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Inverse Heat Conduction Problem in a Semi-Infinite Cylinder and its Thermal Stresses by Quasi-Static Approach

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Abstract

The present paper deals with the determination of unknown temperature and thermal stresses on the curved surface of a semi-infinite circular cylinder defined as $0 \le r \le a,\ 0 \le z < \infty$. The circular cylinder is subjected to an arbitrary known temperature under unsteady state condition. Initially, the cylinder is at zero temperature and temperature at the lower surface is held fixed at zero. The governing heat conduction equation has been solved by using the integral transform method. The results are obtained in series form in terms of Bessel's functions. A mathematical model has been constructed for aluminum material and illustrates the results graphically.

Keywords: Inverse thermoelastic problem; Thermal stresses; Circular cylinder

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1. Introduction

The inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at the some points of the solid under consideration.

Sabherwal (1965, 1966), studied inverse problem in heat conduction. Cialkowski and Graya (1980), Graya and Cialkowski (1981), Graya and Kozlowski (1982), investigated one dimensional transient thermoelastic problem and derived the heating temperature and heat flux on the surface of isotropic infinite slab. Noda (1998) studied inverse problem of coupled thermal stress field in a thick plate. Ashida et al. (1994) studied the inverse problem of two-dimensional piezothermoelasticity in an orthotropic plate exhibiting crystal class mm2. Ashida et al. (2006) attempted an inverse thermoelastic problem in an isotropic plate associated with a Piezoelectric ceramic plate. Deshmukh and Wankhede (1996) solved an inverse problem of thermoelasticity in a thin circular plate by determining the temperature on the curved surface of the plate, displacement and thermal stresses in the plate by using quasi-static approach by employing integral transform techniques. Khobragade and Deshmukh (1998) studied an inverse axially symmetric quasi-static problem of thermoelasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate and suitable heat exchange conditions are met on the upper and lower surfaces of the plate is solved with the help of a generalized integral transform technique. Recently, Kulkarni and Deshmukh (2008) studied an inverse quasi-static steady state thermal stresses in a thick circular plate. Very recently, Deshmukh et al. (2010) studied inverse heat conduction problem in a semi-infinite circular plate and its thermal deflection by quasi-static approach.

In this paper we consider the inverse heat conduction problem studied by Deshmukh et al. (2010) and discuss the thermal stresses in a semi- infinite circular cylinder. A circular cylinder is subjected to arbitrary known temperature under unsteady state condition. Initially, the cylinder is at zero temperature and the lower surface is at zero temperature. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel's functions. Mathematical model has been constructed of a semi-infinite circular cylinder with the help of numerical illustration. No one previously studied such type of problem. This is a new contribution to the field.

The inverse problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines, and the role of the rolling mill. Also arise in the quenching studies, the analysis of experimental data and measurement of aerodynamic heating.

2. Formulation of the Problem

Consider a circular cylinder defined by $0 \le r \le a, 0 \le z < \infty$. Let the cylinder be subjected to arbitrary known interior temperature $f(z,t)$ within the region $0 \le r \le a$, with lower surface $z = 0$ is at zero temperature. Under these more realistic prescribed conditions, the unknown temperature on the curved surface of cylinder at $r = a$ and quasi-static thermal stresses due to unknown temperature $g(z,t)$ are required to be determined.

The displacement equation of thermoelasticity has the form

$$U_{i,kk} + \left(\frac{1+v}{1-v}\right) e,_{i} = 2\left(\frac{1+v}{1-v}\right) \cdot a_t \cdot T,_{i} \tag{1}$$

$$e = U_{k,k}; \quad k, i = 1,2, \tag{2}$$

where

U_i – Displacement component,
e – Dilatation,
T – Temperature,
v – Poisson's ratio, and
a_t – the linear coefficient of thermal expansion of the circular cylinder.

Introducing

$$U_i = \psi_{,i} \ , \ i = 1,2,$$

we have

$$\nabla_1{}^2 \psi = (1+v) a_t T \tag{3}$$

$$\nabla_1{}^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}$$

$$\sigma_{ij} = 2\mu\left(\psi_{,ij} - \delta_{ij}\psi_{,kk}\right), \qquad i, j, k = 1,2, \tag{4}$$

where μ is the Lamé constant and δ_{ij} is the Kronecker symbol.
In the axially-symmetric case

$$\psi = \psi(r,z,t), \ T = T(r,z,t)$$

and the differential equation governing the displacement potential function $\psi(r,z,t)$ is as

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = (1+v) a_t T \,. \tag{5}$$

The stress function σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\psi}{\partial r} \tag{6}$$

and

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2} \tag{7}$$

Also in the plane state of stress within the circular cylinder

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0. \tag{8}$$

Initially

$$T = \psi = \sigma_{rr} = \sigma_{\theta\theta} = 0 \quad \text{at} \quad t = 0. \tag{9}$$

The temperature of the circular cylinder satisfies the heat conduction equation as in Ozisik (1968)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \quad \text{in} \quad 0 \le r \le a,\ 0 \le z < \infty,\ t > 0 \tag{10}$$

with the conditions

$$T(r,z,t) = g(z,t) \quad \text{(unknown) at } r = a, 0 \le z < \infty, \tag{11}$$

$$T(r,z,t) = 0 \quad \text{at} \quad z = 0, \quad 0 \le r \le a, \tag{12}$$

$$T(r,z,t) = 0 \quad \text{at} \quad z = \infty, \quad 0 \le r \le a, \tag{13}$$

$$T(r,z,t) = f(z,t) \quad \text{(known) at } r = \xi,\ 0 < \xi < a,\ 0 \le z < \infty, \tag{14}$$

and the initial condition

$$T(r,z,t) = 0 \quad \text{when} \quad t = 0, \tag{15}$$

where k is thermal diffusivity of the semi-infinite circular cylinder.

Equations (1)-(15) constitute the mathematical formulation of the inverse thermoelastic problem in a semi-infinite circular cylinder.

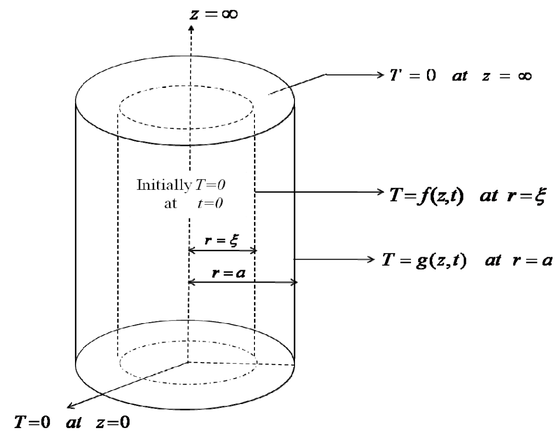

Figure 1. The geometry of the problem

3. The Solution

By applying the integral transform to the heat conduction equation, one obtains the expression for temperature distribution function and unknown heating temperature as

$$T(r,z,t) = \frac{2k}{\xi}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\lambda_n \frac{J_0(\lambda_n r)}{J_1(\lambda_n \xi)}\int_{\beta=0}^{\infty}\sin(\beta z)\left[\int_{t'=0}^{t}\overline{f}(\beta,t')e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}\,dt'\right]d\beta \tag{16}$$

and the unknown temperature $g(z,t)$ can be obtained as

$$g(z,t) = \frac{2k}{\xi}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\lambda_n \frac{J_0(\lambda_n a)}{J_1(\lambda_n \xi)}\int_{\beta=0}^{\infty}\sin(\beta z)\left[\int_{t'=0}^{t}\overline{f}(\beta,t')e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}\,dt'\right]d\beta, \tag{17}$$

where λ_n are the positive roots of the transcendental equation $J_0(\lambda_n\xi) = 0$.

Displacement Potential

Using equation (16) in (5), we have

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = (1+v)a_t \times \frac{2k}{\xi}\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\lambda_n\frac{J_0(\lambda_n r)}{J_1(\lambda_n \xi)}$$
$$\times \int_{\beta=0}^{\infty}\sin\beta z\left[\int_{t'=0}^{t}\overline{f}(\beta,t')e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}\,dt'\right]d\beta. \tag{18}$$

Solving equation (18) by using the result

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)J_0(\lambda_n r)=-\lambda_n^2 J_0(\lambda_n r),$$

one obtains the displacement potential function as

$$\psi(r,z,t)=-2\sqrt{\frac{2}{\pi}}\left(1+v\right)a_t\times\frac{k}{\xi}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{J_0(\lambda_n r)}{J_1(\lambda_n \xi)}$$
$$\times\int_{\beta=0}^{\infty}\sin\beta z\left[\int_{t'=0}^{t}\overline{f}(\beta,t')\,e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}dt'\right]d\beta. \tag{19}$$

Thermal Stresses

Substituting the equation of displacement potential function (19) into the equations (6) and (7), one obtains the expressions of radial stress function and angular stress function as

$$\sigma_{rr}=-4\mu\left(1+v\right)a_t\sqrt{\frac{2}{\pi}}\times\frac{k}{\xi}\sum_{n=1}^{\infty}\frac{1}{r}\frac{J_1(\lambda_n r)}{J_1(\lambda_n \xi)}$$
$$\times\int_{\beta=0}^{\infty}\sin\beta z\left[\int_{t'=0}^{t}\overline{f}(\beta,t')e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}dt'\right]d\beta. \tag{20}$$

$$\sigma_{\theta\theta}=-4\mu\left(1+v\right)a_t\sqrt{\frac{2}{\pi}}\times\left\{\frac{k}{\xi}\sum_{n=1}^{\infty}\lambda_n\left[J_0(\lambda_n r)-\frac{J_1(\lambda_n r)}{\lambda_n r}\right]\right.$$
$$\left.\times\int_{\beta=0}^{\infty}\sin\beta z\left[\int_{t'=0}^{t}\overline{f}(\beta,t')e^{-k\left[\beta^2+\lambda_n^2\right](t-t')}dt'\right]d\beta\right\}. \tag{21}$$

4. Special Case and Numerical Calculations

To study the mathematical thermoelastic behavior of a semi-infinite circular cylinder, we consider the following functions and parameters:

Set $f(z,t)=(1-e^{-\omega t})\,z\,e^{-z}$, with $\omega>0$, $t\rightarrow t'=5$ sec.

Dimensions

Radius of the circular cylinder $a = 1\ m$
Thickness of a circular cylinder $z = 0.2\ m$

Material Properties

The numerical calculation is done for an aluminum (pure) circular cylinder with the material properties as: thermal diffusivity $k = 84.18 \times 10^{-6} (m^2 s^{-1})$, density $\rho = 2707\ \ kg / m^3$, specific heat $c_p = 896\ J / kgK$, Poisson ratio $\nu = 0.35$, coefficient of linear thermal expansion $a_t = 22.2 \times 10^{-6}\ \frac{1}{K}$, and Lamé constant $\mu = 26.67$.

Roots of the Transcendental Equation

$\lambda_1 = 2.4048,\ \lambda_2 = 5.5201,\ \lambda_3 = 28.6537,\ \lambda_4 = 11.7915,\ \lambda_5 = 14.9309, \lambda_6 = 18.0711,$
$\lambda_7 = 21.2116,\ \lambda_8 = 24.3525,\ \lambda_9 = 27.4935,$ and $\lambda_{10} = 30.6346$ are the positive roots of the transcendental equation $J_0(\lambda_n \xi) = 0$.

We set, for convenience, $X = 2\sqrt{\frac{2}{\pi}}(1 + \nu)a_t$ and $Y = 4\sqrt{\frac{2}{\pi}}\mu(1 + \nu)a_t$, which assume to be the constants. The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000 and graphs are plotted with the help of Excel (MS Office-2003).

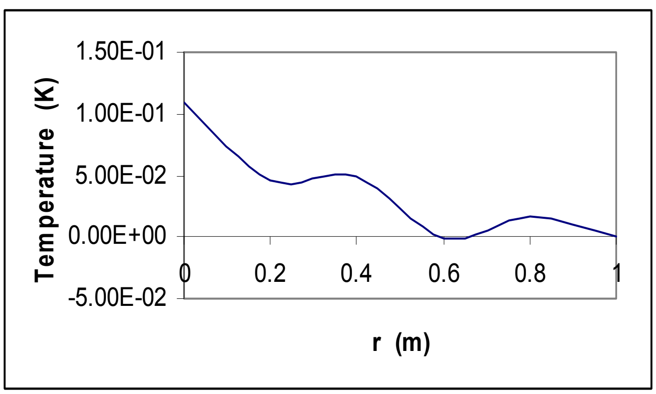

Figure 2. Temperature Distribution

From Figure 2, it is observed that, the temperature decreases from the centre $r = 0$ of a cylinder to the circular boundary $r = 1$. It is zero at the outer circular boundary $r = 1$.

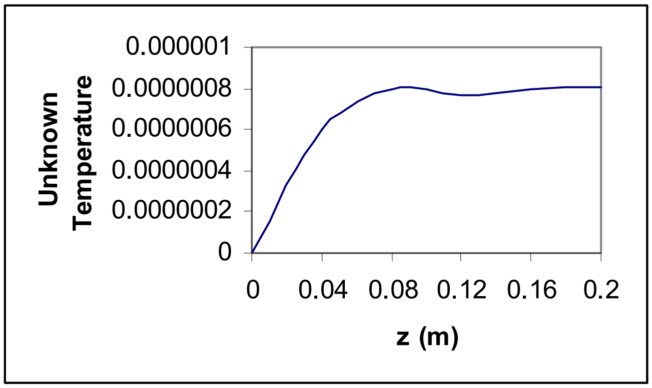

Figure 3. Unknown Temperature Distribution

From Figure 3, it is observed that the unknown temperature increases in the axial direction at $z = 0.08$ and then remains constant thereafter.

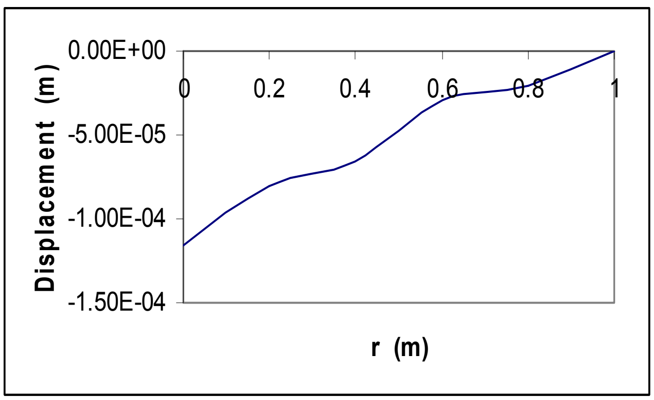

Figure 4. Displacement Function

From Figure 4, it is observed that the displacement is zero around the outer circular boundary $r = 1$ and it is maximum at the centre. Also, observe that the displacement is inversely proportional to the temperature.

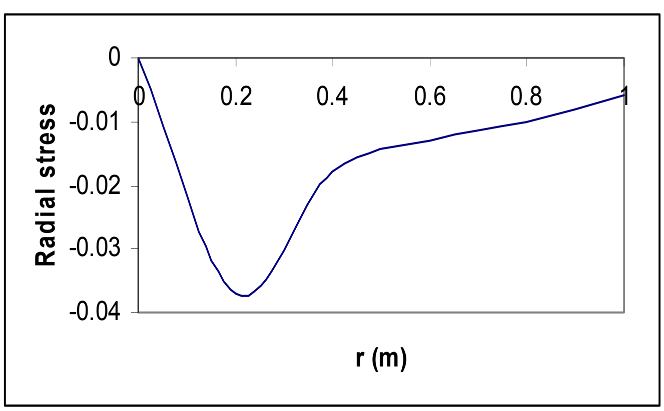

Figure 5. Radial Stress Function

From Figure 5, it is observed that the radial stress function develops compressive stresses in the radial direction.

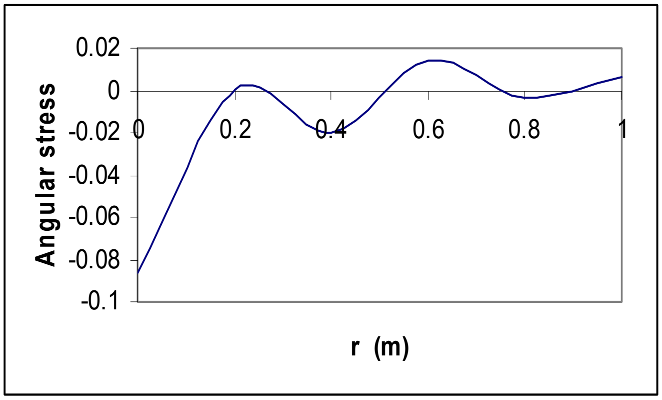

Figure 6. Angular Stress Function

Finally from Figure 6, it is observed that the angular stress function is increases from the centre to the outer circular edge.

5. Discussion

In this paper we consider the inverse heat conduction problem by Deshmukh et al. (2010) and determined thermal stresses on the curved surface of a semi- infinite circular cylinder.

As a special case, a mathematical model is constructed for aluminum (pure) semi-infinite circular cylinder with the material properties specified as above.

From Figures 2, 3, 4, 5 and 6, we observe that the temperature, unknown heating temperature, displacement, radial and angular stresses become negligible at infinite length. Moreover, variations take place in the temperature, unknown heating temperature, displacement, and radial and angular stresses across every cross-section due to an internal heat source.

This type of problem has applications in various industrial machines such as lathe machine and turbine. Also any particular case of special interest may be derived by assigning suitable values to the parameter and function in the expressions (16), (17), (19), (20) and (21).

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