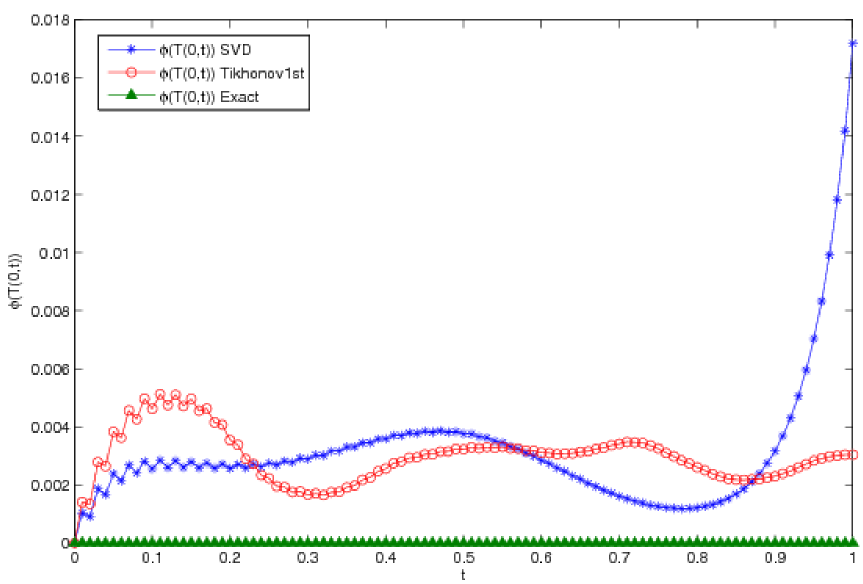

Figure 2. The comparison between the exact results and Tikhonov 1st and SVD of the problem (40)-(44) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

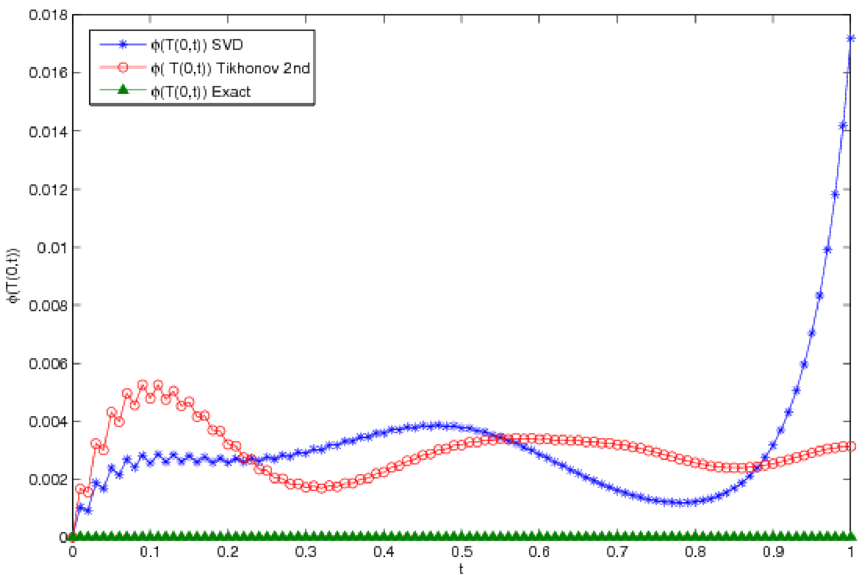

Figure 3. The comparison between the exact results and Tikhonov 2nd and SVD of the problem (40)-(44) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

Example 5.2. In this example let us consider the following one-dimensional inverse problem, for estimating unknown boundary condition $\phi(T(0,t))$ when $a = 0.1$.

$$T_t(x,t) = T_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_M, \tag{45}$$

$$T(x,0) = \cos(x-1), \qquad 0 \le x \le 1, \tag{46}$$

$$T_x(0,t) = \phi(T(0,t)) - e^{-t}(\sin(-1) + \cos(-1)) - e^{-2t}(\cos(-1))^2, \quad 0 \le t \le t_M, \tag{47}$$

$$T_x(1,t) = 0, \qquad 0 \le t \le t_M, \tag{48}$$

and the overspecified condition

$$T(0.1,t) = e^{-t}\cos(-0.9), \quad 0 \le t \le t_M. \tag{49}$$

The exact solution of this problem is

$$T(x,t) = e^{-t}\cos(x-1), \qquad 0 \le x \le 1, 0 < t < t_M,$$

and

$$\phi(T(0,t)) = (T(0,t))^2 + T(0,t), \qquad 0 \le x \le 1, 0 < t < t_M.$$

Table 3 shows the comparison between the exact solution and approximate solution result from our method by Tikhonov regularization 0th, 1st and 2nd and SVD regularization with noiseless data. Table 4 and figures 4, 5, 6 show this comparison with noisy data ($\text{noisy data} = \text{input data} + (0.001)\text{rand}(1)$). Finally, we compare two methods with computation total error by (39).

Table 3. The comparison between exact and Tikhonov and SVD solutions for $\phi(T(0,t))$ with noiseless data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

t	Exact	SVD	Tikhonov 0 th	Tikhonov 1st	Tikhonov 2nd
0.0	0.832229	0.832229	0.832229	0.832229	0.832229
0.1	0.727895	0.728347	0.728347	0.728347	0.728347
0.2	0.638046	0.638325	0.638325	0.638325	0.638325
0.3	0.560478	0.560524	0.560524	0.560524	0.560524
0.4	0.493347	0.493168	0.493168	0.493168	0.493168
0.5	0.435104	0.434729	0.434729	0.434729	0.434729
0.6	0.384451	0.383909	0.383909	0.383909	0.383909
0.7	0.340294	0.339613	0.339613	0.339613	0.339613
0.8	0.301712	0.300914	0.300914	0.300914	0.300914
0.9	0.267926	0.267030	0.267030	0.267030	0.267030
1	-0.071286	-0.072142	-0.071152	-0.071276	-0.071286
S		$5.61e-004$	$5.61e-004$	$5.61e-004$	$5.61e-004$

Table 4. The comparison between exact and Tikhonov and SVD solutions for $\phi(T(0,t))$ with noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

t	Exact	SVD	Tikhonov 0th	Tikhonov 1st	Tikhonov 2nd
0.0	0.832229	0.832229	0.832229	0.832229	0.832229
0.1	0.727895	0.723165	0.812798	0.732954	0.730531
0.2	0.638046	0.639324	0.601871	0.638822	0.640095
0.3	0.560478	0.563648	0.507716	0.564287	0.563256
0.4	0.493347	0.499448	0.475097	0.495693	0.494678
0.5	0.435104	0.427903	0.447593	0.431216	0.432604
0.6	0.384451	0.381417	0.459008	0.379929	0.378869
0.7	0.340294	0.335610	0.430699	0.334565	0.334364
0.8	0.301712	0.293088	0.241095	0.298053	0.299987
0.9	0.267926	0.272220	0.239352	0.272049	0.272890
1	-0.071286	0.296443	0.252615	0.244601	0.243964
S		$3.7557e-002$	$9.2607e-002$	$3.2015e-002$	$3.1757e-002$

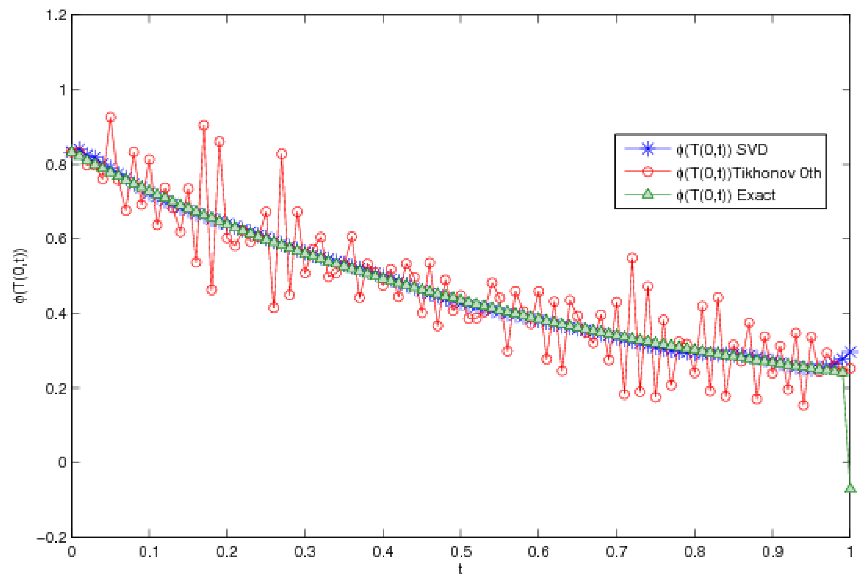

Figure 4. The comparison between the exact results and Tikhonov 0th and SVD of the problem (45)-(49) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

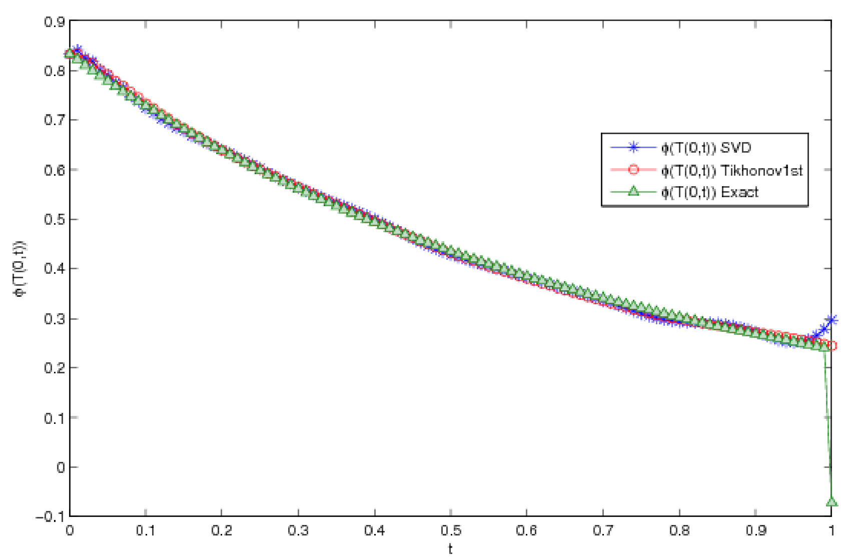

Figure 5. The comparison between the exact results and Tikhonov 1st and SVD of the problem (45)-(49) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

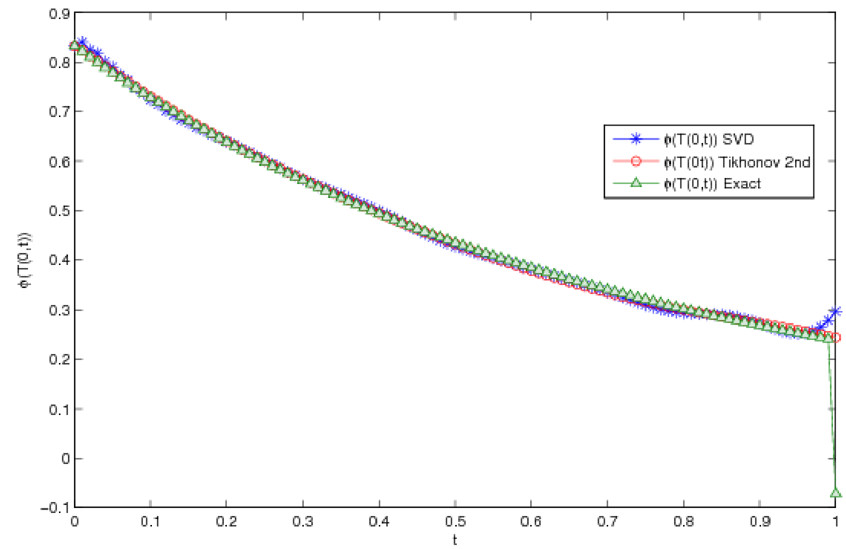

Figure 6. The comparison between the exact results and Tikhonov 2nd and SVD of the problem (45)-(49) with discrete noisy data when $\Delta t = 0.01$ and $\Delta x = 1/7$.

6. Conclusion

A numerical method, to estimate unknown radiation term is proposed for these kinds of IHCPs and the following results are obtained.

1. The present study, successfully applies the numerical method to IHCPs.
2. The Tikhonov regularization 0th, 1st and 2nd and SVD regularization give very similar results with noiseless data.
3. Numerical results show that, heat flux evolutions estimated by the Tikhonov regularization 1st and 2nd are accurate that those obtained by the SVD regularization with noisy data.
4. Numerical results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium(R) 4 CPU 3.20 GHz.
5. The present method has been found stable with respect to small perturbation in the input data.

REFERENCES

Alifanov, O.M. (1994). Inverse Heat Transfer Problems, Springer, NewYork.

Beck, J.V., Blackwell B. and ClairSt C.R. (1985). Inverse Heat Conduction: IllPosed Problems, Wiley-Interscience, NewYork.

Beck, J.V., Blackwell, B. and Haji-sheikh, A. (1996). Comparison of some inverse heat conduction methods using experimental data, Internat. J. Heat Mass Transfer, Vol.3, pp. 3649-3657.

Beck, J.V. and Murio, D.C. (1986). Combined function specification-regularization procedure for solution of inverse heat condition problem, AIAA J., Vol.24, pp. 80-185.

Cabeza, J.M.G., Garcia, J.A.M, and Rodriguez, A.C. (2005). A Sequential Algorithm of Inverse Heat Conduction Problems Using Singular Value Decomposition, International Journal of Thermal Sciences, Vol.44, pp. 235-244.

Cannon, J.R. (1984). One Dimensional Heat Equation, Addison Wesley, Cambridge University Press.

Cannon, J.R. and Duchateau, P. (1973). Determining unknown coefficients in a nonlinear heat conduction problem, SIAM J. Appl. Math., Vol.24, pp. 298-314.

Cannon, J.R. and Duchateau, P. (1980). An inverse problem for a nonlinear diffusion equation, SIAM J. Appl. Math., Vol.39, pp. 272-289.

Cannon, J.R. and Zachmann, D. (1982). Parameter determination in parabolic partial differential equations from overspecified boundary data, Int. J. Engng. Sci., Vol.20, pp. 779-788.

Dowding, K.J. and Beck, J.V. (1999). A Sequential Gradient Method for the Inverse Heat Conduction Problems, J. Heat Transfer, Vol.121, pp. 300-306.

Duchateau, P. (1981). Monotonicity and uniqueness results in identifying an unknown coefficient in a nonlinear diffusion equation, SIAM J. Appl. Math., Vol.41, pp. 310-323.

Elden, L. (1984). A Note on the Computation of the Generalized Cross-validation Function for Ill-conditioned Least Squares Problems, BIT, Vol.24, pp. 467-472.

Golub, G.H. and Van Loan, C.F. (1983). Matrix computations, John Hopkins university press, Baltimore, MD.

Golub, G.H., Heath, M. and Wahba, G. (1979). Generalized Cross-validation as a Method for Choosing a Good Ridge Parameter, Technometrics, Vol.21, pp. 215-223.

Hansen, P.C. (1992). Analysis of Discrete Ill-posed Problems by Means of the L-curve, SIAM Rev, Vol.34, pp. 561-80.

Lawson, C.L. and Hanson, R.J. (1974). Solving Least Squares Problems, Philadelphia, PA: SIAM, 1995. First published by Prentice-Hall.

Martin, L., Elliott, L., Heggs, P.J., Ingham, D.B., Lesnic, D. and Wen, X. (2006). Dual Reciprocity Boundary Element Method Solution of the Cauchy Problem for Helmholtz-type Equations with Variable Coefficients, Journal of sound and vibration, Vol.297, pp. 89-105.

Molhem, H. and Pourgholi, R. (2008). A Numerical Algorithm for Solving a One-Dimensional Inverse Heat Conduction Problem, Journal of Mathematics and Statistics, Vol.4, pp. 60-63.

Murio, D.A. (1993). The Mollification Method and the Numerical Solution of Ill-Posed Problems, Wiley-Interscience, NewYork.

Murio, D.C. and Paloschi, J.R. (1988). Combined mollification-future temperature procedure for solution of inverse heat conduction problem, J. comput. Appl. Math., Vol.23, pp. 235-244.

Pourgholi, R. and Rostamian, M. (2010). A numerical technique for solving IHCPs using Tikhonov regularization method, Appl. Math. Modell., Vol.34, pp. 2102-2110.

Pourgholi, R., Azizi, N., Gasimov, Y.S., Aliev, F. and Khalafi, H.K. (2009). Removal of Numerical Instability in the Solution of an Inverse Heat Conduction Problem, Communications in Nonlinear Science and Numerical Simulation, Vol.14, pp. 2664-2669.

Shidfar A. and Azary, H. (1996). An inverse problem for a nonlinear diffusion equation, Nonlinear Analysis: Theory, Methods and Applications, Vol.28, pp. 589-593.

Shidfar A. and Azary, H. (1997). Nonlinear Parabolic Problems, Nonlinear Analysis: Theory, Methods and Applications, Vol.30, pp. 4823-4832.

Shidfar, A. and Nikoofar, H. R. (1989). An inverse problem for a linear diffusion equation with nonlinear boundary condition, Applied Mathematics Letters, Vol.2, pp. 385-388.

Shidfar, A., Pourgholi, R. and Ebrahimi, M. (2006). A Numerical Method for Solving of a Nonlinear Inverse Diffusion Problem, Comp. Math. Appl., Vol.52, pp. 1021-1030.

Tikhonov, A.N. and Arsenin, V.Y. (1977). On the Solution of Ill-posed Problems, New York, Wiley.

Tikhonov, A.N. and Arsenin, V.Y. (1977). Solution of Ill-Posed Problems, V. H. Winston and Sons, Washington, DC.

Wahba, G. (1990). Spline Models for Observational Data, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol.59, SIAM, Philadelphia.