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The Dynamics of Stage Structured Prey-Predator Model Involving Parasitic Infectious Disease

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Abstract

In this paper a prey-predator model involving parasitic infectious disease is proposed and analyzed. It is assumed that the life cycle of predator species is divided into two stages immature and mature. The analysis of local and global stability of all possible subsystems is carried out. The dynamical behaviors of the model system around biologically feasible equilibria are studied. The global dynamics of the model are investigated with the help of Suitable Lyapunov functions. Conditions for which the model persists are established. Finally, to nationalize our analytical results, numerical simulations are worked out for a hypothetical set of parameter values.

Keywords: Stage structure, Prey-Predator model, Parasitic Infectious disease, Stability, Lyapunov function, Persistence

MSC 2010: 92D25, 92D30, 34D20

1. Introduction

It is well known that, in nature species do not exist in exclusion. In fact, any given habitat may contain dozens or hundreds of species, sometimes thousands. Since any species has at least the potential to interact with any other species in its habitat, the spread of a disease in a community may rapidly become astronomical as the number of infected species in the habitat increases. Therefore, it is more of a biological significance to study the effect of disease on the dynamical behavior of interacting species. In the last two decades, some prey- predator models with infectious disease have been considered. Freedman (1990) discussed a prey-predator model with parasitic infection and obtained conditions for persistence. Anderson and May (1986) remarked that a new strain of parasite could change the dynamics of a resident prey-predator or host-parasite system. Mukherjee (1998) studied a generalized prey-predator system with parasitic infection and derived a condition for persistence and impermanence. Xiao and Chen (2001) investigated a prey-predator system with disease in the prey. They showed boundedness of solutions, the nature of equilibria, permanence and global stability, and observed Hopf-bifurcation when the delay increases. Mukherjee (2003) considered delayed prey-predator model with disease in the prey and derived persistence conditions. Recently, Mukherjee (2006) analyzed a prey-predator model in which some members of a prey population and all predators are subjected to infection by a parasite. All these studies converged at one conclusion: that disease may cause vital changes in the dynamics of an ecosystem.

On the other hand, it is well known that, the age factor is importance for the dynamics and evolution of many mammals. The rate of survival, growth and reproduction almost depend on age or development stage and it has been noticed that the life history of many species is composed of at least two stages, immature and mature, and the species in the first stage may often neither interact with other species nor reproduce, being raised by their mature parents. Most of classical prey-predator models of two species in the literature assumed that all predators are able to attack their prey and reproduce, ignoring the fact that the life cycle of most animals consists of at least two stages (immature and mature). In the last three decades several of the prey-predator models with stage-structure are proposed and analyzed [Wang (1997), Wang and Chen (1997), Wang et al (2001), Xiao and Chen (2003), Georgescu and Hsieh (2007)].

In this paper, a prey-predator model involving both stage structure and parasitic infectious disease is proposed and analyzed. The effects of the parasite infection disease and the stage structure on the dynamical behavior of prey-predator model are considered analytically as well as numerically.

2. Mathematical Model

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density is denoted by $X(t)$, interacting with predator whose total population density is denoted by $Y(t)$. It is assumed that some members of the prey population and all predators are subjected to infection by a parasite. In addition, the formulation of the proposed eco-epidemiological model depends on the following assumptions:

1. In the absence of parasites and predators, the prey population grows logistically with carrying capacity $k\ (k > 0)$ and an intrinsic birth rate constant $r\ (r > 0)$. Then the evolution of the prey can be represented as:

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$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k}\right) \; ; \; X(0) \geq 0 \; . \tag{1}$$

2. In the presence of parasites, the total prey population is divided into two classes, namely, the susceptible prey $s(t)$ and the infected prey $i(t)$, so that at any time t we have $X(t) = s(t) + i(t)$. Further, susceptible prey becomes infected, due to existence of parasites, at a specific infection rate of β_0 $(\beta_0 > 0)$.

3. It is assumed that only susceptible prey are capable of reproducing logistically and that the infected prey population dies, with specific death rate constant c $(c > 0)$, before having the chance to reproduce. However, the infected prey population still contributes along with susceptible prey population to population growth towards the carrying capacity. Hence, the evolution equations of prey population become

$$\begin{aligned} \frac{ds}{dt} &= rs\left(1 - \frac{s+i}{k}\right) - \beta_0 s \quad ; \; s(0) \geq 0 \\ \frac{di}{dt} &= \beta_0 s - ci \quad\quad\quad\quad ; \; i(0) \geq 0 \end{aligned} \; . \tag{2}$$

4. In the case of the existence predator, it is assumed that, the predator population is divided in to two classes, immature predator, whose total population density is denoted by $y_1(t)$ and the other mature, whose total population density is denoted by $y_2(t)$. Moreover, only the mature predator can attack prey population (without distinguish between infected i and healthy s prey) according to Holling type-Π functional response with maximum attack rate a $(a > 0)$ and half saturation constant b $(b > 0)$, and have reproduction ability. While, the immature predator does not attack prey and has no reproductive ability, instead of that, it depends completely on the food supplied by mature predator.

5. In addition to the above, it is assumed that, the immature predator becomes mature predator at a specific rate constant D $(D > 0)$, and the predator populations (immature and mature) decrease due to the natural death rates d_1 $(d_1 > 0)$ and d_2 $(d_2 > 0)$ respectively.

6. Finally, it is assumed that, susceptible prey individuals that infected by mature predators parasites are removed from the susceptible class at a specific rate proportional with mature predator population (i.e. $\beta_1 y_2$ where $\beta_1 > 0$ is infection rate due to predator) and an equivalent number of prey are added to the infected class.

Consequently, the dynamics of a stage structured prey-predator model with parasitic infection can be represented in the following set of equations:

$$
\begin{aligned}
\frac{ds}{dt} &= rs\left(1-\frac{s+i}{k}\right)-(\beta_0+\beta_1 y_2)s-\frac{asy_2}{b+s+i} \quad ; s(0)\geq 0 \\
\frac{di}{dt} &= (\beta_0+\beta_1 y_2)s-ci-\frac{aiy_2}{b+s+i} \quad ; i(0)\geq 0 \\
\frac{dy_1}{dt} &= -d_1 y_1 - Dy_1 + m\left(\frac{s+i}{b+s+i}\right)y_2 \quad ; y_1(0)\geq 0 \\
\frac{dy_2}{dt} &= Dy_1 - d_2 y_2 \quad ; y_2(0)\geq 0.
\end{aligned} \tag{3}
$$

In order to simplifying the proposed model represented by system (3), the following dimensionless variables and parameters are used:

$$
T = rt,\ S=\frac{s}{k},\ I=\frac{i}{k},\ Y_1=\frac{y_1}{k},\ Y_2=\frac{\beta_1}{r}y_2, w_1=\frac{\beta_0}{r}, w_2=\frac{a}{\beta_1 k},
$$
$$
w_3=\frac{b}{k}, w_4=\frac{c}{r}, w_5=\frac{d_1}{r}, w_6=\frac{D}{r}, w_7=\frac{m}{\beta_1 k},\ w_8=\frac{Dk\beta_1}{r^2}, w_9=\frac{d_2}{r}.
$$

Consequently, the proposed model can be written in the following dimensionless form:

$$
\begin{aligned}
\frac{dS}{dT} &= S\ [1-(S+I)] - w_1 S - SY_2 - \frac{w_2 SY_2}{w_3+S+I} = S\ f_1(S,I,Y_1,Y_2) \\
\frac{dI}{dT} &= w_1 S + SY_2 - w_4 I - \frac{w_2 IY_2}{w_3+S+I} = I\ f_2(S,I,Y_1,Y_2) \\
\frac{dY_1}{dT} &= -w_5 Y_1 - w_6 Y_1 + w_7\left(\frac{S+I}{w_3+S+I}\right)Y_2 = Y_1 f_3(S,I,Y_1,Y_2) \\
\frac{dY_2}{dT} &= w_8 Y_1 - w_9 Y_2 = Y_2 f_4(S,I,Y_1,Y_2).
\end{aligned} \tag{4}
$$

The initial condition for system (4) may be taken as any point in the region $R_+^4 = \{(S,I,Y_1,Y_2)\in R^4 : S\geq 0, I\geq 0, Y_1\geq 0, Y_2\geq 0\}$. Obviously, the interaction functions in the right hand side of system (4) are continuously differentiable functions on R_+^4, hence they are Lipschitizian. Therefore the solution of system (4) exists and is unique. Further, all the solutions of system (4) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem 1. All the solutions of system (4), which initiate in R_+^4 are uniformly bounded.

Proof:
Let $(S(T), I(T), Y_1(T), Y_2(T))$ be any solution of system (4) with non negative initial condition $(S_0, I_0, Y_{10}, Y_{20})$. Since we have

$$\frac{dS}{dT} \le S(1-S) .$$

Then according to the theory of differential inequality [Birkhoof and Rota (1982)], we have

$$Sup\ S(T) \le M, \quad \forall T \ge 0 \text{, where } M = \max\{S_0, 1\} .$$

Now, consider the function: $W(T) = S(T) + I(T) + Y_1(T) + Y_2(T)$. Then the time derivative of $W(T)$ along the solution of the system (4) is:

$$\frac{dW}{dT} \le 2S - S - w_4 I - (w_5 + w_6 - w_8) Y_1 - w_9 Y_2 .$$

Note that since the parameters w_5 and w_6 stand for the natural death rate of immature predator and the grown up rate of immature predator to mature predator respectively. While, w_8 represents the conversion rate from immature predator to mature predator. Therefore the following condition always satisfied.

$$w_5 + w_6 \ge w_8 . \qquad (5)$$

Hence, we obtain:

$$\frac{dW}{dT} + m\ W \le 2M ,$$

where $m = \min\{1, w_4, w_5 + w_6 - w_8, w_9\}$.

Again, due to the theory of differential inequalities we obtain

$$W(T) \le \frac{2M}{m} - \frac{2M}{me^{mT}} + \frac{W_0}{e^{mT}} \text{, where } W_0 = (S(0), I(0), Y_1(0), Y_2(0)) .$$

Thus, $\forall T \ge 0$ we have that $0 \le W(T) \le M_1$, where $M_1 = \max\left\{\frac{2M}{m}, W_0\right\}$. Thus all solutions of system (4) are uniformly bounded, and then the proof is complete.

Keeping the above in view, since the dynamical system is said to be dissipative if and only if it is uniformly bounded, then system (4) is dissipative.

3. Stability Analysis of 2D Predator Free Subsystem

It is well known that according to the prey-predator interaction, the prey species can survive in the absence of predator, and since the prey population is divided into two classes namely susceptible prey population S and the infected prey population I. Hence, the following 2D predator free subsystem is obtained:

$$\begin{aligned}\frac{dS}{dT} &= S\left[1 - S - I - w_1\right] = g_1(S,I)\\ \frac{dI}{dT} &= I\left[\frac{w_1 S}{I} - w_4\right] \quad = g_2(S,I)\end{aligned} \quad . \tag{6}$$

The analysis of local and global stability of subsystem (6) is carried out and the following results are obtained:

1. The vanishing equilibrium point $P_0 = (0,0)$ always exists and its locally asymptotically stable provided that

   $$w_1 > 1 \; . \tag{7}$$

2. There is no axial equilibrium point such as $P_1 = (\hat{S},0)$; $\hat{S} > 0$ on the $S-$axis due to the fact that $w_1 > 0$. However $P_1 = (1,0)$ exists if we set $w_1 = 0$.

3. The interior equilibrium point $P_2 = (\widetilde{S}, \widetilde{I})$ in the $Int.R^2_{+(SI)}$, where

   $$\widetilde{S} = \frac{w_4(1-w_1)}{w_1 + w_4}\; ; \; \widetilde{I} = \frac{w_1(1-w_1)}{w_1 + w_4}, \tag{8}$$

   exists under the following condition:

   $$w_1 < 1. \tag{9}$$

Obviously, from a biological point of view the above condition $w_1 (= \frac{\beta_0}{r}) < 1$, which means that the disease contact rate β_0 is less than the reproduction rate r, is the necessary condition for the existence of (endemic) equilibrium point P_2 otherwise (i.e. when $\beta_0 > r$) $\frac{dS}{dT} < 0$ and hence the prey species will face extinction. In addition, the equilibrium point P_2 is locally asymptotically stable whenever it exists. Further, the global dynamics of P_2 is carried out in the next theorem.

Theorem 2. The equilibrium point $P_2 = (\widetilde{S}, \widetilde{I})$ is globally asymptotically stable in the $Int.R^2_{+(SI)} = \left\{(S,I) \in R^2; S > 0, I > 0\right\}$.

Proof:

Consider the function $H(S,I)=\frac{1}{SI}$; clearly $H(S,I)$ is C^1 function and is positive for all $(S,I) \in Int.R^2_{+(SI)}$. Note that

$$\Delta(S,I)=\frac{\partial(Hg_1)}{\partial S}+\frac{\partial(Hg_2)}{\partial I}=\frac{-(I+w_1)}{I^2}<0.$$

Hence $\Delta(S,I)$ does not change sign and is not identically zero in the $Int.R^2_{+(SI)}$. Then according to Bendixon-Dulic criterion, there is no periodic solution in the $Int.R^2_{+(SI)}$. So, since all the solutions of the subsystem (6) are uniformly bounded and P_2 is unique equilibrium point in the $Int.R^2_{+(SI)}$. Hence by using the Poincare-Bendixon theorem P_2 is a globally asymptotically stable and hence the proof is complete.

4. Stability Analysis of 3D Disease Free Subsystem

In the absence of disease the prey population contains only susceptible individuals and the following 3D disease free subsystem of system (4) appears:

$$\begin{aligned}
\frac{dS}{dT} &= S\left[1-S-w_1-Y_2-w_2\frac{Y_2}{w_3+S}\right]= S\,J_1(S,Y_1,Y_2)\\
\frac{dY_1}{dT} &= Y_1\left[-(w_5+w_6)+\frac{w_7SY_2}{(w_3+S)Y_1}\right]= Y_1J_2(S,Y_1,Y_2) \qquad (10)\\
\frac{dY_2}{dT} &= Y_2\left[\frac{w_8Y_1}{Y_2}-w_9\right]= Y_2J_3(S,Y_1,Y_2).
\end{aligned}$$

Our analysis of subsystem (10) gave the following results:

1. The vanishing equilibrium point $F_0=(0,0,0)$ always exists and is unstable saddle point provided that condition (9) holds, while it is locally asymptotically stable provided that condition (7) holds.

2. The axial equilibrium point $F_1=(1-w_1,0,0)$, exists under condition (9) and is locally asymptotically stable in the $R^3_{+(SY_1Y_2)}$ if and only if the following condition holds:

$$w_9(w_5+w_6)>\frac{w_7w_8(1-w_1)}{w_3+(1-w_1)}. \qquad (11a)$$

While it is unstable saddle point provided that

$$w_9(w_5+w_6) < \frac{w_7 w_8(1-w_1)}{w_3+(1-w_1)}. \tag{11b}$$

3. The interior equilibrium point $F_2 = (\bar{S}, \bar{Y}_1, \bar{Y}_2)$, where

$$\bar{S} = \frac{w_3 w_9(w_5+w_6)}{w_7 w_8 - w_9(w_5+w_6)},\ \bar{Y}_1 = \frac{w_9}{w_8}\bar{Y}_2,\ \bar{Y}_2 = \frac{(w_3+\bar{S})(1-\bar{S}-w_1)}{w_2+w_3+\bar{S}}. \tag{12}$$

exists in the $Int.R^3_{+(SY_1Y_2)} = \{(S, Y_1, Y_2) \in R^3 : S > 0, Y_1 > 0, Y_2 > 0\}$ under the following conditions:

$$w_7 w_8 > w_9(w_5+w_6) \tag{13a}$$

$$(1-\bar{S}-w_1) > 0 \Leftrightarrow \bar{S}+w_1 < 1\ . \tag{13b}$$

In addition it is locally asymptotically stable in the $Int.R^3_{+(SY_1Y_2)}$ provided that:

$$\bar{Y}_2 < \min\left\{\frac{\delta_1^2}{w_2}, \frac{w_9\delta_2^2\delta_3 + \delta_1\delta_2\delta_3^2}{w_3 w_7 w_8 w_9^2 \delta_1^2 (w_2+\delta_1)}\right\}, \tag{14}$$

where $\delta_1 = w_3 + \bar{S}$, $\delta_2 = \bar{S}(\delta_1^2 - w_2\bar{Y}_2)$ and $\delta_3 = w_7 w_8 \bar{S} + w_9^2\delta_1$. Moreover, the global stability of F_2 is investigated in the following theorem.

Theorem 3. Assume that $F_2 = (\bar{S}, \bar{Y}_1, \bar{Y}_2)$ is locally asymptotically stable in the $Int.R^3_{+(SY_1Y_2)}$, and let the following conditions are hold

$$a > 0\ , \tag{15a}$$

$$\frac{w_3 w_7 \bar{Y}_2}{\delta\delta_1 Y_1 \sqrt{\frac{w_7 w_8 \bar{S}}{\delta_1 w_9 Y_1}}} \le \sqrt{\frac{a}{\delta\delta_1}} \le \frac{\delta + w_2}{\delta\sqrt{\frac{w_9}{Y_2}}}, \tag{15b}$$

$$\frac{w_7 S Y_2 + \delta w_8 Y_1}{\delta Y_1 Y_2} \le \sqrt{\frac{w_7 w_8 \bar{S}}{\delta_1 w_9 Y_1}}\ \sqrt{\frac{w_9}{Y_2}}\ , \tag{15c}$$

where $a = (\delta\delta_1 - w_2\bar{Y}_2)$ and $\delta = (w_3 + S)$. Then F_2 is globally asymptotically stable in the $IntR^3_{+(SY_1Y_2)}$.

Proof:

Consider the following function:

$$U(S,Y_1,Y_2)=\left[S-\overline{S}-\overline{S}\ln\frac{S}{\overline{S}}\right]+\left[Y_1-\overline{Y}_1-\overline{Y}_1\ln\frac{Y_1}{\overline{Y}_1}\right]+\left[Y_2-\overline{Y}_2-\overline{Y}_2\ln\frac{Y_2}{\overline{Y}_2}\right].$$

Clearly $U:R^3_{+(SY_1Y_2)}\to R$, and is a C^1 positive definite function. Now, the derivative of U along the trajectory of subsystem (10) can be written as

$$\begin{aligned}\frac{dU}{dT}=&-\left[\left(\frac{\delta\delta_1-w_2\overline{Y}_2}{2\delta\delta_1}\right)(S-\overline{S})^2+\left(\frac{\delta+w_2}{\delta}\right)(S-\overline{S})(Y_2-\vec{Y}_2)+\frac{w_9}{2Y_2}(Y_2-\overline{Y}_2)^2\right]\\&-\left[\left(\frac{\delta\delta_1-w_2\overline{Y}_2}{2\delta\delta_1}\right)(S-\overline{S})^2-\frac{w_3w_7\overline{Y}_2}{\delta\delta_1Y_1}(S-\overline{S})(Y_1-\overline{Y}_1)+\frac{w_7w_8\overline{S}}{2\delta_1w_9Y_1}(Y_1-\overline{Y}_1)^2\right]\\&-\left[\frac{w_7w_8\overline{S}}{2\delta_1w_9Y_1}(Y_1-\overline{Y}_1)^2-\left(\frac{w_7SY_2+\delta w_8Y_1}{\delta Y_1Y_2}\right)(Y_1-\overline{Y}_1)(Y_2-\overline{Y}_2)+\frac{w_9}{2Y_2}(Y_2-\overline{Y}_2)^2\right].\end{aligned}$$

Therefore, according to conditions (15a-15c) we obtain that:

$$\begin{aligned}\frac{dU}{dT}\le&-\left[\sqrt{\tfrac{a}{2\delta\delta_1}}(S-\overline{S})+\sqrt{\tfrac{w_9}{2Y_2}}(Y_2-\overline{Y}_2)\right]^2-\left[\sqrt{\tfrac{a}{2\delta\delta_1}}(S-\overline{S})-\sqrt{\tfrac{w_7w_8\overline{S}}{2\delta_1w_9Y_1}}(Y_1-\overline{Y}_1)\right]^2\\&-\left[\sqrt{\tfrac{w_7w_8\overline{S}}{2\delta_1w_9Y_1}}(Y_1-\overline{Y}_1)-\sqrt{\tfrac{w_9}{2Y_2}}(Y_2-\overline{Y}_2)\right]^2.\end{aligned}$$

Hence, $\frac{dU}{dT}<0$, and then U is strictly Lyapunov function. Therefore, F_2 is globally asymptotically stable in the $Int.R^3_{+(SY_1Y_2)}$.

5. Local Stability Analysis of System (4)

In this section, the existence and local stability analysis of all possible equilibrium points of system (4) are discussed and the following results are obtained:

1. The vanishing equilibrium point $E_0=(0,0,0,0)$ always exists.
2. There is no axial equilibrium point such as $E_1=(\hat{S},0,0,0)$; $\hat{S}>0$ on the S-axis due to the fact that $w_1>0$. However $E_1=(1,0,0,0)$ exists if we set $w_1=0$.
3. The predator free equilibrium point $E_2=(\widetilde{S},\widetilde{I},0,0)$; where $\widetilde{S}$ and $\widetilde{I}$ are given in Equation(8), exists under the condition (9) .

4. The disease free equilibrium point $E_3 = (\overline{S}, 0, \overline{Y}_1, \overline{Y}_2)$; where $\overline{S}$, $\overline{Y}_1$ and $\overline{Y}_2$ are given in Equation(12), exists under the conditions (13a-13b).
5. The positive equilibrium point $E_4 = (S^*, I^*, Y_1^*, Y_2^*)$; where

$$S^* = \frac{h(w_4\delta_1 + w_2 Y_2^*)}{M_1},\ I^* = \frac{h\delta_1(w_1 + Y_2^*)}{M_1},\ Y_1^* = \frac{w_9}{w_8}Y_2^*,\ Y_2^* = \frac{\delta_1\left(1 - \overline{S} - w_1\right)}{M_2}, \tag{16}$$

with $M_1 = \delta_1(w_1 + w_4 + Y_2^*) + w_2 Y_2^*$, $M_2 = \left(w_2 + \delta_1\right)$, $\overline{S} = (S^* + I^*)$ and $\overline{S}$ is given in Equation (12), exists uniquely in the $Int.R_+^4$ provided that conditions (13a-13b) hold.

In addition, it is observed that, the eigenvalues of the Jacobian matrix of system (4) at E_0, say $J(E_0)$, are:

$$\lambda_{0S} = 1 - w_1,\ \lambda_{0I} = -w_4 < 0,\ \lambda_{0Y_1} = -(w_5 + w_6) < 0,\ \lambda_{0Y_2} = -w_9 < 0\ . \tag{17}$$

Therefore, E_0 is unstable saddle point with locally stable manifold in the $R^3_{+(IY_1Y_2)}$ (i.e. $\dim(\omega^s) = 3$) and with locally unstable manifold in the $S-$direction (i.e. $\dim(\omega^u) = 1$) provided that condition (9) holds. However, it is locally asymptotically stable provided that condition (7) holds.
The eigenvalues of Jacobian matrix of system (4) at E_2, say $J(E_2)$, satisfy the following relations:

$$\lambda_{2S} + \lambda_{2I} = -(\widetilde{S} + w_4) < 0 \tag{18a}$$

$$\lambda_{2S}.\lambda_{2I} = (w_1 + w_4)\widetilde{S} > 0 \tag{18b}$$

$$\lambda_{2Y_1} + \lambda_{2Y_2} = -(w_5 + w_6 + w_9) < 0 \tag{18c}$$

$$\lambda_{2Y_1}.\lambda_{2Y_2} = w_9(w_5 + w_6) - \tfrac{w_7 w_8(1 - w_1)}{w_3 + (1 - w_1)}\ . \tag{18d}$$

Note that, it is easy to verify that, according to Eqs. (18a-18d) all the eigenvalues of $J(E_2)$ have negative real parts and hence E_2 is locally asymptotically stable in the R_+^4 if and only if condition (11a) holds. However, E_2 is an unstable saddle point in the R_+^4 with locally unstable manifold of dimension one (i.e. $\dim \omega^u = 1$) and with locally stable manifold of dimension three (i.e. $\dim \omega^s = 3$) if condition (11b) holds.

Now, the Jacobian matrix at the disease free equilibrium point $E_3 = (\bar{S}, 0, \bar{Y}_1, \bar{Y}_2)$ can be written as:

$$J(E_3) = [b_{ij}]_{4\times 4} \ ; \ i, j = 1,2,3,4, \tag{19}$$

where $b_{11} = 1 - 2\bar{S} - w_1 - \frac{(\delta_1^2 + w_2 w_3)\bar{Y}_2}{\delta_1^2}$, $b_{12} = \bar{S}\left(-1 + \frac{w_2 \bar{Y}_2}{\delta_1^2}\right)$, $b_{13} = 0$, $b_{14} = -\bar{S}\left(1 + \frac{w_2}{\delta_1}\right) < 0$, $b_{21} = w_1 + \bar{Y}_2 > 0$, $b_{22} = -\left(w_4 + \frac{w_2 \bar{Y}_2}{\delta_1}\right) < 0$, $b_{23} = 0$, $b_{24} = \bar{S} > 0$, $b_{31} = b_{32} = \frac{w_3 w_7 \bar{Y}_2}{\delta_1^2} > 0$, $b_{33} = -(w_5 + w_6) < 0$, $b_{34} = \frac{w_7 \bar{S}}{\delta_1} > 0$, $b_{41} = b_{42} = 0$, $b_{43} = w_8 > 0$, $b_{44} = -w_9 < 0$. Then the characteristic equation of the Jacobian matrix $J(E_3)$ is given by:

$$\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0 \tag{20a}$$

Here, $A_1 = -(\sigma_1 + \sigma_2)$, $A_2 = -\sigma_3 + \sigma_1 \sigma_2$, $A_3 = \sigma_2 \sigma_3 - b_{31} b_{43} \sigma_4$, $A_4 = b_{31} b_{43} (\sigma_5 + \sigma_6)$ with $\sigma_1 = b_{11} + b_{22}$, $\sigma_2 = b_{33} + b_{44}$, $\sigma_3 = b_{12} b_{21} - b_{11} b_{22}$, $\sigma_4 = b_{14} + b_{24}$, $\sigma_5 = b_{14} b_{22} - b_{12} b_{24}$ and $\sigma_6 = b_{11} b_{24} - b_{14} b_{21}$.

Note that, according to the elements of $J(E_3)$, it is easy to verify that:

$$\sigma_1 = \frac{1}{\delta_1^2}\left[\left(\delta_1^2 (1 - 2\bar{S} - w_1) - (\delta_1^2 + w_2 w_3)\bar{Y}_2\right) - (w_4 \delta_1 + w_2 \bar{Y}_2)\delta_1\right],$$

$$\sigma_2 = -(w_5 + w_6 + w_9) < 0,$$

$$\sigma_3 = \frac{1}{\delta_1^2}\left[\bar{S}(w_2 \bar{Y}_2 - \delta_1^2)(w_1 + \bar{Y}_2) + \left(\delta_1^2 (1 - 2\bar{S} - w_1) - (\delta_1^2 + w_2 w_3)\bar{Y}_2\right)(w_4 \delta_1 + w_2 \bar{Y}_2)\right],$$

$$\sigma_4 = -\frac{w_2 \bar{S}}{\delta_1} < 0,$$

$$\sigma_5 = \frac{\bar{S}}{\delta_1^2}\left[(\delta_1 + w_2)(w_4 \delta_1 + w_2 \bar{Y}_2) - \bar{S}(w_2 \bar{Y}_2 - \delta_1^2)\right],$$

$$\sigma_6 = \frac{\bar{S}}{\delta_1^2}\left[\delta_1^2 (1 - 2\bar{S}) + w_1 w_2 \delta_1 + w_2 \bar{S} \bar{Y}_2\right].$$

Further

$$\begin{aligned}\Delta &= A_1 A_2 A_3 - A_3^2 - A_1^2 A_4 \\ &= \left[\sigma_3^2 - \sigma_1\sigma_2\sigma_3 + b_{31}b_{43}\sigma_1\sigma_4 - \sigma_2^2\sigma_3 + b_{31}b_{43}\sigma_2\sigma_4\right]\sigma_1\sigma_2 \\ &\quad + \left[-\sigma_1\sigma_3\sigma_4 + (\sigma_2\sigma_3 - b_{31}b_{43}\sigma_4)\sigma_4 - (\sigma_1+\sigma_2)^2(\sigma_5+\sigma_6)\right]b_{31}b_{43}.\end{aligned} \tag{20b}$$

Now, from the Routh-Hurwitz criterion all the eigenvalues of the $J(E_3)$ have roots with negative real parts and hence E_3 is locally asymptotically stable in the R_+^4, if and only if $A_i (i=1,3,4) > 0$ and $\Delta > 0$.

Straight forward computations show that, σ_1 and σ_3 are negative while σ_5 and σ_6 are positive under the following condition:

$$\max\left\{\frac{(1-2\overline{S}-w_1)\delta_1^2}{\delta_1^2 + w_2 w_3}, \frac{\delta_1[(2\overline{S}-1)\delta_1 - w_1 w_2]}{w_2\overline{S}}, 0\right\} < \overline{Y}_2 < \frac{\delta_1^2}{w_2}. \tag{21a}$$

Consequently, we obtain that $A_i > 0$ for $i = 1,3,4$. Moreover, in addition to condition (21a), it is observed that $\Delta > 0$ provided that:

$$\sigma_1 < \frac{1}{\sigma_3\sigma_4}\left[(\sigma_2\sigma_3 - b_{31}b_{43}\sigma_4)\sigma_4 - (\sigma_1+\sigma_2)^2(\sigma_5+\sigma_6)\right]. \tag{21b}$$

Hence, from to the above analysis, the following theorem can be proved easily.

Theorem 4. Assume that the disease free equilibrium point $E_3 = (\overline{S}, 0, \overline{Y}_1, \overline{Y}_2)$ exists in the R_+^4, then E_3 is locally asymptotically stable if and only if conditions (21a-21b) are hold.

Finally, the Jacobian matrix of system (4) at the positive equilibrium point $E_4 = (S^*, I^*, Y_1^*, Y_2^*)$ can be written as:

$$J(E_4) = [c_{ij}]_{4\times 4}; i,j = 1,2,3,4, \tag{22}$$

where $c_{11} = c_{12} = S^*\left(-1 + \frac{w_2 Y_2^*}{\delta_1^2}\right)$, $c_{13} = 0$, $c_{14} = \frac{-S^*}{\delta_1} M_2 < 0$, $c_{21} = w_1 + Y_2^* + \frac{w_2 Y_2^* I^*}{\delta_1^2} > 0$, $c_{22} = \frac{-(w_1 + Y_2^*)S^*}{I^*} + \frac{w_2 Y_2^* I^*}{\delta_1^2}$, $c_{23} = 0$, $c_{24} = S^* - \frac{w_2 I^*}{\delta_1}$, $c_{31} = c_{32} = \frac{w_3 w_7 Y_2^*}{\delta_1^2} > 0$, $c_{33} = \frac{-w_7 w_8 \overline{S}}{w_9 \delta_1} < 0$, $c_{34} = \frac{w_7 \overline{S}}{\delta_1} > 0$, $c_{41} = c_{42} = 0$, $c_{43} = w_8 > 0$, $c_{44} = -w_9 < 0$. Accordingly the characteristic equation of $J(E_4)$ is given by:

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0\,. \tag{23}$$

Here, $B_1 = -(\gamma_1 + \gamma_2)$, $B_2 = -\gamma_3 + \gamma_1\gamma_2$, $B_3 = \gamma_2\gamma_3 - c_{31}c_{43}\gamma_4$, $B_4 = c_{31}c_{43}(\gamma_5 + \gamma_6)$, with $\gamma_1 = c_{11} + c_{22}$, $\gamma_2 = c_{33} + c_{44}$, $\gamma_3 = c_{12}c_{21} - c_{11}c_{22}$, $\gamma_4 = c_{14} + c_{24}$, $\gamma_5 = c_{14}c_{22} - c_{12}c_{24}$, $\gamma_6 = c_{11}c_{24} - c_{14}c_{21}$, and

$$\begin{aligned}\Delta^* &= B_1B_2B_3 - B_3^2 - B_1^2B_4\\ &= \left[\gamma_3^2 - \gamma_1\gamma_2\gamma_3 + c_{31}c_{43}\gamma_1\gamma_4 - \gamma_2^2\gamma_3 + c_{31}c_{43}\gamma_2\gamma_4\right]\gamma_1\gamma_2\\ &\quad + \left[-\gamma_1\gamma_3\gamma_4 + (\gamma_2\gamma_3 - c_{31}c_{43}\gamma_4)\gamma_4 - (\gamma_1+\gamma_2)^2(\gamma_5+\gamma_6)\right]c_{31}c_{43}.\end{aligned}$$

So, due to the elements of $J(E_4)$, it is easy to verify that:

$$\gamma_1 = \frac{S^*}{\delta_1^2}\left(w_2Y_2^* - \delta_1^2\right) - \frac{(w_1+Y_2^*)S^*}{I^*} + \frac{w_2I^*Y_2^*}{\delta_1^2},\ \gamma_2 = -\frac{w_7w_8\bar{S}}{w_9\delta_1} - w_9 < 0\,,$$

$$\gamma_3 = S^*\left(-1 + \frac{w_2Y_2^*}{\delta_1^2}\right)\left(w_1 + Y_2^* + \frac{(w_1+Y_2^*)S^*}{I^*}\right),\ \gamma_4 = -\frac{w_2\bar{S}}{\delta_1} < 0\,,$$

$$\gamma_5 = -\frac{S^*M_2}{\delta_1}\left(\frac{w_2Y_2^*I^*}{\delta_1^2} - \frac{(w_1+Y_2^*)S^*}{I^*}\right) - S^*\left(-1 + \frac{w_2Y_2^*}{\delta_1^2}\right)\left(S^* - \frac{w_2I^*}{\delta_1}\right),$$

$$\gamma_6 = S^*\left(S^* - \frac{w_2I^*}{\delta_1}\right)\left(-1 + \frac{w_2Y_2^*}{\delta_1^2}\right) + \frac{S^*M_2}{\delta_1}\left(w_1 + Y_2^* + \frac{w_2I^*Y_2^*}{\delta_1^2}\right).$$

Therefore, in the following theorem, the local stability conditions for the positive equilibrium point E_4 in the $Int.R_+^4$ is established.

Theorem 5. Assume that $E_4 = (S^*, I^*, Y_1^*, Y_2^*)$ exists in the $Int.R_+^4$ and the following conditions are satisfied:

$$S^* < \frac{w_2I^*}{\delta_1}\,, \tag{24a}$$

$$\max\left\{\frac{\delta_1{}^2S^* - \delta_1[w_1(w_2 + \bar{S}) + w_2(I^* + w_1)]}{w_2\bar{S} + \delta_1M_2}, 0\right\} < Y_2^* < \frac{\delta_1{}^2}{w_2}\,, \tag{24b}$$

$$\gamma_1 < \frac{1}{\gamma_3\gamma_4}\left[(\gamma_2\gamma_3 - b_{31}b_{43}\gamma_4)\gamma_4 - (\gamma_1+\gamma_2)^2(\gamma_5+\gamma_6)\right]. \tag{24c}$$

Then, E_4 is locally asymptotically stable in the $Int.R_+^4$.

Proof:

According to the Routh-Hurwitz criterion the proof follows if and only if $B_i(i=1,3,4)>0$ and $\Delta^*>0$. Now, straightforward computations show that under the conditions (24a-24b) we obtain that γ_1 and γ_3 are negative while γ_5 and γ_6 are positive.

Consequently, due to the coefficients of Equation(23) and elements of $J(E_4)$, we get that $B_i>0$ for $i=1,3,4$. Moreover, it is easy to verify that $\Delta^*>0$ if and only if in addition to the conditions (24a-24b), condition (24c) holds too. Hence, $J(E_4)$ have eigenvalues with negative real parts. Therefore E_4 is locally asymptotically stable in the $Int.R_+^4$ and the proof is complete.

6. Global Dynamical Behavior of System (4)

In this section the global stability for the equilibrium points of system (4) is investigated using the Lyapunov method as shown in the following theorems.

Theorem 6. Assume that the vanishing equilibrium point E_0 of system (4) is locally asymptotically stable in the R_+^4. Then E_0 is globally asymptotically stable in the R_+^4.

Proof:

Consider the function

$$V_0(S,I,Y_1,Y_2)=c_1S+c_2I+c_3Y_1+c_4Y_2\,. \tag{25}$$

Clearly $V_0:R_+^4\to R$ is C^1 positive definite function, where $c_i,(i=1,2,3,4)$ are nonnegative constants to be determined. Now, since the derivative of V_0 along the trajectory of system (4) can be written as:

$$\begin{aligned}\tfrac{dV_0}{dT}&=(c_1-c_1w_1+c_2w_1)S-c_1S^2-c_1SI-(c_1-c_2)SY_2-\tfrac{c_1w_2-c_3w_7}{\alpha}SY_2\\&\quad-c_2w_4I-\tfrac{c_1w_2-c_3w_7}{\alpha}IY_2-\big(c_3(w_5+w_6)-c_4w_8\big)Y_1-c_4w_9Y_2,\end{aligned}$$

where $\alpha=(w_3+S+I)$. Now choosing the positive constants $c_1=1$, $c_2=0$, $c_3=\frac{w_2}{w_1w_7}$ and $c_4=\frac{w_2(w_5+w_6)}{w_1w_7w_8}$, yield that

$$\frac{dV_0}{dT} < (1 - w_1)S - \frac{w_2}{\alpha}\left(1 - \frac{1}{w_1}\right)(SY_2 + IY_2).$$

Clearly $\frac{dV_0}{dT} < 0$ under the local stability condition (7), hence V_0 is strictly Lyapunov function. Therefore E_0 is globally asymptotically stable in the R_+^4.

Now, in the following theorem we will study the global behavior of E_2.

Theorem 7. Assume that the predator free equilibrium point E_2 is locally asymptotically stable in the R_+^4, and let the following conditions:

$$\frac{w_1 - I}{I} \le 2\sqrt{\frac{w_4}{I}} \ , \tag{26}$$

$$\frac{\widetilde{S}(w_2 + \alpha) + w_2 \widetilde{I}}{\alpha} \le \frac{w_2 w_9}{w_7} \tag{27}$$

hold, then E_2 is globally asymptotically stable in the R_+^4.

Proof:

Let $V_2(S, I, Y_1, Y_2) = \left(S - \widetilde{S} - \widetilde{S}\ln\frac{S}{\widetilde{S}}\right) + \left(I - \widetilde{I} - \widetilde{I}\ln\frac{I}{\widetilde{I}}\right) + \frac{w_2}{w_7}Y_1 + \frac{w_2}{w_7}Y_2$. Clearly $V_2 : R_+^4 \to R$ is C^1 positive definite function. Now, according to conditions (26) it is easy to verify that:

$$\frac{dV_2}{dT} \le -\left(\left(S - \widetilde{S}\right) - \sqrt{\frac{w_4}{I}}\left(I - \widetilde{I}\right)\right)^2 + \left(\widetilde{S} + \frac{w_2(\widetilde{S} + \widetilde{I})}{\alpha} - \frac{w_2 w_9}{w_7}\right)Y_2 - \frac{\widetilde{I}SY_2}{I} - \frac{w_2(w_5 + w_6 - w_8)}{w_7}Y_2.$$

Therefore, $\frac{dV_2}{dT} < 0$ under condition (27), and hence V_2 is strictly Lyapunov function. Therefore, E_2 is globally asymptotically stable in the R_+^4.

Theorem 8. Assume that the disease free equilibrium point E_3 is locally asymptotically stable in the R_+^4. Then E_3 is globally asymptotically stable in the following region

$$\Phi = \left\{(S, I, Y_1, Y_2) : S > \overline{S}, I > \frac{(w_1 + Y_2)\overline{S}}{w_4}, Y_1 < \overline{Y}_1, Y_2 > \frac{(w_2 + \delta_1)\overline{Y}_2}{\delta_1}\right\}.$$

Proof:

Consider the following function:

$$V_3(S,I,Y_1,Y_2)=c_1\left(S-\overline{S}-\overline{S}\ln\tfrac{S}{\overline{S}}\right)+c_2I+c_3\left(Y_1-\overline{Y}_1-\overline{Y}_1\ln\tfrac{Y_1}{\overline{Y}_1}\right)+c_4\left(Y_2-\overline{Y}_2-\overline{Y}_2\ln\tfrac{Y_2}{\overline{Y}_2}\right).$$

Clearly $V_3:R_+^4\to R$ is C^1 positive definite function, where $c_i\,(i=1,2,3,4)$ are positive constants to be determined. Note that, simple mathematical manipulations give that:

$$\begin{aligned}\tfrac{dV_3}{dT}=&-c_1\left(S-\overline{S}\right)^2-\left[c_1I-c_2(w_1+Y_2)\right]S+\left[c_1\overline{S}-c_2w_4\right]I-\tfrac{c_3w_7w_8\overline{S}}{w_9\delta_1Y_1}\left(Y_1-\overline{Y}_1\right)^2-\tfrac{c_2w_2}{\alpha}IY_2\\&-\tfrac{c_4w_9}{Y_2}\left(Y_2-\overline{Y}_2\right)^2-c_1\left[Y_2\left(1+\tfrac{w_2}{\alpha}\right)-\left(\tfrac{\delta_1+w_2}{\delta_1}\right)\overline{Y}_2\right]\left(S-\overline{S}\right)+\tfrac{c_3w_3w_7IY_2}{\alpha\delta_1Y_1}\left(Y_1-\overline{Y}_1\right)\\&+\left[\tfrac{c_3w_7(\delta_1S+\overline{S}I)}{\alpha\delta_1Y_1}+\tfrac{c_4w_8}{Y_2}\right]\left(Y_1-\overline{Y}_1\right)\left(Y_2-\overline{Y}_2\right)+\tfrac{c_3w_3w_7\overline{Y}_2}{\alpha\delta_1Y_1}\left(S-\overline{S}\right)\left(Y_1-\overline{Y}_1\right).\end{aligned}$$

So, choosing the positive constants as $c_1=w_4$, $c_2=\overline{S}$, $c_3=\frac{\delta_1}{w_7}$ and $c_4=1$ gives:

$$\begin{aligned}\tfrac{dV_3}{dT}\le&-w_4\left(S-\overline{S}\right)^2-\left[w_4I-\overline{S}(w_1+Y_2)\right]S-\tfrac{w_8\overline{S}}{w_9Y_1}\left(Y_1-\overline{Y}_1\right)^2\\&-\tfrac{w_9}{Y_2}\left(Y_2-\overline{Y}_2\right)^2-w_4\left[Y_2\left(1+\tfrac{w_2}{\alpha}\right)-\left(\tfrac{\delta_1+w_2}{\delta_1}\right)\overline{Y}_2\right]\left(S-\overline{S}\right)+\tfrac{w_3IY_2}{\alpha Y_1}\left(Y_1-\overline{Y}_1\right)\\&+\left[\tfrac{(\delta_1S+\overline{S}I)Y_2+w_8\alpha Y_1}{\alpha Y_1Y_2}\right]\left(Y_1-\overline{Y}_1\right)\left(Y_2-\overline{Y}_2\right)+\tfrac{w_3\overline{Y}_2}{\alpha Y_1}\left(S-\overline{S}\right)\left(Y_1-\overline{Y}_1\right).\end{aligned}$$

Clearly $\frac{dV_3}{dT}<0$ in Φ , and then V_3 is a strictly Lyapunov function. Therefore E_3 is globally asymptotically stable in the region Φ .

Obviously, Φ in the above theorem represents the basin of attraction for E_3 in R_+^4. Finally, in the following theorem, the conditions of global asymptotic stability of the positive equilibrium point E_4 are established.

Theorem 9. Assume that the positive equilibrium point $E_4=(S^*,I^*,Y_1^*,Y_2^*)$ is locally asymptotically stable in the $Int.R_+^4$ with

$$e_i>0;\qquad \forall i=1,2 \tag{28a}$$

$$\frac{\alpha\delta_1I-2w_2Y_2^*-(w_1+Y_2^*)}{\alpha SI}\ge 2\sqrt{\frac{e_1}{3\alpha\delta_1}}\sqrt{\frac{e_2}{3II^*\alpha\delta_1}} \tag{28b}$$

$$\frac{(S\alpha-w_2I)}{\alpha I\sqrt{\frac{e_2}{3II^*\alpha\delta_1}}}\le 2\sqrt{\frac{w_9}{3Y_2}}\le\frac{(\alpha+w_2)}{\alpha\sqrt{\frac{e_1}{3\alpha\overline{S}}}} \tag{28c}$$

$$\max\left\{\frac{w(S+I)Y_2+w_8\alpha Y_1}{\alpha Y_1 Y_2\sqrt{\frac{w_9}{3Y_2}}},\frac{w_3 w_7 Y_2^*}{\alpha\delta_1 Y_1\sqrt{\frac{e_1}{3\alpha\delta_1}}},\frac{w_3 w_7 Y_2^*}{\alpha\delta Y\sqrt{\frac{e_2}{3II^*\alpha\delta_1}}}\right\}\le 2\sqrt{\frac{w_7 w_8\bar{S}}{3w_9\delta_1 Y_1}}\,, \tag{28d}$$

where $e_1=(\alpha\delta_1-w_2Y_2^*)$ and $e_2=\left(\alpha\delta_1 S^*(w_1+Y_2^*)-w_2 II^* Y_2^*\right)$. Then E_4 is globally asymptotically stable in the $Int.R_+^4$.

Proof:

Consider the following function

$$V_4(S,I,Y_1,Y_2)=\left(S-S^*-S^*\ln\tfrac{S}{S^*}\right)+\left(I-I^*-I^*\ln\tfrac{I}{I^*}\right)$$
$$+\left(Y_1-Y_1^*-Y_1^*\ln\tfrac{Y_1}{Y_1^*}\right)+\left(Y_2-Y_2^*-Y_2^*\ln\tfrac{Y_2}{Y_2^*}\right).$$

Clearly $V_4:R_+^4\to R$ is C^1 positive definite function. Since

$$\frac{dV_4}{dT}=-\left(\tfrac{e_1}{\alpha\delta_1}\right)\left(S-S^*\right)^2-\left(\tfrac{e_2}{II^*\alpha\delta_1}\right)\left(I-I^*\right)^2-\left(\tfrac{w_7w_8\bar{S}}{w_9\delta_1Y_1}\right)\left(Y_1-Y_1^*\right)^2-\tfrac{w_9}{Y_2}\left(Y_2-Y_2^*\right)^2$$
$$-\left(1-\tfrac{2w_2Y_2^*}{\alpha\delta_1}-\tfrac{(w_1+Y_2^*)}{I}\right)\left(S-S^*\right)\left(I-I^*\right)-\left(\tfrac{\alpha+w_2}{\alpha}\right)\left(S-S^*\right)\left(Y_2-Y_2^*\right)$$
$$+\left(\tfrac{S\alpha-w_2I}{I\alpha}\right)\left(I-I^*\right)\left(Y_2-Y_2^*\right)+\left(\tfrac{w_7(S+I)Y_2+w_8\alpha Y_1}{\alpha Y_1Y_2}\right)\left(Y_1-Y_1^*\right)\left(Y_2-Y_2^*\right)$$
$$+\left(\tfrac{w_3w_7Y_2^*}{\alpha\delta_1Y_1}\right)\left(S-S^*\right)\left(Y_1-Y_1^*\right)+\left(\tfrac{w_3w_7Y_2^*}{\alpha\delta_1Y_1}\right)\left(I-I^*\right)\left(Y_1-Y_1^*\right).$$

Therefore, according to conditions (28a-28d), it is easy to verify that:

$$\frac{dV_4}{dT}\le-\left(\sqrt{\tfrac{e_1}{3\alpha\delta_1}}(S-S^*)+\sqrt{\tfrac{e_2}{3II^*\alpha\delta_1}}(I-I^*)\right)^2-\left(\sqrt{\tfrac{e_1}{3\alpha\delta_1}}(S-S^*)+\sqrt{\tfrac{w_9}{3Y_2}}(Y_2-Y_2^*)\right)^2$$
$$-\left(\sqrt{\tfrac{e_2}{3II^*\alpha\delta_1}}(I-I^*)-\sqrt{\tfrac{w_9}{3Y_2}}(Y_2-Y_2^*)\right)^2-\left(\sqrt{\tfrac{w_7w_8\bar{S}}{3w_9\delta_1Y_1}}(Y_1-Y_1^*)-\sqrt{\tfrac{w_9}{3Y_2}}(Y_2-Y_2^*)\right)^2$$
$$-\left(\sqrt{\tfrac{e_1}{3\alpha\delta_1}}(S-S^*)-\sqrt{\tfrac{w_7w_8\bar{S}}{3w_9\delta_1Y_1}}(Y_1-Y_1^*)\right)^2-\left(\sqrt{\tfrac{e_2}{3II^*\alpha\delta_1}}(I-I^*)-\sqrt{\tfrac{w_7w_8\bar{S}}{3w_9\delta_1Y_1}}(Y_1-Y_1^*)\right)^2.$$

So, $\frac{dV_4}{dT}<0$, and then V_4 is strictly Lyapunov function. Therefore, E_4 is globally asymptotically stable in the $Int.R_+^4$.

In the following section, the persistence condition for system (4) is established.

7. Persistence Analysis

In this section, the persistence of system (4) is studied. It is well known that the system is said to persist if and only if each species persists. Mathematically, this means that, system (4) persists if the solution of the system with positive initial condition does not have omega limit set on the boundary of its domain [see Xiao and Chen (2003)]. In the following theorem the persistence conditions of system (4) are established.

Theorem 10. Assume that the equilibrium points E_2 and E_3 are globally asymptotically stable in the interior of $R^2_{+(SI)}$ and $R^3_{+(SY_1Y_2)}$ respectively. In addition, if conditions (9) and (11b) are hold together with the following condition:

$$\lambda_{3I} > 0 \ . \tag{29}$$

Here, λ_{3I} represents the eigenvalue of $J(E_3)$ that describe the dynamics in the positive direction of I. Then system (4) persists.

Proof:

Suppose that q is a point in the $Int.R^4_+$ and $o(q)$ is the orbit through q. Let $\Omega(q)$ is the omega limit set of the $o(q)$. Note that $\Omega(q)$ is bounded, due to the boundedness of system (4).

We first claim that $E_0 \notin \Omega(q)$. Assume the contrary, then since E_0 is a saddle point due to condition (9), thus E_0 cannot be the only point in $\Omega(q)$, and hence by Butler-McGhee lemma [Freedman and Waltman (1984)] there is at least one other point, say p, such that $p \in \omega^s(E_0) \cap \Omega(q)$, where $\omega^s(E_0)$ is the stable manifold of E_0. Now, since $\omega^s(E_0)$ is the space $R^3_{+(IY_1Y_2)}$ and the entire orbit through p, which denoted by $o(p)$, is contained in $\Omega(q)$. Then, if $p \in \partial R^3_{+(IY_1Y_2)}$ (i.e. on the boundary axes of $R^3_{+(IY_1Y_2)}$), we obtain that the positive specific axis (that containing p) is contained in $\Omega(q)$ contradicting its boundedness. Now, let $p \in Int.R^3_{+(IY_1Y_2)}$. Since there is no equilibrium point in the $Int.R^3_{+(IY_1Y_2)}$, the orbit through p which is contained in $\Omega(q)$ must be unbounded. Giving a contradiction too, this shows that $E_0 \notin \Omega(q)$.

Now, we show that $E_1 = (1 - w_1, 0,0,0)$ cannot be in $\Omega(q)$. Since E_1 is saddle point under condition (11b). Then again by Butler-McGhee Lemma $\exists\ p_1 \in \omega^s(E_1) \cap \Omega(x)$, Also, since $\omega^s(E_1)$ could be either the space $R^3_{+(SIY_1)}$ or the space $R^3_{+(SIY_2)}$. Suppose that $\omega^s(E_1)$ is the space $R^3_{+(SIY_1)}$ (similar proof for the space $R^3_{+(SIY_2)}$). Note that, if $p_1 \in \partial R^3_{+(SIY_1)}$, then we get contradiction as in the first part of proof. Let now, $p_1 \in Int.R^3_{+(SIY_1)}$, again since there is no equilibrium point in the $Int.R^3_{+(SIY_1)}$, then the $o(p_1) \subset \Omega(q)$ is unbounded, which gives a contradiction to the boundedness of $\Omega(q)$. Thus, $E_1 \notin \Omega(q)$.

Now, since the points E_2 and E_3 are saddle points in the R^4_+ under the conditions (11b) and (29) respectively. Then by using argument completely analogous to the above yields E_2, E_3 cannot contained in $\Omega(q)$. Thus $\Omega(q)$ must be in the $Int.R^4_+$, which proves persistence of the system (4).

8. Numerical Simulation

In this section the global dynamics of system (4) is studied numerically by solving it, for different sets of parameters and for different sets of initial conditions, using predictor-corrector method with six order Runge-Kutta method, and then the time series for the trajectories of system (4) are drown. Note that, we will use solid line type for S; dash line type for I; dot line type for Y_1 and dash-dot line type for Y_2 in the all of the following figures. Now, for the following set of hypothetical set of parameter values:

$$w_1 = 0.35, w_2 = 0.2, w_3 = 0.2, w_4 = 0.4, w_5 = 0.1, w_6 = 0.3, w_7 = 0.1, w_8 = 0.3, w_9 = 0.1 \qquad (30)$$

It is observed that the equilibrium points E_3 and E_4 do not exist, while E_2 is a global asymptotically stable point, as shown in Figure (1).

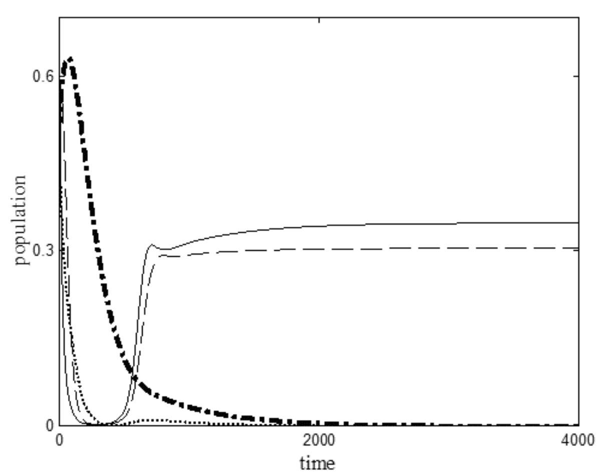

Figure 1: Time series of the trajectories of system (4), for data given in Equation (30), which shows Global asymptotically stable point (0.346, 0.3, 0, 0).

In order to investigate the effect of the infection rate (i.e., parameter w_1) on the dynamics of system (4) in case of existence of E_4, the system is solved numerically for different values of w_1 with $w_2 = 0.4$ and $w_7 = 0.3$, while the rest of parameters kept constant as given in Equation (30) and then the trajectories of system (4) are drown in the Figure (2a-2d).

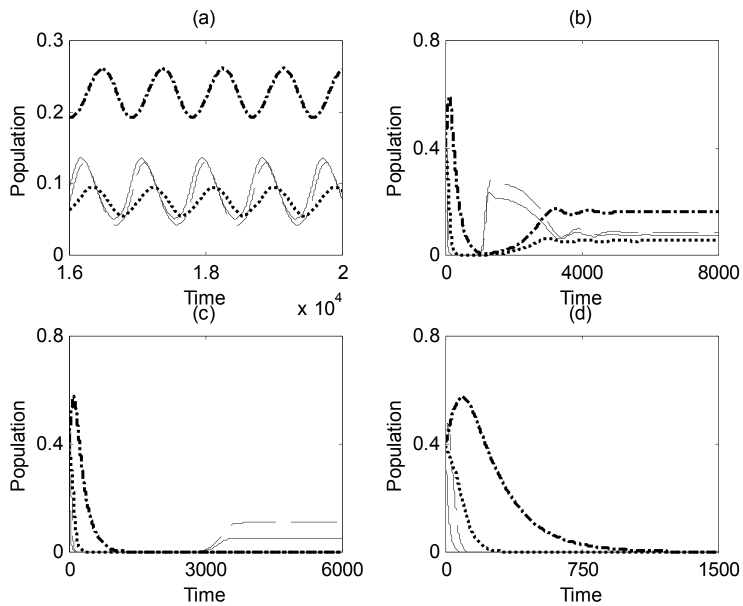

Figure. 2: Time series of the trajectories of system (4), for data given in Equation (30) with $w_2 = 0.4$ and $w_7 = 0.3$, which shows that: (a) Periodic attractor for $w_1 = 0.35$ (b) Global asymptotically stable point $(0.08, 0.078, 0.069, 0.2)$ for $w_1 = 0.5$. (c) Global asymptotically stable point $(0.037, 0.08, 0, 0)$ for $w_1 = 0.84$. (d) Global asymptotically stable point $(0,0,0,0)$ for $w_1 = 1$.

According to the above results, it is observed that the trajectory of system (4) approaches periodic dynamics for $w_1 < 0.39$ as shown in the typical Figure(2a), while it approaches globally asymptotically stable point in the $Int.R_+^4$ for $0.39 \le w_1 < 0.84$ as shown in Figure (2b). However for the $0.84 \le w_1 < 1$ and $1 < w_1$, system (4) losses the persistence and the trajectory approaches to the equilibrium point $(\widetilde{S}, \widetilde{I}, 0, 0) = (0.037, 0.08, 0, 0)$ in the $Int.R_{+(SI)}^2$ and the vanishing equilibrium point $(0,0,0,0)$ respectively as shown in Figures (2c-2d).

Finally, the effect of the conversion rate, at which the immature predator becomes mature (i.e. parameter w_8), on the dynamics of system (4) is investigated. Here the system is solved numerically for the values $w_1 = 0.5$, $w_2 = 0.4$, $w_7 = 0.3$ while the rest of parameters kept constant as given in Equation (30) and then the trajectories of system (4) are drown in the Figures (3) and (4a-4b) for $w_8 = 0.18$, $w_8 = 0.34$ and $w_8 = 0.38$ respectively.

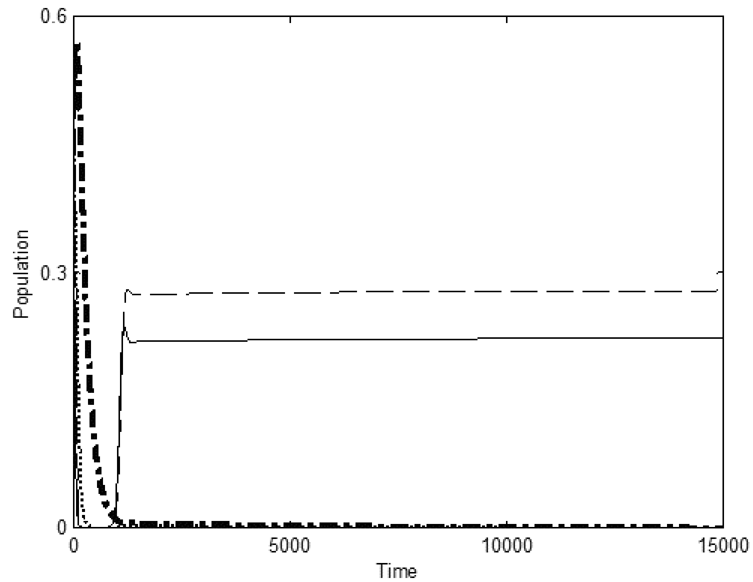

Figure 3: Time series of the trajectories of system (4) with $w_8 = 0.18$, which shows global asymptotically stable point (0.22, 0.27, 0, 0).

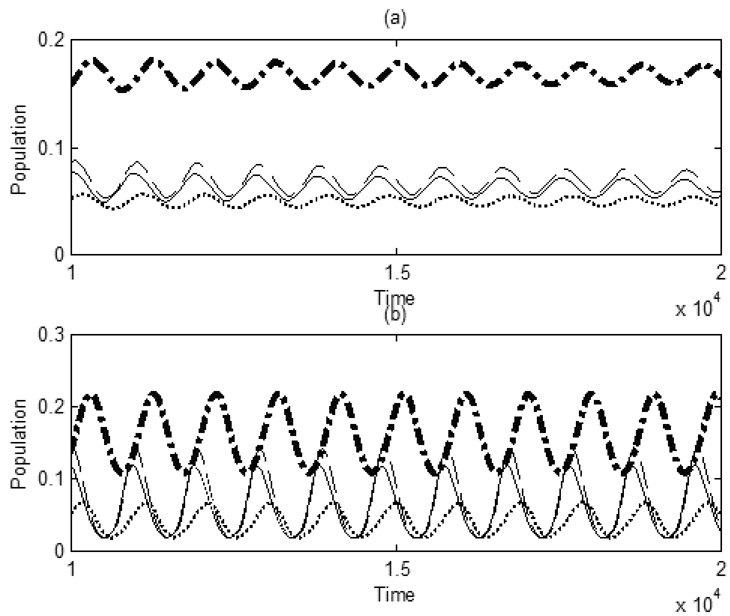

Figure 4: Time series for the trajectories of system (4) which shows that: (a) Periodic attractor for $w_8 = 0.34$. (b) Periodic attractor for $w_8 = 0.38$.

Obviously, as w_8 increases, our numerical analysis shows that for $w_8 \leq 0.18$ the trajectory of system (4) approaches to the equilibrium point $(\widetilde{S}, \widetilde{I}, 0, 0) = (0.22, 0.27, 0, 0)$ in the $Int.R^2_{+(SI)}$ as shown in Figure (3) and hence system (4) is not persists. However the trajectory of system (4) approaches to globally asymptotically stable point for $0.18 < w_8 \le 0.33$ as shown in Figure (2b), while it is approaches to periodic dynamics for $w_8 > 0.33$ as shown in Figures (4a-4b).

9. Conclusions and Discussion

In this paper, an eco-epidemiological model has been proposed and analyzed. Our main objective is to understand the effect of parasite infection disease and stage structure on the dynamics of the prey-predator system. As such, the dynamical behavior of system (4) and all possible subsystems have been investigated analytically. The persistence conditions of system (4) have been derived. Moreover, the effects of changing the parameters on the dynamical behavior of system (4) are discussed numerically and the following results are obtained:

1. For the effect of varying w_1, keeping other parameters fixed as in Equation (30) with $w_2 = 0.4, w_7 = 0.3$, it is obtained that, for small value of infection rate say ($w_1 < 0.39$) the trajectory of system (4) approaches a periodic dynamics in the $Int.R_+^4$. As the infection rate increases $0.39 \le w_1 < 0.84$ the trajectory of system (4) approaches a globally asymptotically stable point in the $Int.R_+^4$. Finally, system (4) losses persistence for $w_1 \ge 0.84$. Consequently, for $w_1 < 0.84$ the disease is under control and the system persists, for $0.84 \le w_1 < 1$ the disease is still under control but the system losses its persistence; finally for $w_1 \ge 1$ the disease is uncontrolled.
2. For the effect of varying w_8, keeping other parameters fixed as, in Equation (30) with $w_1 = 0.5, w_2 = 0.4, w_7 = 0.3$, it is obtained that, for small value of conversion rate say $w_8 \le 0.18$ system (4) losses the persistence, as the conversion rate increases $0.18 < w_8 \le 0.33$ the trajectory of system (4) approaches to globally asymptotically stable point in the $Int.R_+^4$ finally the trajectory of system (4) transfers to periodic dynamics in the $Int.R_+^4$ for $w_8 > 0.33$. Consequently, for $w_8 \le 0.18$, system (4) is not persists, while for $w_8 > 0.18$ the system is persists.

3. Keeping the above in view, mathematically it is observed that, as the infection rate w_1 decreases and passing through the value $w_1 = 0.39$ the positive equilibrium points losses its global stability and the trajectory of system (4) approaches periodic dynamics in the $Int.R_+^4$ and hence a hopf bifurcation occurs at this value. Similarly as shown in Figure (4), a hopf bifurcation occurs as the value of conversion rate passes through $w_8 = 0.33$.

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